KYBERNETIKA — VOLUME 39 (2003), NUMBER 6, PAGES 719–729

CENTRAL LIMIT THEOREM FOR RANDOM MEASURES GENERATED BY STATIONARY PROCESSES OF COMPACT SETS

ZBYNĚK PAWLAS

Random measures derived from a stationary process of compact subsets of the Euclidean space are introduced and the corresponding central limit theorem is formulated. The result does not require the Poisson assumption on the process. Approximate confidence intervals for the intensity of the corresponding random measure are constructed in the case of fibre processes.

Keywords: central limit theorem, fibre process, point process, random measure, space of compact sets

AMS Subject Classification: 60D05, 60F05, 60G57

1. INTRODUCTION

Stochastic geometry is a part of mathematics which deals with random geometrical structures. Point processes play a fundamental role in stochastic geometry. Replacing ordinary points by compact sets, we obtain processes of compact sets. Random patterns of more complicated geometrical objects can be studied in this way. It is possible to associate a measure with compact sets. The sum of contributions of this measure of all observable sets defines a random measure.

Only stationary processes are considered in this paper. A process is stationary if its characteristics are invariant under translations. The simplest parameter of the random measure derived from the stationary point process is its intensity. We mention two unbiased estimators of the intensity and study their asymptotic properties as the observation window expands to the whole space.

A central limit theorem was established in the case of the stationary Poisson process of compact sets in [8]. The aim of this work is to formulate a similar theorem, which does not require the Poisson assumption. It is shown that the central limit theorem for a stationary process of compact sets follows from the asymptotic normality of the underlying point process of reference points.

A suitable tool for establishing the central limit theorem for the number of points of a point process is provided by verifying mixing conditions and using a central limit theorem for stationary mixing random fields (see [5], [6], [7]).

Statistical applications of the main theorem are discussed at the end of the paper. In the special case of a stationary fibre process, asymptotic approximate confidence intervals are constructed.

2. STATIONARY INDEPENDENTLY MARKED POINT PROCESSES

In this section we summarize basic definitions from the theory of point processes and random measures, for more details see [1] and [10].

By $(\mathbb{R}^d, \mathcal{B}^d)$ denote the d-dimensional Euclidean space with Borel σ-algebra. We write $\mathcal{B}_0^d$ for a family of bounded Borel sets in $\mathbb{R}^d$. Let $\mathcal{M}$ be the space of locally finite Borel measures μ on $\mathbb{R}^d$ (i.e. $\mu(B) < \infty$ for every $B \in \mathcal{B}_0^d$) and let $\mathfrak{M}$ be the smallest σ-algebra on $\mathcal{M}$ making the mappings $\mu \mapsto \mu(B)$ measurable for all $B \in \mathcal{B}^d$. A random measure on $\mathbb{R}^d$ is a random element in $(\mathcal{M}, \mathfrak{M})$, i.e. a measurable mapping $\Psi : (\Omega, \mathcal{A}, P) \to (\mathcal{M}, \mathfrak{M})$, where $(\Omega, \mathcal{A}, P)$ is an abstract probability space. Note that $\Psi(B)$ is a random variable for each fixed $B \in \mathcal{B}^d$. The distribution $Q = P\Psi^{-1}$ of the random measure Ψ is the induced probability measure on $(\mathcal{M}, \mathfrak{M})$ such that $Q(U) = P(\Psi \in U)$. The intensity measure of Ψ is a Borel measure on $\mathbb{R}^d$ defined as $\Lambda(B) = \mathrm{E}\Psi(B)$.

Further, let

$$\mathcal{N} = \{\mu \in \mathcal{M} : \mu(B) \in \mathbb{N} \cup \{0, \infty\},\ B \in \mathcal{B}^d\}$$

be the space of locally finite counting measures equipped with σ-algebra $\mathfrak{N}$ which is defined as the trace of $\mathfrak{M}$, i.e. $\mathfrak{N} = \{\mathfrak{M} \cap \mathcal{N} : \mathfrak{M} \in \mathfrak{M}\}$. A random element Φ in the space $(\mathcal{N}, \mathfrak{N})$ is called a point process. Obviously, the point process is a special case of the random measure. A point process is called simple if $P(\Phi \in \mathcal{N}^*) = 1$, where

$$\mathcal{N}^* = \{\nu \in \mathcal{N} : \nu(\{x\}) \leq 1\ \forall x \in \mathbb{R}^d\}.$$

The moment measures of higher orders for point processes can be introduced as follows. The kth-order factorial moment measure of the point process Φ is defined by

$$M^{(k)}(B) = \mathrm{E}\Phi^k(\{(x_1, \dots, x_k) \in B : x_i \neq x_j \text{ for } i \neq j\}), \quad B \in (\mathcal{B}^d)^k.$$

Further, the kth-order factorial cumulant measures are given by (see (5.5.15) in [1])

$$\gamma^{(k)}(A_1 \times \dots \times A_k) = \sum_{j=1}^{k} (-1)^{j-1} (j-1)! \sum_{\mathcal{T} \in \mathcal{P}_{jk}} \prod_{i=1}^{j} M^{(|S_i(\mathcal{T})|)}(A_{i,1} \times \dots \times A_{i,|S_i(\mathcal{T})|}),$$

where $\mathcal{T} \in \mathcal{P}_{jk}$ is the partition of the set $\{1, \dots, k\}$ into j sets $S_1(\mathcal{T}), \dots, S_j(\mathcal{T})$. The first-order factorial moment measure and the first-order factorial cumulant measure coincide with the intensity measure. The second-order factorial cumulant measure is called the factorial covariance measure.

For $x \in \mathbb{R}^d$ let t_x be the shift operator on $\mathcal{M}$:

$$t_x \mu(B) = \mu(B - x),\ B \in \mathcal{B}^d.$$

A random measure is stationary if its distribution Q is translation invariant, i.e. $Qt_x^{-1} = Q$ for all $x \in \mathbb{R}^d$. If the intensity measure of a stationary random measure is locally finite then it is a multiple of d-dimensional Lebesgue measure. This multiple is called the intensity of the random measure.

For a stationary point process Φ with intensity λ, the kth-order reduced factorial cumulant measure $\gamma_{red}^{(k)}$ is defined by the desintegration (see [1], Lemma 10.4.III)

$$\int_{(\mathbb{R}^d)^k} f(x_1, \dots, x_k)\gamma^{(k)}(\mathrm{d}x_1, \dots, \mathrm{d}x_k) =$$

$$= \lambda \int_{\mathbb{R}^d} \int_{(\mathbb{R}^d)^{k-1}} f(x, x + y_1, \dots, x + y_{k-1})\gamma_{red}^{(k)}(\mathrm{d}y_1, \dots, \mathrm{d}y_{k-1})\,\mathrm{d}x,$$

where f is an arbitrary bounded measurable function with bounded support. The total variation of the signed measure $\gamma_{red}^{(k)}$ is denoted by $|\gamma_{red}^{(k)}|$.

Let $(\mathcal{K}', d_H)$ be the space of non-empty compact subsets of $\mathbb{R}^d$ endowed with the Hausdorff metric

$$d_H(K, L) = \max\left\{\sup_{x\in K} d(x, L), \sup_{y\in L} d(y, K)\right\}, \quad K, L \in \mathcal{K}',$$

where $d(x, L) = \inf_{z\in L} \|x - z\|$ is the distance from the point x to the set L. Further, let $\mathcal{K}'_0$ be the space of sets from $\mathcal{K}'$ which have the lexicographic minimum point at the origin, $\mathcal{K}'_0$ is the closed subset of $\mathcal{K}'$. Throughout the paper, by a stationary process of compact sets we will mean the marked point process (see Chapter 4.2 in [10])

$$\Phi_m = \sum_{i:i\geq 1} \delta_{(x_i, K_i)},$$

such that the corresponding process of unmarked points $\Phi = \sum_{i:i\geq 1} \delta_{x_i}$ is a simple stationary point process with a finite intensity $\lambda_\Phi > 0$ and the marks $\{K_i, i \geq 1\}$ are independent identically distributed copies of a random compact set K_0 (random element in the space $\mathcal{K}'_0$), independent of the process Φ. The distribution of K_0 will be denoted by Λ_0 (called a distribution of the typical mark). For notational simplicity, we write $\mathrm{E}_{\Lambda_0} f(K_0) = \int_{\mathcal{K}'_0} f(K_0)\,\Lambda_0(\mathrm{d}K_0)$, where f is an arbitrary measurable function.

Let ζ be an arbitrary translation invariant Borel measure on $\mathbb{R}^d$ such that $K \mapsto \zeta(K)$ is a measurable mapping from $\mathcal{K}'$.

Put

$$\Psi(B) = \sum_{i:i\geq 1} \zeta((x_i + K_i) \cap B), \quad B \in \mathcal{B}_0^d. \tag{1}$$

Assume that $\mathrm{E}_{\Lambda_0}\zeta(K_0) < \infty$. Then Ψ is a stationary random measure on $\mathbb{R}^d$ with the intensity

$$\lambda_\Psi = \lambda_\Phi \int \zeta(K_0)\,\Lambda_0(\mathrm{d}K_0) = \lambda_\Phi \mathrm{E}_{\Lambda_0}\zeta(K_0). \tag{2}$$

This formula can be easily deduced from Campbell's theorem for marked point processes (see (4.2.4) in [10]) together with the translation invariance of ζ and Fubini's theorem.

3. CENTRAL LIMIT THEOREMS FOR RANDOM SUMS

Let $\xi_1, \xi_2, \ldots$ be independent identically distributed random variables with the mean μ and the finite variance σ^2. Denote $S_n = \sum_{i=1}^{n} \xi_i$, $n \in \mathbb{N}$.

The well-known Lévy–Lindeberg central limit theorem states that

$$\frac{S_n - n\mu}{\sqrt{n}} \stackrel{n\to\infty}{\longrightarrow} N(0, \sigma^2) \quad \text{in distribution.}$$

When the number N_n of summands is random such that $N_n \stackrel{n\to\infty}{\longrightarrow} \infty$ in probability, the convergence to a Gaussian limit was first considered in the classical work of H. Robbins [9]. More general versions of limit theorems for normalized random variables S_{N_n} can be found in [3]. We will use the central limit theorem for random sums in the following form:

Theorem 1. Let N_n be integer positive random variables independent of the sequence ξ_i for every $n \in \mathbb{N}$. Let a_n be a sequence of real numbers such that $a_n \stackrel{n\to\infty}{\longrightarrow} \infty$ and

$$\frac{N_n}{a_n} \stackrel{n\to\infty}{\longrightarrow} \theta \quad \text{in probability, and} \quad \frac{N_n - \mathrm{E}N_n}{\sqrt{a_n}} \stackrel{n\to\infty}{\longrightarrow} N(0, \sigma_N^2) \quad \text{in distribution,}$$

where $\theta > 0$ is a real constant. Then

$$\frac{S_{N_n} - \mu \mathrm{E}N_n}{\sqrt{a_n}} \stackrel{n\to\infty}{\longrightarrow} N(0, \theta\sigma^2 + \mu^2\sigma_N^2) \quad \text{in distribution.}$$

4. THE CENTRAL LIMIT THEOREM

We are now in the position to formulate and to prove the main result of this work. Let Ψ be a random measure generated by a stationary process of compact sets Φ_m. The unbiased estimator of the intensity λ_Ψ is $\frac{\Psi(W)}{|W|}$, where $|W|$ denotes the d-dimensional Lebesgue measure of W. The asymptotic normality of this estimator is guaranteed by the central limit theorem for the unmarked point process Φ together with conditions on second-order properties of the marked process Φ_m, which ensure the existence of the variance of the estimator.

We consider that the window W expands to the whole space in a regular way. The sequence of bounded sampling windows $W_n \in \mathcal{B}_0^d$ is a convex averaging sequence if it satisfies the following three conditions (see Definition 10.2.I in [1]):

1. W_n are convex,
2. $W_n \subseteq W_{n+1}$,
3. $\sup\{r : W_n \text{ contains a ball of radius } r\} \stackrel{n\to\infty}{\longrightarrow} \infty$.

Theorem 2. Let $W_n \subseteq \mathbb{R}^d$ be the convex averaging sequence. Assume that

$$\mathrm{E}_{\Lambda_0}\zeta(K_0)^2 = \int_{\mathcal{K}_0'} \zeta(K_0)^2\, \Lambda_0(\mathrm{d}K_0) < \infty \tag{3}$$

and the reduced factorial covariance measure $\gamma_{red}^{(2)}$ has bounded total variation, i.e.

$$|\gamma_{red}^{(2)}|(\mathbb{R}^d) < \infty. \tag{4}$$

If

$$\sqrt{|W_n|}\left(\frac{\Phi(W_n)}{|W_n|} - \lambda_\Phi\right) \overset{n\to\infty}{\longrightarrow} N(0,\sigma_\Phi^2) \quad \text{in distribution,} \tag{5}$$

where $\sigma_\Phi^2 = \lambda_\Phi(1+\gamma_{red}^{(2)}(\mathbb{R}^d))$, then we have

$$\sqrt{|W_n|}\left(\frac{\Psi(W_n)}{|W_n|} - \lambda_\Psi\right) \overset{n\to\infty}{\longrightarrow} N(0,\sigma_\Psi^2) \quad \text{in distribution,}$$

where $\sigma_\Psi^2 = \lambda_\Phi \left(\mathrm{E}_{\Lambda_0}\zeta(K_0)^2 + (\mathrm{E}_{\Lambda_0}\zeta(K_0))^2\gamma_{red}^{(2)}(\mathbb{R}^d)\right)$.

Proof. From the definition of moment measures we get

$$\begin{aligned}\operatorname{var}\Phi(W_n) &= \lambda_\Phi|W_n| + \gamma^{(2)}(W_n\times W_n) = \lambda_\Phi|W_n| + \lambda_\Phi\int_{W_n}\gamma_{red}^{(2)}(W_n - x)\,\mathrm{d}x\\ &\le \lambda_\Phi|W_n|(1+|\gamma_{red}^{(2)}|(\mathbb{R}^d)).\end{aligned}$$

Applying Chebyshev's inequality it is easily shown that $\frac{\Phi(W_n)}{|W_n|} \overset{n\to\infty}{\longrightarrow} \lambda_\Phi$, in probability. Thus, using Theorem 1 we obtain

$$\frac{1}{\sqrt{|W_n|}}\left(\sum_{i=1}^{\Phi(W_n)}\zeta(K_i) - \lambda_\Psi|W_n|\right) \overset{n\to\infty}{\longrightarrow} N(0,\sigma_\Psi^2), \quad \text{in distribution,}$$

where $\sigma_\Psi^2 = \lambda_\Phi \operatorname{var}\zeta(K_0) + (\mathrm{E}\zeta(K_0))^2\sigma_\Phi^2 = \lambda_\Phi(\mathrm{E}\zeta(K_0)^2 + (\mathrm{E}\zeta(K_0))^2\gamma_{red}^{(2)}(\mathbb{R}^d)$.

By Slutzky's theorem it remains to show that

$$\frac{1}{\sqrt{|W_n|}}\sum_{i\ge 1}\Big(\zeta((x_i+K_i)\cap W_n) - \mathbf{1}_{W_n}(x_i)\zeta(K_i)\Big) \overset{n\to\infty}{\longrightarrow} 0, \quad \text{in probability.}$$

The left-hand side can be rewritten as

$$\frac{1}{\sqrt{|W_n|}}\sum_{i\ge1}\mathbf{1}_{W_n^c}(x_i)\zeta((x_i+K_i)\cap W_n) - \frac{1}{\sqrt{|W_n|}}\sum_{i\ge1}\mathbf{1}_{W_n}(x_i)\zeta((x_i+K_i)\cap W_n^c) = X_n - Y_n.$$

The expectation of the difference $X_n - Y_n$ is equal to zero. In order to accomplish the proof, it suffices to verify that $\operatorname{var} X_n + \operatorname{var} Y_n \overset{n\to\infty}{\longrightarrow} 0$. Campbell's theorem and the definition of reduced factorial cumulant measures yields

$$\begin{aligned}
\operatorname{var} X_n &= \frac{1}{\sqrt{|W_n|}} \sum_{x_i \in W_n^c} \zeta((x_i + K_i) \cap W_n) \\
&= \frac{1}{|W_n|} \Bigg(\mathrm{E} \sum_{x_i \in W_n^c} \zeta((x_i + K_i) \cap W_n)^2 \\
&\qquad + \mathrm{E} \sum_{x_i \neq x_j \in W_n^c} \zeta((x_i + K_i) \cap W_n) \zeta((x_j + K_j) \cap W_n) \\
&\qquad - \Bigg(\mathrm{E} \sum_{x_i \in W_n^c} \zeta((x_i + K_i) \cap W_n) \Bigg)^2 \Bigg) \\
&= \frac{\lambda_\Phi}{|W_n|} \Bigg(\mathrm{E}_{\Lambda_0} \int_{W_n^c} \zeta((x + K_0) \cap W_n)^2 \,\mathrm{d}x \\
&\quad + \mathrm{E}_{\Lambda_0} \int_{W_n^c} \int_{W_n^c - x} \zeta((x + K_0) \cap W_n) \mathrm{E}_{\Lambda_0} \zeta((x + y + K_1) \cap W_n) \,\gamma_{red}^{(2)}(\mathrm{d}y) \,\mathrm{d}x \Bigg) \\
&\le \lambda_\Phi \mathrm{E}_{\Lambda_0} \iint \mathbf{1}_{K_0}(y_1) \mathbf{1}_{K_0}(y_2) \frac{|W_n^c \cap (W_n - y_1) \cap (W_n - y_2)|}{|W_n|} \,\zeta(\mathrm{d}y_1) \,\zeta(\mathrm{d}y_2) \\
&\quad + \lambda_\Phi \mathrm{E}_{\Lambda_0} \mathrm{E}_{\Lambda_0} \iiint \mathbf{1}_{K_0}(y_1) \mathbf{1}_{K_1}(y_2) \frac{|W_n^c \cap (W_n - y_1) \cap (W_n - y - y_2)|}{|W_n|} \,\gamma_{red}^{(2)}(\mathrm{d}y) \,\zeta(\mathrm{d}y_1) \,\zeta(\mathrm{d}y_2)
\end{aligned}$$

Making use of the assumptions (3) and (4) and the fact that (see [2])

$$\frac{|W_n \cap (W_n - x)|}{|W_n|} \overset{n\to\infty}{\longrightarrow} 1 \quad \text{for any fixed } x \in \mathbb{R}^d,$$

an immediate consequence of Lebesgue dominated convergence theorem is that $\operatorname{var} X_n \overset{n\to\infty}{\longrightarrow} 0$. Quite similar arguments lead to $\operatorname{var} Y_n \overset{n\to\infty}{\longrightarrow} 0$. This completes the proof. □

The assumption (5) is fulfilled for the stationary Poisson point process Φ (random variables $\Phi(W_n)$ have the Poisson distribution). For general stationary point process Φ, the validity of the central limit theorem (5) is ensured if Φ satisfies strong mixing conditions (e.g. β-mixing [5], [6] or Brillinger-mixing [7]). Under mild additional assumptions it is known that this is the case for quite a few classes of point processes – processes derived from Poisson point process, Gibbs point processes under Dobrushin's uniqueness condition, point processes generated by a Voronoi tesselation (e.g. vertices or midpoints of edges).

We will consider a doubly stochastic Poisson process (Cox process). Let Λ be a random measure on $\mathbb{R}^d$ with a distribution Q on $(\mathcal{M}, \mathfrak{M})$ and P_μ be a distribution of the Poisson process with the intensity measure μ. Then the Cox process Φ with driving random measure Λ has distribution (see Chapter 5.2 in [10])

$$Q_\Phi(U) = \int P_\mu(U) \, Q(\mathrm{d}\mu), \quad U \in \mathcal{N}.$$

The intensity measure of Cox process and the intensity measure of Λ are equal. If Λ is stationary, Φ is stationary as well.

The following theorem shows that the condition (5) follows from the central limit theorem for a driving measure Λ, see [3] and [4] for a one-dimensional version.

Theorem 3. Let Φ be a stationary Cox process controlled by a random measure Λ with intensity λ_Λ. Assume that

$$\frac{\Lambda(W_n) - \lambda_\Lambda |W_n|}{\sqrt{|W_n|}} \overset{n\to\infty}{\longrightarrow} N(0, \sigma_\Lambda^2), \quad \text{in distribution.} \tag{6}$$

Then

$$\frac{\Phi(W_n) - \lambda_\Lambda |W_n|}{\sqrt{|W_n|}} \overset{n\to\infty}{\longrightarrow} N(0, \sigma_\Phi^2), \quad \text{in distribution,}$$

where $\sigma_\Phi^2 = \sigma_\Lambda^2 + \lambda_\Lambda$.

P r o o f. The proof is based on the formula for the characteristic function of Poisson process (see (6.4.6) and (7.4.10) in [1])

$$\int \mathrm{e}^{it\nu(W)}\, P_\mu(\mathrm{d}\nu) = \exp\{\mu(W)(\mathrm{e}^{it} - 1)\}, \quad W \in \mathcal{B}^d.$$

Then, for the Cox process we have

$$\begin{aligned} \mathrm{E}\mathrm{e}^{it\Phi(W_n)} &= \int \mathrm{e}^{it\nu(W_n)}\, Q_\Phi(\mathrm{d}\nu) = \int\int \mathrm{e}^{it\nu(W_n)}\, P_\mu(\mathrm{d}\nu)\, Q(\mathrm{d}\mu) \\ &= \mathrm{E}\exp\{\Lambda(W_n)(\mathrm{e}^{it} - 1)\} = \varphi_n\left(\frac{\mathrm{e}^{it} - 1}{i}\right), \end{aligned}$$

where φ_n is the characteristic function of $\Lambda(W_n)$. From (6) it follows that

$$\mathrm{e}^{-it\lambda_\Lambda\sqrt{|W_n|}}\varphi_n\left(\frac{t}{\sqrt{|W_n|}}\right) \overset{n\to\infty}{\longrightarrow} \mathrm{e}^{-\frac{t^2\sigma_\Lambda^2}{2}}.$$

Consequently,

$$\exp\{-\lambda_\Lambda|W_n|\left(\mathrm{e}^{\frac{it}{\sqrt{|W_n|}}} - 1\right)\}\varphi_n\left(\frac{\mathrm{e}^{\frac{it}{\sqrt{|W_n|}}} - 1}{i}\right) \overset{n\to\infty}{\longrightarrow} \mathrm{e}^{-\frac{t^2\sigma_\Lambda^2}{2}}.$$

Next, we use a Taylor expansion of $\exp\left\{\frac{it}{\sqrt{|W_n|}}\right\}$ and get

$$\mathrm{e}^{-it\lambda_\Lambda\sqrt{|W_n|}}\varphi_n\left(\frac{\mathrm{e}^{\frac{it}{\sqrt{|W_n|}}} - 1}{i}\right) \overset{n\to\infty}{\longrightarrow} \exp\{-\frac{1}{2}t^2\left(\sigma_\Lambda^2 + \lambda_\Lambda\right)\}.$$

This completes the proof because the term on left-hand side is equal to $\mathrm{E}\exp\{it\frac{\Phi(W_n) - \lambda_\Lambda|W_n|}{\sqrt{|W_n|}}\}$. □

5. FIBRE PROCESS

If $\Phi_m = \sum_{i:i\geq 1} \delta_{(x_i,K_i)}$ is a marked point process with a mark space $\mathcal{K}_0'$, the corresponding set-theoretic union

$$\Xi = \bigcup_{i:i\geq 1} (x_i + K_i)$$

is called a germ-grain model (see Chapter 6.4 in [10]). The points x_i are called germs and the compact sets K_i are called grains. If the point process of germs is Poisson, the germ-grain model is the well-known Boolean model. A central limit theorem for the random measure associated with the Boolean model is derived in [8].

For this statistical analysis, only an observation of the germ-grain model in a sampling window is available. Typically, grains overlap and it is not possible to evaluate the associated random measure Ψ defined by (1). Therefore, we restrict our considerations to lower-dimensional grains. The most usual examples are fibre and surface processes (see [10], Chapter 9).

In what follows we consider fibre processes. The measure ζ is taken to be the one-dimensional Hausdorff measure H^1. By a fibre K we mean a compact connected set K such that $H^1(K) < \infty$. Suppose that Φ_m is a stationary fibre process and Ψ is the associated random measure. Then Ψ is the total sum of lengths of fibres observable in the sampling window. The intersection of any two different shifted grains (fibres) has ζ-measure zero. Thus, Ψ can be evaluated. From Theorem 2 we know that $\Psi(B)$ is asymptotically normal distributed.

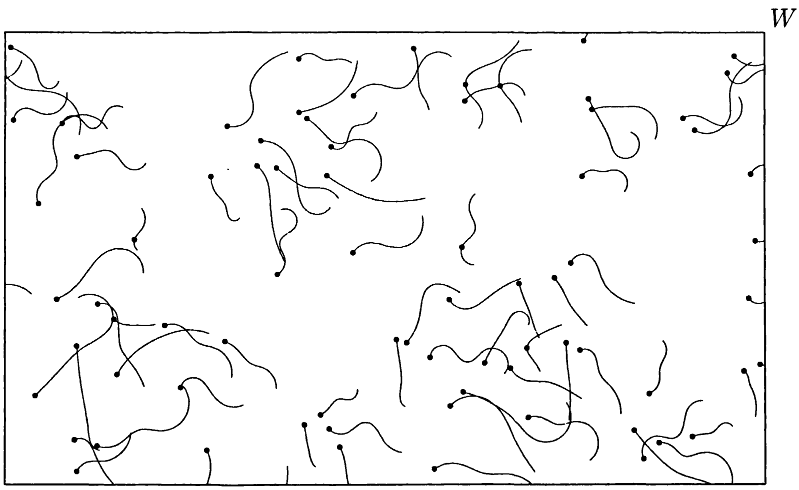

Fig. 1. An example of a realization of a stationary fibre process in a planar window W with denoted reference points.

The intensity λ_Ψ of Ψ is called the length intensity of a stationary fibre process. Recall that by (2) $\lambda_\Psi = \lambda_\Phi \mathrm{E}_{\Lambda_0} H^1(K_0)$ is the product of the intensity of the process and the mean length of fibre. The usual unbiased estimator of the length intensity is

$$\hat{\lambda}^{(1)}_{\Psi,n} = \frac{\Psi(W_n)}{|W_n|}.$$

Under the assumptions of Theorem 2 it follows

$$\sqrt{|W_n|}\left(\hat{\lambda}^{(1)}_{\Psi,n} - \lambda_\Psi\right) \overset{n\to\infty}{\longrightarrow} N(0,\sigma_1^2), \quad \text{in distribution,} \tag{7}$$

where $\sigma_1^2 = \lambda_\Phi \operatorname{var} H^1(K_0) + (\mathrm{E}_{\Lambda_0} H^1(K_0))^2 \sigma_\Phi^2$.

If $\mathrm{E}_{\Lambda_0} H^1(K_0)$ is known, it suffices to estimate λ_Φ. Then we can define another unbiased estimator of λ_Ψ which is based on the number of germs (reference points) lying in the sampling window

$$\hat{\lambda}^{(2)}_{\Psi,n} = \frac{\Phi(W_n)}{|W_n|}\mathrm{E}_{\Lambda_0} H^1(K_0).$$

Since we assume (5), we have

$$\sqrt{|W_n|}\left(\hat{\lambda}^{(2)}_{\Psi,n} - \lambda_\Psi\right) \overset{n\to\infty}{\longrightarrow} N(0,\sigma_2^2), \quad \text{in distribution,} \tag{8}$$

where $\sigma_2^2 = \sigma_\Phi^2(\mathrm{E}_{\Lambda_0} H^1(K_0))^2$.

6. STATISTICAL APPLICATIONS

Central limit theorems enable the construction of the asymptotic confidence intervals or the testing of hypotheses. These require the asymptotic variances to be known. In (7) and (8), asymptotic variances of the estimators $\hat{\lambda}^{(i)}_{\Psi,n} (i = 1,2)$ are unknown. Our aim is to construct asymptotically unbiased and consistent estimators for $\sigma_i^2, i = 1,2$.

We assume in this section that sampling windows have the form $W_n = [-n,n]^d$. Let $G : \mathbb{R}^d \to \mathbb{R}^1$ be a symmetric non-negative bounded function with the support in W_1 and $\lim_{\|x\|\to 0} G(x) = G(0) = 1$. Assume that b_n is a sequence of positive numbers (bandwidths) such that $b_1 = 1$, $b_n \searrow 0$, $nb_n \to \infty$ and $n^{d-1}b_n^d \to 0$. Put

$$G_n = (nb_n)^d \int_{\mathbb{R}^d} G(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} G\left(\frac{x}{nb_n}\right)\,\mathrm{d}x.$$

In addition to (4), assume that the reduced factorial cumulant measures $\gamma^{(3)}_{red}$ and $\gamma^{(4)}_{red}$ are also of bounded variation.

The sequence of estimators of the variance σ_Φ^2 was introduced in [5], namely

$$\hat{\sigma}^2_{\Phi,n} = \sum_{x,y\in \operatorname{supp}\Phi\cap W_n} \frac{G(\frac{x-y}{nb_n})}{|(W_n - x)\cap(W_n - y)|} - (nb_n)^d \int_{\mathbb{R}^d} G(x)\,\mathrm{d}x \cdot \left(\frac{\Phi(W_n)}{|W_n|}\right)^2.$$

It was shown that under the above assumptions these estimators are asymptotically unbiased and

$$\mathrm{E}\left(\hat{\sigma}^2_{\Phi,n} - \sigma^2_\Phi\right)^2 \overset{n\to\infty}{\longrightarrow} 0.$$

Using the same idea, we can construct the sequence of estimators of the asymptotic variance σ_1^2

$$\hat{\sigma}_{1,n}^2 = \sum_{x_i, x_j \in \operatorname{supp} \Phi} \frac{G(\frac{x_i - x_j}{nb_n}) \mathbf{1}_{W_n}(x_i) \mathbf{1}_{W_n}(x_j)}{|(W_n - x_i) \cap (W_n - x_j)|} H^1(K_i \cap W_n) H^1(K_j \cap W_n) - G_n \cdot \left(\frac{\Psi(W_n)}{|W_n|} \right)^2 .$$

A lengthy calculation yields that $\hat{\sigma}_{1,n}$ are again asymptotically unbiased and

$$\mathrm{E} \left(\hat{\sigma}_{1,n}^2 - \sigma_1^2 \right)^2 \overset{n \to \infty}{\longrightarrow} 0.$$

Since $\hat{\sigma}_{1,n}^2$ and $\hat{\sigma}_{2,n}^2 = \hat{\sigma}_{\Phi,n}^2 (\mathrm{E}_{\Lambda_0} H^1(K_0))^2$ are consistent estimators for σ_1^2 and σ_2^2, respectively, we obtain from (7) and (8)

$$\sqrt{\frac{|W_n|}{\hat{\sigma}_{i,n}^2}} \left(\hat{\lambda}_{\Psi,n}^{(i)} - \lambda_\Psi \right) \overset{n \to \infty}{\longrightarrow} N(0,1), \quad \text{in distribution, } i = 1, 2.$$

This yields the approximate $100(1 - \alpha)\%$ confidence intervals for the unknown intensity λ_Ψ

$$\left(\hat{\lambda}_{\Psi,n}^{(i)} - u_{\alpha/2} \frac{\hat{\sigma}_{i,n}}{\sqrt{|W_n|}}, \hat{\lambda}_{\Psi,n}^{(i)} + u_{\alpha/2} \frac{\hat{\sigma}_{i,n}}{\sqrt{|W_n|}} \right), \quad i = 1, 2,$$

where the quantile $u_{\alpha/2}$ is determined such that $P(|X| \leq u_{\alpha/2}) = 1 - \alpha$ and X has standard Gaussian distribution $N(0,1)$.

ACKNOWLEDGEMENT

I would like to thank Professor Lothar Heinrich for his help and useful suggestions. The research was supported by the Grant Agency of the Academy of Sciences of the Czech Republic under Grant IAA 1075201 "Asymptotics of Spatial Random Processes" and by the Ministry of Education, Youth and Sports of the Czech Republic under Grant MSM 113200008 "Mathematical Methods in Stochastics".

(Received April 24, 2003.)

REFERENCES

[1] D. J. Daley and D. Vere-Jones: An Introduction to the Theory of Point Processes. Springer-Verlag, New York 1988.

[2] J. Fritz: Generalization of McMillan's theorem to random set functions. Studia Sci. Math. Hungar. *5* (1970), 369–394.

[3] B. V. Gnedenko and V. Y. Korolev: Random Summation: Limit Theorems and Applications. CRC Press, Boca Raton 1996.

[4] J. Grandell: Doubly Stochastic Poisson Processes. (Lecture Notes in Mathematics 529.) Springer-Verlag, Berlin 1976.

[5] L. Heinrich: Normal approximation for some mean-value estimates of absolutely regular tessellations. Math. Methods Statist. *3* (1994), 1–24.

[6] L. Heinrich and I. S. Molchanov: Central limit theorem for a class of random measures associated with germ-grain models. Adv. in Appl. Probab. *31* (1999), 283–314.

[7] L. Heinrich and V. Schmidt: Normal convergence of multidimensional shot noise and rates of this convergence. Adv. in Appl. Probab. *17* (1985), 709–730.

[8] Z. Pawlas and V. Beneš: On the central limit theorem for the stationary Poisson process of compact sets. Math. Nachr. (2003), to appear.

[9] H. Robbins: The asymptotic distribution of the sum of a random number of random variables. Bull. Amer. Math. Soc. *54* (1948), 1151–1161.

[10] D. Stoyan, W. S. Kendall, and J. Mecke: Stochastic Geometry and Its Applications. Second edition. Wiley, New York, 1995.

Mgr. Zbyněk Pawlas, Department of Probability and Mathematical Statistics, Faculty of Mathematics and Physics – Charles University, Sokolovská 83, 186 75 Praha 8, and Institute of Information Theory and Automation – Academy of Sciences of the Czech Republic, Pod Vodárenskou věží 4, 182 08 Praha 8. Czech Republic.
e-mail: pawlas@karlin.mff.cuni.cz