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THE $\mathrm{d}X(t) = Xb(X)\,\mathrm{d}t + X\sigma(X)\,\mathrm{d}W$ EQUATION AND FINANCIAL MATHEMATICS II

JOSEF ŠTĚPÁN AND PETR DOSTÁL

This paper continues the research started in [5]. Considering a stock price $X(t)$ born by the above semilinear SDE with $\sigma(x,t) = \tilde{\sigma}(x(t))$, we suggest two methods how to compute the price of a general option $g(X(T))$. The first, a more universal one, is based on a Monte Carlo procedure while the second one provides explicit formulas. We in this case need an information on the two dimensional distributions of $\mathcal{L}(Y(s), \tau(s))$ for $s \geq 0$, where Y is the exponential of Wiener process and $\tau(s) = \int \tilde{\sigma}^{-2}(Y(u))\,\mathrm{d}u$. Both methods are compared for the European option and the special choice $\tilde{\sigma}(y) = \sigma_2 I_{(-\infty, y_0]}(y) + \sigma_1 I_{(y_0, \infty)}(y)$.

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1. INTRODUCTION

In [5] any weak solution of the above SDE that is unique in law is called a (b,σ)-stock price. Assuming that b and σ are bounded $C(\mathbb{R}^+)$-progressive processes such that $\sigma \geq \varepsilon > 0$ holds and such that the corresponding equation is unique in law, [5] provides instruments that enable to remove the drift part of the equation (Corollary 3.2) and to prove that the price of an arbitrary integrable option and its valuation exist as stochastic invariants (Theorems 5.4 and 5.5). Most importantly, if $\sigma(x,t) = \tilde{\sigma}(x(t))$, [5] delivers the following information (Summary 4.4):

Any $(0,\sigma)$-stock price X is generated by $X = Y(\varphi)$, where Y is a $(0,1)$-stock price and $\varphi(t)$ is the inversion of the process $\tau(s) = \int_0^s \tilde{\sigma}^{-2}(Y(u))\,\mathrm{d}u$.

Remark 4.6 provides explicit formulas for $Ef(X(T))$ if $f \in C^2(R)$ that need a not always simply available information about the family of probability distributions $\mathcal{L}(Y(s), \tau(s))$ and this is the point from where the present paper continues [5].

We propose two methods how to compute $Eg(X(T))$. Both methods need to assume that the set of all points of discontinuity of $\tilde{\sigma}$ has Lebesgue measure zero. The first one is based on a Monte Carlo approximation of $X(T)$ (see Theorem 2.1). The procedure, generally simple enough, recovers $Eg(X(T))$ with a satisfactory precision for a continuous bounded g and not so quite accurately for a continuous unbounded g. In case of an absolutely continuous $\mathcal{L}(X(T))$, we may weaken the continuity requirement on g assuming that $\lambda(D_g) = 0$, where D_g denotes the set of all discontinuities of g. We believe that $X(T)$ has an absolutely continuous distribution generally, but it still remains to be an open problem. The second method offers an explicit, though a rather complex, formula for $Eg(X(T))$ that necessarily calls for a numerical integration treatment (see Formula (2.10) in Theorem 2.2). The procedure requires to know the probability distributions $\mathcal{L}(Y(s), \tau(s))$ for $s > 0$ and assumes that each $\tau(s)$ possesses a density that is continuous at T for almost every $s > 0$ and that $\varphi(T)$ has a density, the assumption not being satisfied when $X(0) = x$ and $\tilde{\sigma}$ is a constant in a neighbourhood of x, for example, when choosing $\tilde{\sigma}(y) = \sigma_2 I_{(-\infty,y_0]}(y) + \sigma_1 I_{(y_0,\infty)}(y)$. This is also an open problem yet to be solved whether the absolute continuity assumption can be removed or not. We are inclined to believe that the answer is the positive one.

Our idea to hide a stochastic volatility $\sigma(t)$ into the classical Black-Scholes model with the volatility $\sigma = 1$ by means of a random time change $\varphi(t)$ is just another instant of a random time technology applied in recent developments in mathematical finance. See, for example, H. Geman, D. B. Madan and M. Yor [2].

2. RESULTS

Through the rest of our presentation, we shall consider $x > 0$, a Borel function $\tilde{\sigma} : \mathbb{R} \to \mathbb{R}$ such that $0 < \varepsilon \leq \tilde{\sigma} \leq C < \infty$ and keep the following notation:

$B = (B(s), s \geq 0)$ is a Wiener process on a complete $(\Omega, \mathcal{F}, P)$, (2.1)

$(\mathcal{F}_s^B)$ its augmented filtration,

$$Y(s) = x \exp\{B(s) - s/2\}, \quad \tau(s) = \int_0^s \tilde{\sigma}^{-2}(Y(u))\,\mathrm{d}u, \tag{2.2}$$

$$\varphi(t) = \inf\{s \geq 0, \tau(s) > t\} \quad \text{and} \quad X(t) = Y(\varphi(t)). \tag{2.3}$$

Recall Summary 4.4 in [5] and observe that X can be interpreted as a weak solution on $(\Omega, \mathcal{F}, P)$ to

$$\mathrm{d}X(t) = X(t)\tilde{\sigma}(X(t))\,\mathrm{d}W(t), \; X(0) = x \tag{2.4}$$

that is an SDE which is unique in law. Denote by $\mu_\sigma = \mathcal{L}(X)$ the probability distribution of X (and of any other weak solution to (2.4)) in $C(\mathbb{R}^+)$.

If b is a bounded $C(\mathbb{R}^+)$-progressive process[1], then

$$\mathrm{d}Z(t) = Z(t)b(Z,t)\,\mathrm{d}t + Z(t)\tilde{\sigma}(Z(t))\,\mathrm{d}W(t), \; Z(0) = x \tag{2.5}$$

is an equation that possesses a weak solution and it is unique in law (see Corollary 3.3 in [5]) and any its weak solution Z is called a (b, σ)-stock price. It follows by

[1]The process $b : C(\mathbb{R}^+) \times \mathbb{R}^+ \to \mathbb{R}$ is $C(\mathbb{R}^+)$-progressive if there are Borel measurable functions $b_t : C([0,t]) \to \mathbb{R}, t \geq 0$, such that $b_t(x(s), s \leq t) = b(x,t)$ holds for all $x \in C(\mathbb{R}^+)$ and $t \geq 0$.

Theorem 5.4 in [5] that if $g : \mathbb{R} \to \mathbb{R}^+$ is a Borel function such that $g(X(T)) \in L_1(P)$, then

$$Eg(X(T)) = \int_{C(\mathbb{R}^+)} g(y(T))\mu_\sigma(\mathrm{d}y) \tag{2.6}$$

is the price of financial claim $g(X(T))$ in the market, where an investment process $C = (C(t), t \geq 0)$ is defined by

$$\mathrm{d}C(t) = p(Z,t)\,\mathrm{d}Z(t), \;\; C \geq 0, \tag{2.7}$$

where Z is a (b, σ)-stock market and p a $C(\mathbb{R}^+)$-progressive portfolio such that

$$\int_0^t p^2(z,s)\,\mathrm{d}s < \infty \;\; \forall\, t \geq 0 \;\; \text{for } \mu_\sigma\text{-almost every } z. \tag{2.8}$$

We propose two methods how to either approximate or compute $Eg(X(T))$ stressing the choice of $g(y) = (y-K)^+, K > 0$ ($Eg(X(T))$ being the price of the European call option in the market (2.7)) and simple volatilities as

$$\tilde{\sigma}(y) := \sigma_1 I_{(-\infty, y_0]}(y) + \sigma_2 I_{(y_0, +\infty)}(y) \text{ perhaps with } y_0 = x, \tag{2.9}$$

where x is the initial value of the process X at time $t = 0$.

The approximation method is suggested by

Theorem 2.1. Assume that $\lambda(D_{\tilde{\sigma}}) = 0$, where $D_{\tilde{\sigma}}$ denotes the set of discontinuities of $\tilde{\sigma}$. Then for any $t \geq 0$

$$X_n(t) := Y(\varphi_n(t)) \to Y(\varphi(t)) =: X(t) \text{ a.s. as } n \to \infty,$$

where

$$\varphi_n(t) = \inf\{s \geq 0, I_n(s) \geq t\} \text{ and } I_n(s) := \frac{1}{n}\sum_{k=1}^{\lfloor sn \rfloor} \tilde{\sigma}^{-2}\left(Y\left(\frac{k}{n}\right)\right),$$

where $\lfloor z \rfloor$ denotes the integer part of a real number z.

We apply Theorem 2.1 as follows: Choose $g(y) = (y-K)^+$, observe that X is a martingale that starts from x and compute

$$q_X := E(X(T)-K)^+ = (x-K) + \int_{C(\mathbb{R}^+)} (y(T)-K)^- \mu_\sigma(\mathrm{d}y).$$

The strong law of large numbers supports the idea to approximate q_X by

$$(x-K) + \frac{1}{m}\sum_{i=1}^{m}(\zeta_{ni} - K)^-,$$

where $(\zeta_{ni}, i \leq m)$ are i.i.d. with $\mathcal{L}(\zeta_{ni}) = \mathcal{L}(X_n(T))$ for $1 \leq i \leq n$. Since $y \in (0,\infty) \mapsto (y-K)^- \in [0,K]$ is bounded by K, the standard deviation of sample mean obtained from simulation is less or equal to $K/\sqrt{m}$.

On the other hand, a general compact formula for $Eg(X(T))$ is available provided that we are able to establish two-dimensional distributions $G_t = \mathcal{L}(Y(t), \tau(t))$. Compare with Corollary 4.5 and Remark 4.6 in [5].

Theorem 2.2. Consider a $\tilde{\sigma}$ as in Theorem 2.1. Assume that $T > 0$ and x are such that r.v.'s $\tau(s)$ have densities $\vartheta_s(y)$ that are continuous at T for almost every $s > 0$ and such that $\mathcal{L}(\varphi(T)) \ll \lambda$, where λ denotes the Lebesgue measure on the real line $\mathbb{R}$. Then for arbitrary Borel $g : \mathbb{R} \to \mathbb{R}^+$

$$Eg(X(T)) = \int_0^\infty \int_{\mathbb{R}} g(xe^{b-\frac{1}{2}s})\tilde{\sigma}^{-2}(xe^{b-\frac{1}{2}s})\beta_T^s(\mathrm{d}b)\vartheta_s(T)\,\mathrm{d}s, \tag{2.10}$$

where $\beta_t^s = \mathcal{L}(B(s)|\tau(s) = t)$ is a regular conditional distribution of $B(s)$ given $\tau(s)$. If, moreover, $\mathcal{L}(B(s), \tau(s)) \ll \lambda^2$ for all $s > 0$ and $\xi_s(b,t)$ denotes the corresponding densities, we get (2.10) in the form

$$Eg(X(T)) = \int_0^\infty \int_{\mathbb{R}} g(xe^{b-\frac{1}{2}s})\tilde{\sigma}^{-2}(xe^{b-\frac{1}{2}s})\xi_s(b,T)\,\mathrm{d}b\,\mathrm{d}s. \tag{2.11}$$

Remark 2.3. Consider a $\tilde{\sigma}$ as in Theorem 2.1. Assume that $T > 0$ and x are such that r.v.'s $\int_0^s f(B(u))\,\mathrm{d}u$, where $f(b) = \tilde{\sigma}^{-2}(xe^b)$, have densities $\tilde{\vartheta}_s(y)$ that are continuous at T for almost every $s > 0$ and such that the function $P(\int_0^s f(B(u))\,\mathrm{d}u > T)$ is locally absolutely continuous in s. If $\mathcal{L}(B(s), \int_0^s f(B(u))\,\mathrm{d}u|P) \ll \lambda^2$ for all $s > 0$ and $\tilde{\xi}_s(b,t)$ denote the corresponding densities, then

$$Eg(X(T)) = \int_0^\infty \int_{\mathbb{R}} (g\tilde{\sigma}^{-2})(xe^w)e^{-\frac{1}{2}(w+\frac{1}{4}s)}\tilde{\xi}_s(w,T)\,\mathrm{d}w\,\mathrm{d}s. \tag{2.12}$$

Since the fair price of the option $g(X(T))$ is by [5, Theorems 5.4, 5.5] stochastically invariant and $\mathcal{F}_\infty^X = \mathcal{F}_\infty^B$ (see [5, Theorem 4.3]), we may suppose that $\Omega = C(\mathbb{R}^+), B(\omega) = \omega$. Put $W(s) := B(s) - s/2$ and define a probability measure Q on $\sigma_\infty(B) = \sigma_\infty(W)$ by

$$\frac{\mathrm{d}Q|\sigma_s(B)}{\mathrm{d}P|\sigma_s(B)} = \exp\left\{\frac{1}{2}\left(B(s) - \frac{1}{4}s\right)\right\} = \exp\left\{\frac{1}{2}\left(W(s) + \frac{1}{4}s\right)\right\}, \tag{2.13}$$

where $\sigma_s(B)$ and $\sigma_s(W)$ denote the canonical filtrations of the processes B and W, respectively. The left-hand side of (2.13) denotes the Radon-Nikodym derivative of corresponding measures. Then W is a Wiener process under Q (see [4, Theorem 38.9, Chapter IV]) and the densities of $(W(s), \tau(s))$ under Q by definition (2.2) of τ exist and they are equal to $\tilde{\xi}_s(w,t)$. It is not difficult to verify that $(W(s), \tau(s))$ have by (2.13) densities $\xi_s(w,t)$ under P such that

$$\tilde{\xi}_s(w,t) = Q(W(s) \in \mathrm{d}w, \tau(s) \in \mathrm{d}t) = \exp\left\{\frac{1}{2}\left(w + \frac{1}{4}s\right)\right\}\xi_s(b,t), \tag{2.14}$$

where $b = w + \frac{1}{2}s$. Hence, we are able to derive (2.12) from (2.11) under the assumptions of Remark 2.3.

Obviously, $\varphi(T)$ has an absolutely continuous distribution function under P if and only if $P\varphi(T)^{-1} \ll \lambda$.[2] Since $P|\sigma_s(B) \sim Q|\sigma_s(B)$ for every $s > 0$ and $\varphi(T)$ is

[2]By $Pf^{-1} := \mathcal{L}(f|P)$ we denote the probability distribution of r.v. f under P.

bounded r.v., we get that

$$P\varphi(T)^{-1} = P|\sigma_s(B)\varphi(T)^{-1} \sim Q|\sigma_s(B)\varphi(T)^{-1} = Q\varphi(T)^{-1}$$

holds for s large enough. Hence, $\varphi(T)$ has an absolutely continuous distribution under P if and only if it has an absolutely continuous distribution under Q. Further,

$$Q\left[\varphi(T) < s\right] = Q\left[T < \tau(s)\right] = P\left(\int_0^s f(B(u))\,\mathrm{d}u > T\right). \tag{2.15}$$

Thus, we conclude that the assumption $\mathcal{L}(\varphi(T)|P) \ll \lambda$ in Theorem 2.2 is equivalent to the assumption that the function $P(\int_0^s f(B(u))\,du > T)$ is locally absolutely continuous in s.

Remark 3.1 shows how to modify the proof of Theorem 2.2 so that we could replace the assumption that the densities $\vartheta_s(y)$ are continuous at T for almost every $s > 0$ by the same assumption with $\tilde{\vartheta}_s(y)$ instead of with $\vartheta_s(y)$. The remaining assumptions of Theorem 2.2 are satisfied under the assumptions of Remark 2.3, as we have shown before. Moreover, we are able to get (2.12) from (2.11).

Example 2.4. Obviously, $\tilde{\sigma}(y) = \sigma_2 I_{(-\infty,x]}(y) + \sigma_1 I_{(x,\infty)}(y)$ has only one possible point of discontinuity x and therefore $D_{\tilde{\sigma}} \subseteq \{x\}$ has Lebesgue measure zero. We will compute the density $\tilde{\xi}_s(w,t) = Q(W(s) \in \mathrm{d}w, \tau(s) \in \mathrm{d}t)$ and we will left to the reader to verify that the marginal density $\tilde{\vartheta}_s(t) = Q(\tau(s) \in \mathrm{d}t)$ is continuous at T for almost every $s > 0$ and that the distribution function (2.15) of $\varphi(T)$ is absolutely continuous. Both conditions can be verified easily, since $\tau(s)$ is an affine transform of $\lambda_s^+(W) := \lambda\{u \le s; W(s) > 0\}$ and

$$Q(\lambda_s^+(W) < r) = Q\left(\lambda_1^+(W) < \frac{r}{s}\right) = \frac{2}{\pi}\arcsin\sqrt{\frac{r}{s}} \quad \text{if} \quad 0 < r \le s$$

by the arcsin law. See [1, pp. 97–100].

Theorem 2.5. Consider $\tilde{\sigma}$ defined by (2.9) with $y_0 = x$. Then $\mathcal{L}(W(s), \tau(s)|Q) \ll \lambda^2$ for $s > 0$ and the corresponding density is given as $\tilde{\xi}_s(w,t) =$

$$\sigma^2 s^{-3/2}\left[\int_\varepsilon^1 h(a, w/\sqrt{s})\,\mathrm{d}a I\{w \le 0\} + \int_{1-\varepsilon}^1 h(a, w/\sqrt{s})\,\mathrm{d}a I\{w > 0\}\right] I\{0 < \varepsilon < 1\},$$

where $\varepsilon = \sigma^2(t - \sigma_2^{-2}s)/s, \sigma_1^{-2} = \sigma_2^{-2} + \sigma^{-2}$

$$h(a,v) = \frac{|v|}{2\pi[a(1-a)]^{3/2}} e^{-\frac{v^2}{2(1-a)}}$$

in case $\sigma_2 > \sigma_1$. If $\sigma_2 < \sigma_1$, $\tilde{\xi}_s(w,t)$ equals to

$$\sigma^2 s^{-3/2}\left[\int_\varepsilon^1 h(a, w/\sqrt{s})\,\mathrm{d}a I\{w \ge 0\} + \int_{1-\varepsilon}^1 h(a, w/\sqrt{s})\,\mathrm{d}a I\{w < 0\}\right] I\{0 < \varepsilon < 1\},$$

where $\varepsilon = \sigma^2(t - \sigma_1^{-2}s)/s$ and $\sigma_2^{-2} = \sigma_1^{-2} + \sigma^{-2}$.

Formula 2.6. Assume $\tilde{\sigma}$ as in (2.9) with $y_0 = x$ and consider a Borel $g \geq 0$. Then

$$Eg(X(T)) = \sigma_1^{-2} \iint_{0<1-a<\varepsilon<1} e^{-s/8} \frac{s^{1/2}}{T} I_1(a,\varepsilon)\, da\, d\varepsilon + \sigma_2^{-2} \iint_{0<\varepsilon<a<1} e^{-s/8} \frac{s^{1/2}}{T} I_2(a,\varepsilon)\, da\, d\varepsilon, \tag{2.16}$$

where $\varepsilon = \sigma^2(T - \sigma_2^{-2}s)/s$, $\sigma_1^{-2} = \sigma_2^{-2} + \sigma^{-2}$ and

$$I_1(a,\varepsilon) = \int_0^\infty g(xe^w) e^{-\frac{1}{2}w} h(a, w/\sqrt{s})\, dw, \quad I_2(a,\varepsilon) = \int_{-\infty}^0 g(xe^w) e^{-\frac{1}{2}w} h(a, w/\sqrt{s})\, dw, \tag{2.17}$$

provided that $\sigma_2 > \sigma_1$. If $\sigma_2 < \sigma_1$, then

$$Eg(X(T)) = \sigma_1^{-2} \iint_{0<\varepsilon<a<1} e^{-s/8} \frac{s^{1/2}}{T} I_1(a,\varepsilon)\, da\, d\varepsilon + \sigma_2^{-2} \iint_{0<1-a<\varepsilon<1} e^{-s/8} \frac{s^{1/2}}{T} I_2(a,\varepsilon)\, da\, d\varepsilon, \tag{2.18}$$

where $\varepsilon = \sigma^2(T - \sigma_1^{-2}s)/s$ and $\sigma_2^{-2} = \sigma_1^{-2} + \sigma^{-2}$.

Formula 2.7. If g in Formula 2.6 is defined as $g(y) = (y - K)^+$, then

$$I_1(a,\varepsilon) = A + B\,(xZ_{1,x} + KZ_{1,K}),$$
$$I_2(a,\varepsilon) = A - B\,(xZ_{2,x} + KZ_{2,K})\, I\{x > K\},$$

where

$$A = \frac{(x-K)_+\sqrt{s}}{2\pi a^{3/2}(1-a)^{1/2}}, \quad B = \frac{e^{\frac{1}{8}s(1-a)} s/2}{\sqrt{2\pi} a^{3/2}},$$

and

$$Z_{1,x} = 1 - \Phi\left(\frac{(\ln K/x)_+}{\sqrt{s(1-a)}} - \frac{1}{2}\sqrt{s(1-a)}\right), \tag{2.19}$$

$$Z_{1,K} = 1 - \Phi\left(\frac{(\ln K/x)_+}{\sqrt{s(1-a)}} + \frac{1}{2}\sqrt{s(1-a)}\right), \tag{2.20}$$

$$Z_{2,x} = \Phi\left(-\frac{\sqrt{s(1-a)}}{2}\right) - \Phi\left(\frac{\ln K/x}{\sqrt{s(1-a)}} - \frac{\sqrt{s(1-a)}}{2}\right), \tag{2.21}$$

$$Z_{2,K} = \Phi\left(\frac{\sqrt{s(1-a)}}{2}\right) - \Phi\left(\frac{\ln K/x}{\sqrt{s(1-a)}} + \frac{\sqrt{s(1-a)}}{2}\right). \tag{2.22}$$

Finally, we are also able to handle the volatilities $\tilde{\sigma}$ in (2.9) not assuming $y_0 = x$. Denote

$$C(x,K,t) = \int_{C(\mathbb{R}+)} (y(t) - K)^+ \mu_\sigma(dy). \tag{2.23}$$

Formula 2.8. For arbitrary volatility coefficient $\tilde{\sigma}$ defined by (2.9) $C(x,K,t)$ is equal to

$$\sqrt{\frac{x}{y_0}}\int_0^{\tilde{\sigma}^2(x)t} C\left(y_0,K,t-\tilde{\sigma}^2(x)r\right)e^{-\frac{1}{8}r}\mathcal{L}(\eta|Q)(dr)+E\left(X(t)-K\right)^+ I\{t<\tau(\eta)\},$$

where η is the time of the first entry of Q-Wiener process $W(s):=\mathcal{B}(s)-s/2$ into $\{\ln\frac{y_0}{x}\}$, hence $\tau(\eta)$ is the time of the first entry of process X into $\{y_0\}$.

Formula 2.9. Consider $\tilde{\sigma}$ as in Formula 2.8. Put $E_t=E\left(X(t)-K\right)^+ I\{t<\tau(\eta)\}$ and $s=\tilde{\sigma}^2(x)t$. If $K,x<y_0$, then

$$\begin{aligned}E_t=x&\left[\Phi\left(\frac{\ln y_0/x}{\sqrt{s}}-\frac{1}{2}\sqrt{s}\right)-\Phi\left(\frac{\ln K/x}{\sqrt{s}}-\frac{1}{2}\sqrt{s}\right)\right]-\\ -y_0&\left[\Phi\left(\frac{\ln x/y_0}{\sqrt{s}}-\frac{1}{2}\sqrt{s}\right)-\Phi\left(\frac{\ln Kx/y_0^2}{\sqrt{s}}-\frac{1}{2}\sqrt{s}\right)\right]-\\ -K&\left[\Phi\left(\frac{\ln y_0/x}{\sqrt{s}}+\frac{1}{2}\sqrt{s}\right)-\Phi\left(\frac{\ln K/x}{\sqrt{s}}+\frac{1}{2}\sqrt{s}\right)\right]+\\ +\frac{Kx}{y_0}&\left[\Phi\left(\frac{\ln x/y_0}{\sqrt{s}}+\frac{1}{2}\sqrt{s}\right)-\Phi\left(\frac{\ln Kx/y_0^2}{\sqrt{s}}+\frac{1}{2}\sqrt{s}\right)\right].\end{aligned}$$

If $x<y_0\le K$, $E_t=0$. If $y_0\le K,x$, then

$$\begin{aligned}E_t&=x\Phi\left(\frac{\ln x/K}{\sqrt{s}}+\frac{\sqrt{s}}{2}\right)-y_0\Phi\left(\frac{\ln y_0^2/Kx}{\sqrt{s}}+\frac{\sqrt{s}}{2}\right)-\\ &-K\Phi\left(\frac{\ln x/K}{\sqrt{s}}-\frac{\sqrt{s}}{2}\right)+\frac{Kx}{y_0}\Phi\left(\frac{\ln y_0^2/Kx}{\sqrt{s}}-\frac{\sqrt{s}}{2}\right).\end{aligned}$$

If $x\ge y_0>K$, then

$$\begin{aligned}E_t&=x\Phi\left(\frac{\ln x/y_0}{\sqrt{s}}+\frac{\sqrt{s}}{2}\right)-y_0\Phi\left(\frac{\ln y_0/x}{\sqrt{s}}+\frac{\sqrt{s}}{2}\right)-\\ &-K\Phi\left(\frac{\ln x/y_0}{\sqrt{s}}-\frac{\sqrt{s}}{2}\right)+\frac{Kx}{y_0}\Phi\left(\frac{\ln y_0/x}{\sqrt{s}}-\frac{\sqrt{s}}{2}\right).\end{aligned}$$

3. PROOFS

Proof of Theorem 2.1. Write $f(y)=\tilde{\sigma}^{-2}(xe^y)$ and denote by D_f the set of all points of discontinuity of f. By assumption, it has Lebesgue measure zero. Write $\mathbb{W}_\mu=\mathcal{L}(B(s)+\mu s, s\ge 0|P)$ for the shifted Wiener measure with a shift μ and apply Fubini Theorem to get

$$0=\int_0^s \mathbb{W}_\mu[b;b(u)\in D_f]\,du=\int_0^s\int_{C(\mathbb{R}^+)} I\{b(u)\in D_f\}\mathbb{W}_\mu(db)\,du=\int_{C(\mathbb{R}^+)}\lambda\{u\le s;b(u)\in D_f\}\mathbb{W}_\mu(db)$$

for any fixed $s > 0$. Hence, $\mathbb{W}_\mu(A_s) = 1$, where

$$A_s := \{b \in C(\mathbb{R}^+); \lambda\{u \le s\,; b(u) \in D_f\} = 0\}. \tag{3.1}$$

Further, let $b_n \to b$ uniformly on $[0, s]$ for a b in A_s. Then $f(b_n(u)) \to f(b(u))$ λ-almost everywhere on $[0, s]$ and it follows that

$$\int_0^s f(b_n(u))\,\mathrm{d}u \to \int_0^s f(b(u))\,\mathrm{d}u, \tag{3.2}$$

because b is a bounded function. Obviously,

$$\left(B\left(\frac{\lfloor un\rfloor}{n}\right) - \frac{\lfloor un\rfloor}{2n}, u \ge 0\right) \to \left(B(u) - \frac{1}{2}u, u \ge 0\right)$$

locally uniformly on $\mathbb{R}^+$ and therefore, putting $\mu = -\frac{1}{2}$, we get that outside of a P-null set

$$\int_0^s \tilde{\sigma}^{-2}\left(x\exp\left\{B\left(\frac{\lfloor un\rfloor}{n}\right) - \frac{\lfloor un\rfloor}{2n}\right\}\right)\mathrm{d}u \to \int_0^s \tilde{\sigma}^{-2}\left(x\exp\left\{B(u) - \frac{1}{2}u\right\}\right)\mathrm{d}u \tag{3.3}$$

as $n \to \infty$. Since $\tilde{\sigma} \ge \varepsilon > 0$, the trajectories of the processes in (3.3) are Lipschitz and therefore, (3.3) implies the almost sure convergence of the corresponding processes in $C(\mathbb{R}^+)$. Denoting the left-hand side integral in (3.3) by $J_n(s)$, we observe that

$$|J_n(s) - I_n(s)| \le \frac{c}{n} \quad \text{for a } c > 0 \tag{3.4}$$

and therefore, outside of a P-null set

$$I_n(s) \to \int_0^s \tilde{\sigma}^{-2}\left(x\exp\left\{B(u) - \frac{1}{2}u\right\}\right)\mathrm{d}u$$

locally uniformly on $\mathbb{R}^+$. Finally, consider $b \in C(\mathbb{R}^+)$, denote $F = \int f(b(u))\,\mathrm{d}u$ and assume that F_n are such that $F_n \to F$ locally uniformly on $\mathbb{R}^+$ and observe that for any fixed $t \ge 0$

$$\inf\{s \ge 0, F_n(s) \ge t\} \to \inf\{s \ge 0, F(s) \ge t\} \text{ as } n \to \infty,$$

since $f(y) = \tilde{\sigma}^{-2}(xe^y)$ and $0 < \varepsilon \le \tilde{\sigma} \le C < \infty$. In particular, we get that for any $t \ge 0$ outside of a P-null set

$$\varphi_n(t) := \inf\{s \ge 0, I_n(s) \ge t\} \to \varphi(t) = \inf\left\{s \ge 0, \int_0^s \tilde{\sigma}^{-2}\left(xe^{B(u) - \frac{1}{2}u}\right)\mathrm{d}u\right\}$$

a.s. as $n \to \infty$. □

Proof of Theorem 2.2. We may suppose that g is a bounded and Lipschitz function since both sides in (2.10) are continuous in g w.r.t. the bounded and monotone convergences. Put

$$H(s) = Eg(X(T))I\{T < \tau(s)\}$$

and prove that H is locally absolutely continuous with the derivative

$$H'(s) = \int_{\mathbb{R}} g\left(xe^{b-\frac{1}{2}s}\right)\tilde{\sigma}^{-2}\left(xe^{b-\frac{1}{2}s}\right)\beta_T^s(\mathrm{d}b)\vartheta_s(T)$$

λ-almost everywhere on $(0,\infty)$. Indeed, if $s_1 < s_2$, then

$$H(s_2) - H(s_1) = Eg(X(T))I\{\tau(s_1) \le T < \tau(s_2)\} \le \sup|g|P(s_1 \le \varphi(T) < s_2).$$

Since $\varphi(T)$ is supposed to have an absolutely continuous distribution, we conclude that H is also absolutely continuous. Write

$$H'(s) = \lim_{\Delta\to 0+}\frac{1}{\Delta}Eg(X(T))I\{\tau(s) \le T < \tau(s+\Delta)\} \tag{3.5}$$

and define

$$h(s) := \lim_{\Delta\to 0+}\frac{1}{\Delta}Eg(Y(s))I\{\tau(s) \le T < \tau(s+\Delta)\}.$$

Next, we prove that $H'(s) = h(s)$ λ-almost everywhere on $(0,\infty)$. The Hölder inequality implies that

$$Z_\Delta(s) := \frac{1}{\Delta}E|g(X(T)) - g(Y(s))|I\{\tau(s) \le T < \tau(s+\Delta)\} \le \tag{3.6}$$

$$\left(\frac{1}{\Delta}E|g(X(T))-g(Y(s))|^pI\{\tau(s) \le T < \tau(s+\Delta)\}\right)^{\frac{1}{p}}\left(\frac{1}{\Delta}P(\tau(s) \le T < \tau(s+\Delta))\right)^{\frac{1}{q}}$$

whenever $p,q \ge 1$ are such that $\frac{1}{p}+\frac{1}{q} = 1$. Since g is supposed to be L-Lipschitz, we conclude that

$$\begin{aligned}E|g(X(T)) - g(Y(s))|^pI\{\tau(s) \le T < \tau(s+\Delta)\} \le \\ \le L^pE|Y(\varphi(T)) - Y(s)|^pI\{s \le \varphi(T) < s+\Delta\}.\end{aligned}$$

Denote $\phi(T) = (\varphi(T) \vee s) \wedge (s+\Delta)$. Since $\phi(T)$ is an $\mathcal{F}_t^B$-Markov time, the process $\tilde{Y}(u) := Y([s+u]\wedge\phi(T))$ is a (continuous) $\mathcal{F}_u^{\tilde{Y}}$-local martingale. By Proposition 15.7 in [3], we get that

$$E|Y(\phi(T)) - Y(s)|^p \le C_pE(\langle Y\rangle(\phi(T)) - \langle Y\rangle(s))^{p/2} \le C_p\left(\frac{\Delta}{\varepsilon^2}\right)^{p/2},$$

where C_p is a constant depending only on p. It follows that (3.6) is not greater than

$$\left(\frac{1}{\Delta}L^p C_p \frac{\Delta^{p/2}}{\varepsilon^p}\right)^{1/p} \left(\frac{1}{\Delta}P(\tau(s) \le T < \tau(s+\Delta))\right)^{1/q}.$$

Since $\varphi(T)$ is supposed to have a density, we reason that

$$\lim_{\Delta\to 0+} \frac{1}{\Delta}P[\tau(s) \le T < \tau(s+\Delta)] = \lim_{\Delta\to 0+} \frac{1}{\Delta}P(s \le \varphi(T) < s+\Delta) \tag{3.7}$$

exists in $\mathbb{R}^+$ for almost all $s \in \mathbb{R}^+$. Consider p such that $\frac{p}{2} > 1$ and $q = \frac{p}{p-1}$. Then, $\lim_{\Delta\to 0+} Z_\Delta(s) = 0$ almost everywhere on $\mathbb{R}^+$.

To compute conditional expectation given $\mathcal{F}_s^B$, we need to express $\tau(s+\Delta)$ in a special form. Obviously,

$$\tau(s+\Delta) - \tau(s) = \int_s^{s+\Delta} \tilde{\sigma}^{-2}(x\mathcal{E}_u(B))\,\mathrm{d}u = \int_0^{\Delta} \tilde{\sigma}^{-2}(Y(s)\mathcal{E}_v(V))\,\mathrm{d}v,$$

where $(V(v) := B(s+v) - B(s), v \ge 0)$ is a Wiener process independent of $\mathcal{F}_s^B$ and where $\mathcal{E}_s(V) := \exp\{V(s) - \frac{1}{2}s\}$ denotes the exponential of Wiener processes V. Then

$$E\left(g(Y(s))I\{\tau(s) \le T < \tau(s+\Delta)\}|\mathcal{F}_s^B\right)$$

$$= g(Y(s)) \int I\left\{z \le T < z + \int_0^{\Delta} \tilde{\sigma}^{-2}\left(xe^{b-\frac{1}{2}s}\mathcal{E}_v(w)\right)\mathrm{d}v\right\} \mathbb{W}(\mathrm{d}w)|_{b=B(s)}^{z=\tau(s)},$$

where $\mathbb{W}$ denotes the Wiener measure on $C(\mathbb{R}^+)$. Further,

$$E_\Delta(s) := Eg(Y(s))I\{\tau(s) \le T < \tau(s+\Delta)\} = \tag{3.8}$$

$$\iiint g\left(xe^{b-\frac{1}{2}s}\right)I\left\{z \le T < z+\int_0^{\Delta}\tilde{\sigma}^{-2}\left(xe^{b-\frac{1}{2}s}\mathcal{E}_v(w)\right)\mathrm{d}v\right\}\mathbb{W}(\mathrm{d}w)\beta_T^s(\mathrm{d}b)\vartheta_s(T)\,\mathrm{d}z + r_\Delta \tag{3.9}$$

where

$$|r_\Delta| \le \sup|g| \int I\left\{z \le T < z + \Delta\varepsilon^{-2}\right\}|\vartheta_s(z) - \vartheta_s(T)|\,\mathrm{d}z \tag{3.10}$$

$$\le \sup|g|\frac{\Delta}{\varepsilon^2}\sup\left\{|\vartheta_s(z) - \vartheta_s(T)|, T - \Delta\varepsilon^{-2} \le z \le T\right\}. \tag{3.11}$$

Since $\mathrm{d}z$ integrates in (3.9) a constant over an interval of the length $\int_0^{\Delta}\tilde{\sigma}^{-2}(xe^{b-\frac{1}{2}s}\mathcal{E}_v(w))\,\mathrm{d}v$, we get that $E_\Delta(s)$ is equal to

$$\iint g\left(xe^{b-\frac{1}{2}s}\right)\int_0^{\Delta}\tilde{\sigma}^{-2}\left(xe^{b-\frac{1}{2}s}\mathcal{E}_v(w)\right)\mathrm{d}v\,\beta_T^s(\mathrm{d}b)\vartheta_s(T)\mathbb{W}(\mathrm{d}w) + r_\Delta.$$

Having assumed that ϑ_s is a function continuous at T for almost all $s \in (0,\infty)$, we get that $r_\Delta = o(\Delta)$ with $\Delta \to 0^+$ almost everywhere on $(0,\infty)$. Hence,

$$H'(s) = \lim_{\Delta\to 0^+} \iint g\left(xe^{b-\frac{1}{2}s}\right) \frac{1}{\Delta}\int_0^\Delta \tilde{\sigma}^{-2}\left(xe^{b-\frac{1}{2}s}\mathcal{E}_v(w)\right) \mathrm{d}v\, \mathbb{W}(\mathrm{d}w)\beta_T^s(\mathrm{d}b)\vartheta_s(T) \tag{3.12}$$

$$= \int g\left(xe^{b-\frac{1}{2}s}\right)\tilde{\sigma}^{-2}\left(xe^{b-\frac{1}{2}s}\right)\beta_T^s(\mathrm{d}b)\vartheta_s(T) \tag{3.13}$$

holds for almost every $s > 0$ by the dominated convergence theorem, since

$$\frac{1}{\Delta}\int_0^\Delta \tilde{\sigma}^{-2}\left(xe^{b-\frac{1}{2}s}\mathcal{E}_v(w)\right)\mathrm{d}v \to \tilde{\sigma}^{-2}\left(xe^{b-\frac{1}{2}s}\right),\ \Delta \to 0^+ \quad \mathbb{W}\otimes\lambda\text{-a.e.} \tag{3.14}$$

as we shall see later on. First, we prove that function $v \mapsto \tilde{\sigma}^{-2}(xe^{b-\frac{1}{2}s}\mathcal{E}_v(w))$ is right-continuous at zero $(\mathbb{W}\otimes\lambda)$-almost everywhere on $C(\mathbb{R}^+)\times\mathbb{R}^+$: In fact, what we need to prove is that $xe^{b-\frac{1}{2}s} \notin D_{\tilde{\sigma}}$ for almost all b, because $\mathcal{E}_v(w) \to 1$ as $v \to 0^+$ $\mathbb{W}$-almost everywhere. But $D_{\tilde{\sigma}}$ and therefore also $\frac{1}{2}s + \ln D_{\tilde{\sigma}} - \ln x = \{\frac{1}{2}s + \ln\frac{d}{x}, d \in D_{\tilde{\sigma}}\}$ has Lebesgue measure zero which proves the proclaimed right-continuity. This implies (3.14) and also (3.12), hence we get

$$\begin{aligned} Eg(X(T)) &= \lim_{s\to\infty} H(s) = \int_0^\infty H'(s)\,\mathrm{d}s \\ &= \int_0^\infty\int_{\mathbb{R}} g(xe^{b-\frac{1}{2}s})\tilde{\sigma}^{-2}(xe^{b-\frac{1}{2}s})\beta_T^s(\mathrm{d}b)\vartheta_s(T)\,\mathrm{d}s. \qquad \square \end{aligned}$$

Remark 3.1. Assume that function

$$\tilde{\vartheta}_s(z) = Q(\tau(s)\in \mathrm{d}z) = P\left(\int_0^s f(B(u))\,\mathrm{d}u \in \mathrm{d}z\right)$$

is continuous at $z = T$ for almost every $s > 0$ instead of ϑ_s in Theorem 2.2, where $f(b) = \tilde{\sigma}^{-2}(xe^b)$. Further, assume that

$$\mathcal{L}\left(B(s) - \frac{1}{2}s, \tau(s)\middle| Q\right) = \mathcal{L}\left(B(s), \int_0^s f(B(u))\,\mathrm{d}u\middle| P\right) \ll \lambda^2 \tag{3.15}$$

for all $s > 0$ and denote the corresponding densities

$$\tilde{\xi}_s(b,t) = Q\left(B(s) - \frac{1}{2}s \in \mathrm{d}b, \tau(s)\in\mathrm{d}t\right) = P\left(B(s)\in\mathrm{d}b, \int_0^s f(B(u))\,\mathrm{d}u \in \mathrm{d}t\right). \tag{3.16}$$

Then

$$r_\Delta = \iiint g\left(xe^{b-\frac{1}{2}s}\right)\mathrm{I}\,\mathbb{W}(\mathrm{d}w)[\xi_s(b,z) - \xi_s(b,T)]\,\mathrm{d}b\,\mathrm{d}z, \quad \text{where}$$

$$\mathrm{I} := I\left\{z \le T < z + \int_0^\Delta \tilde{\sigma}^{-2}\left(xe^{b-\frac{1}{2}s}\mathcal{E}_v(w)\right)\mathrm{d}v\right\} \le I\left\{z \le T < z + \Delta\varepsilon^{-2}\right\},$$

and (3.10) and (3.11) reads as

$$|r_\Delta| \le \sup|\tilde g_s| \int I\left\{z \le T < z + \Delta\varepsilon^{-2}\right\} |\tilde\vartheta_s(z) - \tilde\vartheta_s(T)|\,\mathrm{d}z + \tag{3.17}$$

$$+ \iiint \tilde g_s\left(xe^{b-\frac12 s}\right) \mathrm{I}\mathbb{W}(\mathrm{d}w)\xi_s(b,T)\left|1 - \exp\left\{\frac12\left(b - \frac14 s\right)\right\}\right| \mathrm{d}b\,\mathrm{d}z \tag{3.18}$$

$$\le \sup|\tilde g_s| \frac{\Delta}{\varepsilon^2} \max\left\{|\tilde\vartheta_s(z) - \tilde\vartheta_s(T)|, T - \Delta\varepsilon^{-2} \le z \le T\right\} + \tag{3.19}$$

$$+ \frac{\Delta}{\varepsilon^2}\int \tilde g_s\left(xe^{b-\frac12 s}\right)\xi_s(b,T)\left|1 - \exp\left\{\frac12\left(b - \frac14 s\right)\right\}\right|\,\mathrm{d}b, \tag{3.20}$$

where $\tilde g_s(y) = g(y)\exp\{-\frac12(\ln\frac{y}{x} + \frac14)s\}$ is such that

$$\tilde g_s\left(xe^w\right) = g\left(xe^w\right)\exp\left\{-\frac12\left(w + \frac14 s\right)\right\}.$$

Assuming $\sup|\tilde g_s| < \infty$ and $\sup|g| < \infty$, let $\Delta \to 0^+$ to get $r_\Delta \to 0$ holds for almost every $s > 0$.

All this happens for a g that is continuous and supported by a compact. It follows that (2.10) and (2.12) are true for a g that is Lipschitz with a compact support. By a standard procedure, we argument that the same holds also for an arbitrary Borel $g : \mathbb{R} \to \mathbb{R}^+$. □

Proof of Theorem 2.5. By definition

$$\tau(s) = \int_0^s \tilde\sigma^{-2}\left(xe^{W(u)}\right)\mathrm{d}u = \sigma_1^{-2}\lambda_s^+(W) + \sigma_2^{-2}\lambda_s^-(W),$$

where $\lambda_s^+(W) = \lambda\{u \le s, W(s) > 0\}$ and $\lambda_s^-(W) = \lambda\{u \le s, W(s) \le 0\}$ stand for the times spent by a Q-Wiener process W above or below zero, respectively, up to time s. Let us suppose that $\sigma_2 > \sigma_1$. Then

$$\tau(s) = \sigma_2^{-2}(\lambda_s^+(W) + \lambda_s^-(W)) + (\sigma_1^{-2} - \sigma_2^{-2})\lambda_s^+(W) = \sigma_2^{-2}s + \sigma^{-2}\lambda_s^+(W),$$

where $\sigma_1^{-2} = \sigma_2^{-2} + \sigma^{-2}$. Further, $Q(W(s) \in \mathrm{d}w, \tau(s) \in \mathrm{d}t) =$

$$Q(W(s) \in \mathrm{d}w, \lambda_s^+(W) \in (\mathrm{d}t - \sigma_2^{-2}s)\sigma^2) = \sigma^2 Q(W(s) \in \mathrm{d}w, \lambda_s^+(W) \in \mathrm{d}\sigma^2(t - \sigma_2^2 s)).$$

Denote $\tilde\zeta_s(w,c) = Q(W(s) \in \mathrm{d}w, \lambda_s^+(W) \in \mathrm{d}c)$ the density of $(W(s), \lambda_s^+(W))$ under Q. By [1, pp. 97–100]

$$\tilde\zeta_1(v,d) = \left[\int_d^1 h(a,v)\,\mathrm{d}a\, I\{v \le 0\} + \int_{1-d}^1 h(a,v)\,\mathrm{d}a\, I\{v > 0\}\right] I\{0 < d < 1\},$$

where $h(a,v) = \dfrac{|v|}{2\pi[a(1-a)]^{3/2}} e^{-\frac{v^2}{2(1-a)}}$.

Then $Q(W(s) \in \mathrm{d}w, \lambda_s^+(W) \in \mathrm{d}c) =$

$$Q\left(\frac{W(s)}{\sqrt{s}} \in \frac{\mathrm{d}w}{\sqrt{s}}, \frac{\lambda_s^+(W)}{s} \in \frac{\mathrm{d}c}{s}\right) = s^{-3/2} Q\left(W_1 \in \mathrm{d}\frac{w}{\sqrt{s}}, \lambda_1^+(W) \in \mathrm{d}\frac{c}{s}\right).$$

Hence, $\tilde{\xi}_s(w,t) = Q(W(s) \in \mathrm{d}w, \tau(s) \in \mathrm{d}t) =$

$$\sigma^2 s^{-3/2}\left[\int_\varepsilon^1 h(a, w/\sqrt{s})\,\mathrm{d}a\, I\{w \le 0\} + \int_{1-\varepsilon}^1 h(a, w/\sqrt{s})\,\mathrm{d}a\, I\{w > 0\}\right] I\{0 < \varepsilon < 1\},$$

where $\varepsilon = \sigma^2(t - \sigma_2^{-2}s)/s$.

Now, suppose that $\sigma_1 > \sigma_2$. This time, we define σ by $\sigma_2^{-2} = \sigma_1^{-2} + \sigma^{-2}$. Then

$$\begin{aligned}\tau(s) &= \int_0^s \tilde{\sigma}^{-2}\left(xe^{W(s)}\right)\,\mathrm{d}s = \sigma_1^{-2}\lambda_s^+(W) + \sigma_2^{-2}\lambda_s^-(W) = \\ &= \sigma_1^{-2}(\lambda_s^+(W) + \lambda_s^-(W)) + (\sigma_2^{-2} - \sigma_1^{-2})\lambda_s^-(W) = \sigma_1^{-2}s + \sigma^{-2}\lambda_s^-(W).\end{aligned}$$

Further,

$$\begin{aligned}\xi_s(w,t) &= Q(W(s) \in \mathrm{d}w, \tau(s) \in \mathrm{d}t) \\ &= s^{-3/2}\sigma^2 Q\left(W(1) \in \mathrm{d}\frac{w}{\sqrt{s}}, \lambda_1^-(W) \in \mathrm{d}\frac{\sigma^2(t - \sigma_1^{-2}s)}{s}\right) \\ &= s^{-3/2}\sigma^2 Q\left(-W(1) \in \mathrm{d}\left(-\frac{w}{\sqrt{s}}\right), \lambda_1^+(-W) \in \mathrm{d}\frac{\sigma^2(t - \sigma_1^2 s)}{s}\right) \\ &= s^{-3/2}\sigma^2 Q\left(W(1) \in \mathrm{d}\left(-\frac{w}{\sqrt{s}}\right), \lambda_1^+(W) \in \mathrm{d}\frac{\sigma^2(t - \sigma_1^2 s)}{s}\right) \\ &= \sigma^2 s^{-3/2}\left[\int_\varepsilon^1 h(a, w/\sqrt{s})\,\mathrm{d}a\, I\{w \ge 0\}\right. \\ &+ \left.\int_{1-\varepsilon}^1 h(a, w/\sqrt{s})\,\mathrm{d}a\, I\{w < 0\}\right] I\{0 < \varepsilon < 1\},\end{aligned}$$

where $\varepsilon = \sigma^2(t - \sigma_1^{-2}s)/s$. □

Proof of Formula 2.6. First, assume that $\sigma_2 > \sigma_1$. By Theorem 2.2, Remarks 2.3, 3.1, Formula 2.5 and Example 2.4

$$\begin{aligned}&Eg(X(T)) = \\ &= \iint g(xe^w)\left[\sigma_1^{-2}I\{w > 0\} + \sigma_2^{-2}I\{w \le 0\}\right] e^{-\frac{1}{2}(w + \frac{1}{4}s)}\tilde{\xi}_s(w,T)\,\mathrm{d}w\,\mathrm{d}s \\ &= \left(\frac{\sigma}{\sigma_1}\right)^2 \iint_0^\infty g(xe^w)e^{-\frac{1}{2}(w+\frac{1}{4}s)}s^{-3/2}\int_{1-\varepsilon}^1 h(a, w/\sqrt{s})\,\mathrm{d}a\, I\{0 < \varepsilon < 1\}\,\mathrm{d}w\,\mathrm{d}s \\ &+ \left(\frac{\sigma}{\sigma_2}\right)^2 \iint_{-\infty}^0 g(xe^w)e^{-\frac{1}{2}(w+\frac{1}{4}s)}s^{-3/2}\int_\varepsilon^1 h(a, w/\sqrt{s})\,\mathrm{d}a\, I\{0 < \varepsilon < 1\}\,\mathrm{d}w\,\mathrm{d}s,\end{aligned}$$

where $\varepsilon = \sigma^2(T - \sigma_2^{-2}s)/s$ and $\sigma_1^{-2} = \sigma_2^{-2} + \sigma^{-2}$.

Denote

$$I_1(a,\varepsilon) = \int_0^\infty g(xe^w)e^{-\frac{1}{2}w}h(a,w/\sqrt{s})\,\mathrm{d}w, \quad I_2(a,\varepsilon) = \int_{-\infty}^0 g(xe^w)e^{-\frac{1}{2}w}h(a,w/\sqrt{s})\,\mathrm{d}w, \tag{3.21}$$

and observe that

$$\begin{aligned} Eg(X(T)) &= \left(\frac{\sigma}{\sigma_1}\right)^2 \iint_{0<1-a<\varepsilon<1} s^{-3/2}e^{-s/8}I_1(a,\varepsilon)\,\mathrm{d}a\,\mathrm{d}s \\ &+ \left(\frac{\sigma}{\sigma_2}\right)^2 \iint_{0<\varepsilon<a<1} s^{-3/2}e^{-s/8}I_2(a,\varepsilon)\,\mathrm{d}a\,\mathrm{d}s. \end{aligned}$$

Since $\varepsilon = \sigma^2(T - \sigma_2^{-2}s)/s$, we get $s = \frac{T}{\sigma^{-2}\varepsilon + \sigma_2^{-2}}$. Hence,

$$\mathrm{d}s = \frac{\sigma^{-2}T}{(\sigma^{-2}\varepsilon + \sigma_2^{-2})^2}\,\mathrm{d}\varepsilon = \frac{s^2}{\sigma^2 T}\,\mathrm{d}\varepsilon$$

and therefore

$$\begin{aligned} Eg(X(T)) &= \sigma_1^{-2} \iint_{0<1-a<\varepsilon<1} e^{-s/8}\frac{s^{1/2}}{T}I_1(a,\varepsilon)\,\mathrm{d}a\,\mathrm{d}\varepsilon \\ &+ \sigma_2^{-2} \iint_{0<\varepsilon<a<1} e^{-s/8}\frac{s^{1/2}}{T}I_2(a,\varepsilon)\,\mathrm{d}a\,\mathrm{d}\varepsilon. \end{aligned} \tag{3.22}$$

The case $\sigma_2 < \sigma_1$ is treated as follows:

$$Eg(X(T)) = \tag{3.23}$$

$$\iint g(xe^w)e^{-\frac{1}{2}(w+\frac{1}{4}s)}[\sigma_1^{-2}I\{w>0\} + \sigma_2^{-2}I\{w \le 0\}]\tilde{\xi}_s(w,T)\,\mathrm{d}w\,\mathrm{d}s \tag{3.24}$$

$$= \left(\frac{\sigma}{\sigma_1}\right)^2 \iint_0^\infty g(xe^w)e^{-\frac{1}{2}(w+\frac{1}{4}s)}s^{-3/2}\int_\varepsilon^1 h(a,w/\sqrt{s})\,\mathrm{d}a\,I\{0<\varepsilon<1\}\,\mathrm{d}w\,\mathrm{d}s \tag{3.25}$$

$$+ \left(\frac{\sigma}{\sigma_2}\right)^2 \iint_{-\infty}^0 g(xe^w)e^{-\frac{1}{2}(w+\frac{1}{4}s)}s^{-3/2}\int_{1-\varepsilon}^1 h(a,w/\sqrt{s})\,\mathrm{d}a\,I\{0<\varepsilon<1\}\,\mathrm{d}w\,\mathrm{d}s \tag{3.26}$$

$$= \sigma_1^{-2}\iint_{0<\varepsilon<a<1} e^{-s/8}\frac{s^{1/2}}{T}I_1(a,\varepsilon)\,\mathrm{d}a\,\mathrm{d}\varepsilon + \sigma_2^{-2}\iint_{0<1-a<\varepsilon<1} e^{-s/8}\frac{s^{1/2}}{T}I_2(a,\varepsilon)\,\mathrm{d}a\,\mathrm{d}\varepsilon, \tag{3.27}$$

where $I_1(a,\varepsilon)$ and $I_2(a,\varepsilon)$ satisfy (3.21) and $s = \frac{T}{\sigma^{-2}\varepsilon + \sigma_1^{-2}}$. □

Proof of Formula 2.7. We need to compute $I_1(a,\varepsilon), I_2(a,\varepsilon)$ with $g(y) = (y - K)^+$. Integrating per partes, we get that

$$\begin{aligned}
I_1 &= \int_{(\ln K/x)_+}^{\infty} (xe^w - K)\, e^{-\frac{1}{2}w} h(a, w/\sqrt{s})\, dw = \\
&= \int_{(\ln K/x)_+}^{\infty} \left(xe^{\frac{1}{2}w} - Ke^{-\frac{1}{2}w}\right) h(a, w/\sqrt{s})\, dw = \\
&= \left[(xe^{w/2} - Ke^{-w/2})(-1)H(a, w/\sqrt{s})\sqrt{s}\right]_{(\ln K/x)_+}^{\infty} \\
&\quad + \int_{(\ln K/x)_+}^{\infty} \sqrt{s}\left(\frac{x}{2}e^{w/2} + \frac{K}{2}e^{-w/2}\right) H(a, w/\sqrt{s})\, dw,
\end{aligned}$$

where

$$H(a,v) := \frac{\exp\left\{-\frac{v^2}{2(1-a)}\right\}}{2\pi a^{3/2}(1-a)^{1/2}} \quad \text{is such that} \quad \int_v^{\infty} h(a,u)\, du = H(a,v)$$

holds for all $v > 0$. If $K \geq x$, i.e. $(\ln K/x)_+ = \ln K/x$, then

$$I_1(a,\varepsilon) = \int_{\ln K/x}^{\infty} \sqrt{s}\left(\frac{x}{2}e^{\frac{w}{2}} + \frac{K}{2}e^{-\frac{w}{2}}\right) H(a, w/\sqrt{s})\, dw.$$

If $K < x$, i.e. $(\ln K/x)_+ = 0$, then

$$I_1(a,\varepsilon) = \frac{(x-K)\sqrt{s}}{2\pi a^{3/2}(1-a)^{1/2}} + \int_0^{\infty} \sqrt{s}\left(\frac{x}{2}e^{\frac{w}{2}} + \frac{K}{2}e^{-\frac{w}{2}}\right) H(a, w/\sqrt{s})\, dw.$$

Obviously, $I_2(a,\varepsilon) = 0$ if $K \geq x$. If $K < x$, then a per partes integration gives

$$\begin{aligned}
I_2(a,\varepsilon) &= \int_{\ln K/x}^{0} \left(xe^{\frac{w}{2}} - Ke^{-\frac{w}{2}}\right) h(a, w/\sqrt{s})\, dw = \\
&= \left[(xe^{\frac{w}{2}} - Ke^{-\frac{w}{2}})H(a, w/\sqrt{s})\sqrt{s}\right]_{\ln K/x}^{0} - \int_{\ln K/x}^{0} \sqrt{s}\left(\frac{x}{2}e^{\frac{w}{2}} + \frac{K}{2}e^{-\frac{w}{2}}\right) H(a, w/\sqrt{s})\, dw \\
&= \frac{(x-K)\sqrt{s}}{2\pi a^{3/2}(1-a)^{1/2}} - \int_{\ln K/x}^{0} \sqrt{s}\left(\frac{x}{2}e^{\frac{w}{2}} + \frac{K}{2}e^{-\frac{w}{2}}\right) H(a, w/\sqrt{s})\, dw,
\end{aligned}$$

since

$$\int_{-\infty}^{v} h(a,u)\, du = \int_{-v}^{\infty} h(a,u)\, du = H(a,-v) = H(a,v),$$

whenever $v < 0$. We also need to integrate

$$e^{\pm\frac{w}{2}} H(a, w/\sqrt{s}) = \frac{1}{2\pi a^{3/2}(1-a)^{1/2}} e^{-\frac{w^2}{2s(1-a)} \pm \frac{w}{2}}.$$

Since

$$-\frac{w^2}{2s(1-a)} \pm \frac{w}{2} = -\frac{1}{2s(1-a)}\left[w^2 \mp 2w\frac{s(1-a)}{2} + \frac{s^2(1-a)^2}{4}\right] + \frac{1}{8}s(1-a),$$

we get

$$\int e^{\pm\frac{w}{2}} H(a, w/\sqrt{s})\,\mathrm{d}w = \int \frac{\exp\left\{-\left(w \mp \frac{s(1-a)}{2}\right)^2/(2s(1-a))\right\}}{2\pi a^{3/2}(1-a)^{1/2}}\,\mathrm{d}w \cdot e^{\frac{1}{8}s(1-a)}$$
$$= \frac{\sqrt{s}}{\sqrt{2\pi}a^{3/2}}\Phi\left(\frac{w}{\sqrt{s(1-a)}} \mp \frac{1}{2}\sqrt{s(1-a)}\right) \cdot e^{\frac{1}{8}s(1-a)},$$

where Φ denotes the distribution function of the Gaussian $N(0,1)$ distribution. Then

$$I_1(a,\varepsilon) = \frac{(x-K)_+\sqrt{s}}{2\pi a^{3/2}(1-a)^{1/2}} + \frac{e^{\frac{1}{8}s(1-a)}s/2}{\sqrt{2\pi}a^{3/2}} \cdot (xZ_{1,x} + KZ_{1,K}) \text{ and}$$
$$I_2(a,\varepsilon) = \frac{(x-K)_+\sqrt{s}}{2\pi a^{3/2}(1-a)^{1/2}} - \frac{s/2}{\sqrt{2\pi}a^{3/2}} e^{\frac{1}{8}s(1-a)} (xZ_{2,x} + KZ_{2,K})\, I\{x > K\},$$

where $Z_{i,x}, Z_{i,K}, i = 1,2$ are defined in (2.19), (2.20) and (2.21), (2.22), respectively.□

Proof of Formula 2.8. Denoting by η the time of the first entry of process W into $\{\ln \frac{y_0}{x}\}$ $\tau(\eta)$ is the time of the first entry of process X into $\{y_0\}$. Obviously,

$$X(\tau(\eta)) = Y(\varphi(\tau(\eta))) = Y(\eta) = xe^{W(\eta)} = y_0, \text{ on } [\eta < \infty].$$

Further,

$$C(x,K,t) = EE\left[g(X(t))|\mathcal{F}^X_{\tau(\eta)\wedge t}\right] I\{t \geq \tau(\eta)\} + Eg\,(X(t))\, I\{t < \tau(\eta)\}, \quad (3.28)$$

where $g(y) = (y-K)^+$. Recall that X is a solution (that is unique in law) to the equation

$$\mathrm{d}X(t) = X(t)\,\tilde{\sigma}\,(X(t))\,\mathrm{d}W(t), \;\; X(0) = x,$$

where W is an $\mathcal{F}^X_t$-Wiener process. By 18.7 and 18.11 in [3] X has the strong Markov property. Obviously, $\tau(\eta)$ and $\tau(\eta) \wedge t$ are $\mathcal{F}^X_t$-Markov times and $X(\tau(\eta) \wedge t) = y_0$ on $[t \geq \tau(\eta)]$. By the strong Markov property

$$E\left[g(X(t))|\mathcal{F}^X_{\tau(\eta)\wedge t}\right] I\{t \geq \tau(\eta)\} = C(y_0, K, t - \tau(\eta)) I\{t \geq \tau(\eta)\}$$

P-almost surely. Also recall that $X(t) = Y(\varphi(t))$, where $Y(s) = x\exp\{B(s) - \frac{1}{2}s\}$ and where $B(s)$ is a Wiener process. By Remark 2.3, $(\Omega, \mathcal{F}, P, B)$ can be chosen so that $W(s) = B(s) - \frac{1}{2}s$ is a Q-Wiener process, where Q is a measure on $(\Omega, \sigma_\infty(B))$ such that

$$\frac{\mathrm{d}Q}{\mathrm{d}P}(s) := \exp\left\{\frac{1}{2}W(s) + \frac{1}{8}s\right\} = \frac{\mathrm{d}Q|\sigma_s(W)}{\mathrm{d}P|\sigma_s(W)} \quad P\text{-almost surely.}$$

The process $\frac{\mathrm{d}Q}{\mathrm{d}P}$ is obviously a continuous $(P, \sigma_s(W))$-martingale. Hence, by the stopping theorem,

$$\frac{\mathrm{d}Q|\sigma_{\eta\wedge\varphi(t)}(W)}{\mathrm{d}P|\sigma_{\eta\wedge\varphi(t)}(W)} = E\frac{\mathrm{d}Q}{\mathrm{d}P}(\varphi(T))|\sigma_{\eta\wedge\varphi(t)}(B)) = \frac{\mathrm{d}Q}{\mathrm{d}P}(\eta \wedge \varphi(t)) = \sqrt{\frac{y_0}{x}}e^{\frac{1}{8}\eta}$$

on $[\tau(\eta) \le t] = [\eta \le \varphi(t)]$ as $\varphi(t)$ and η are $\sigma_t(W)$-Markov times. It is easily seen that

$$P(\eta \in \mathrm{d}r, \eta \le \varphi(t)) = \sqrt{\frac{x}{y_0}}e^{-\frac{1}{8}r}Q(\eta \in \mathrm{d}r, \eta \le \varphi(t)).$$

Obviously,

$$\tau(\eta) = \int_0^\eta \tilde{\sigma}^{-2}\left(xe^{W(u)}\right)\,\mathrm{d}u = \tilde{\sigma}^{-2}(x)\eta$$

and therefore we get the first term in (3.28) to be equal to

$$\sqrt{\frac{x}{y_0}}\int_0^{\tilde{\sigma}^2(x)t} C\left(y_0, K, t - \tilde{\sigma}^{-2}(x)r\right)e^{-\frac{1}{8}r}\mathcal{L}(\eta|Q)(\mathrm{d}r). \qquad \square$$

Proof of Formula 2.9. We need to compute the right-hand term in (3.28). It can be simplified as

$$Eg(X(t))I\{t < \tau(\eta)\} = Eg(Y(\tilde{\sigma}^2(x)t))I\{\tilde{\sigma}^2(x)t < \eta\}, \tag{3.29}$$

since

$$t = \tau(\varphi(t)) = \int_0^{\varphi(t)} \tilde{\sigma}^{-2}\left(xe^{W(u)}\right)\,\mathrm{d}u = \tilde{\sigma}^{-2}(x)\varphi(t) \text{ on } [\varphi(t) < \eta] = [t < \tau(\eta)].$$

Denoting $s = \tilde{\sigma}^2(x)t$, we get that (3.29) is equal to

$$\int g\left(xe^{W(s)}\right)\frac{\mathrm{d}P}{\mathrm{d}Q}(s)I\{s < \eta\}\,\mathrm{d}Q = \int g(xe^w)\exp\left\{-\frac{1}{2}\left(w + \frac{s}{4}\right)\right\}Q(W(s) \in \mathrm{d}w, s < \eta),$$

since $\frac{\mathrm{d}P}{\mathrm{d}Q}(s) = \exp\{-\frac{1}{2}(W(s) + \frac{s}{4})\}$. Also denote $\delta = \ln y_0/x$ and apply the reflection principle to obtain

$$Q\left(W(s) \in \mathrm{d}w, s < \eta\right) = Q\left(W(s) \in \mathrm{d}w, \max_{u\le s} W(u) < \delta\right) =$$

$$[Q(W(s) \in \mathrm{d}w) - Q(W(s) \in 2\delta - \mathrm{d}w)]I\{w < \delta\} = \frac{I\{w < \delta\}}{\sqrt{s}} \left[\overline{\varphi}\left(\frac{w}{\sqrt{s}}\right) - \overline{\varphi}\left(\frac{2\delta - w}{\sqrt{s}}\right)\right]$$

holds for $\delta > 0$, i.e. for $y_0 > x$, where $\overline{\varphi}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. If $\delta \leq 0$, i.e. if $y_0 \leq x$, then

$$Q(W(s) \in \mathrm{d}w, s < \eta) = Q\left(W(s) \in \mathrm{d}w, \min_{u \leq s} W(u) > \delta\right) =$$

$$[Q(W(s) \in \mathrm{d}w) - Q(W(s) \in 2\delta - \mathrm{d}w)]I\{w > \delta\} = \frac{I\{w > \delta\}}{\sqrt{s}} \left[\overline{\varphi}\left(\frac{w}{\sqrt{s}}\right) - \overline{\varphi}\left(\frac{2\delta - w}{\sqrt{s}}\right)\right]$$

If $y_0 > x$, we have to compute

$$I = \int_{-\infty}^{\delta} g\left(xe^{w}\right) \exp\left\{-\frac{1}{2}\left(w + \frac{s}{4}\right)\right\} \frac{1}{\sqrt{s}} \left[\overline{\varphi}\left(\frac{w}{\sqrt{s}}\right) - \overline{\varphi}\left(\frac{2\delta - w}{\sqrt{s}}\right)\right] \mathrm{d}w.$$

Denoting $g(y) = (y - K)^+$ and substituting $w - \delta = v$, we get $xe^w = y_0 e^v$ and

$$I = \int_{\ln K/y_0}^{0} \left(y_0 e^{v/2} - K e^{-v/2}\right) \frac{e^{-s/8}}{\sqrt{s}} \sqrt{\frac{x}{y_0}} \left[\overline{\varphi}\left(\frac{v + \delta}{\sqrt{s}}\right) - \overline{\varphi}\left(\frac{v - \delta}{\sqrt{s}}\right)\right] \mathrm{d}v.$$

Obviously,

$$y_0 e^{v/2} \frac{e^{-s/8}}{\sqrt{s}} \sqrt{\frac{x}{y_0}} \overline{\varphi}\left(\frac{v + \delta}{\sqrt{s}}\right) = \frac{x}{\sqrt{s}} \overline{\varphi}\left(\frac{v + \delta}{\sqrt{s}} - \frac{1}{2}\sqrt{s}\right) \tag{3.30}$$

$$y_0 e^{v/2} \frac{e^{-s/8}}{\sqrt{s}} \sqrt{\frac{x}{y_0}} \overline{\varphi}\left(\frac{v - \delta}{\sqrt{s}}\right) = \frac{y_0}{\sqrt{s}} \overline{\varphi}\left(\frac{v - \delta}{\sqrt{s}} - \frac{1}{2}\sqrt{s}\right) \tag{3.31}$$

$$K e^{-v/2} \frac{e^{-s/8}}{\sqrt{s}} \sqrt{\frac{x}{y_0}} \overline{\varphi}\left(\frac{v + \delta}{\sqrt{s}}\right) = \frac{K}{\sqrt{s}} \overline{\varphi}\left(\frac{v + \delta}{\sqrt{s}} + \frac{1}{2}\sqrt{s}\right) \tag{3.32}$$

$$K e^{-v/2} \frac{e^{-s/8}}{\sqrt{s}} \sqrt{\frac{x}{y_0}} \overline{\varphi}\left(\frac{v - \delta}{\sqrt{s}}\right) = \frac{Kx}{y_0 \sqrt{s}} \overline{\varphi}\left(\frac{v - \delta}{\sqrt{s}} + \frac{1}{2}\sqrt{s}\right) \tag{3.33}$$

and therefore, I is equal to

$$\begin{aligned}
& x\left[\Phi\left(\frac{\ln y_0/x}{\sqrt{s}} - \frac{1}{2}\sqrt{s}\right) - \Phi\left(\frac{\ln K/x}{\sqrt{s}} - \frac{1}{2}\sqrt{s}\right)\right] \\
-& y_0\left[\Phi\left(\frac{\ln x/y_0}{\sqrt{s}} - \frac{1}{2}\sqrt{s}\right) - \Phi\left(\frac{\ln Kx/y_0^2}{\sqrt{s}} - \frac{1}{2}\sqrt{s}\right)\right] \\
-& K\left[\Phi\left(\frac{\ln y_0/x}{\sqrt{s}} + \frac{1}{2}\sqrt{s}\right) - \Phi\left(\frac{\ln K/x}{\sqrt{s}} + \frac{1}{2}\sqrt{s}\right)\right] \\
+& \frac{Kx}{y_0}\left[\Phi\left(\frac{\ln x/y_0}{\sqrt{s}} + \frac{1}{2}\sqrt{s}\right) - \Phi\left(\frac{\ln Kx/y_0^2}{\sqrt{s}} + \frac{1}{2}\sqrt{s}\right)\right]
\end{aligned}$$

if $K < y_0$, otherwise, $I = 0$. If $y_0 \le x$, we need to compute

$$I = \int_\delta^\infty g(xe^w) \exp\left\{-\frac{1}{2}\left(w + \frac{s}{4}\right)\right\} \frac{1}{\sqrt{s}}\left[\overline{\varphi}\left(\frac{w}{\sqrt{s}}\right) - \overline{\varphi}\left(\frac{2\delta - w)}{\sqrt{s}}\right)\right] dw.$$

The substitution $w - \delta = v$ gives

$$I = \int_0^\infty \left(y_0 e^{v/2} - K e^{v/2}\right)^+ \frac{e^{-s/8}}{\sqrt{s}}\sqrt{\frac{x}{y_0}}\left[\overline{\varphi}\left(\frac{v+\delta}{\sqrt{s}}\right) - \overline{\varphi}\left(\frac{v-\delta}{\sqrt{s}}\right)\right] dv$$

and we have two cases to be considered. If $y_0 \le K$, i.e. if $\ln(K/y_0) \ge 0$, then it follows by (3.30) – (3.33) that I is equal to

$$x\left[1 - \Phi\left(\frac{\ln K/y_0 + \ln y_0/x}{\sqrt{s}} - \frac{1}{2}\sqrt{s}\right)\right] - y_0\left[1 - \Phi\left(\frac{\ln K/y_0 - \ln y_0/x}{\sqrt{s}} - \frac{1}{2}\sqrt{s}\right)\right]$$
$$-K\left[1 - \Phi\left(\frac{\ln K/y_0 + \ln y_0/x}{\sqrt{s}} + \frac{1}{2}\sqrt{s}\right)\right] + \frac{Kx}{y_0}\left[1 - \Phi\left(\frac{\ln K/y_0 - \ln y_0/x}{\sqrt{s}} + \frac{1}{2}\sqrt{s}\right)\right]$$

$$= x\,\Phi\left(\frac{\ln x/K}{\sqrt{s}} + \frac{\sqrt{s}}{2}\right) - y_0\Phi\left(\frac{\ln y_0^2/Kx}{\sqrt{s}} + \frac{\sqrt{s}}{2}\right)$$
$$-K\Phi\left(\frac{\ln x/K}{\sqrt{s}} - \frac{\sqrt{s}}{2}\right) + \frac{Kx}{y_0}\Phi\left(\frac{\ln y_0^2/Kx}{\sqrt{s}} - \frac{\sqrt{s}}{2}\right).$$

If $y_0 > K$, i.e. if $\ln(K/y_0) < 0$, then (3.30) – (3.33) applies to prove that I is equal to

$$x\,\Phi\left(\frac{\ln x/y_0}{\sqrt{s}} + \frac{\sqrt{s}}{2}\right) - y_0\Phi\left(\frac{\ln y_0/x}{\sqrt{s}} + \frac{\sqrt{s}}{2}\right)$$
$$-K\Phi\left(\frac{\ln x/y_0}{\sqrt{s}} - \frac{\sqrt{s}}{2}\right) + \frac{Kx}{y_0}\Phi\left(\frac{\ln y_0/x}{\sqrt{s}} - \frac{\sqrt{s}}{2}\right).$$ □

4. SIMULATIONS

As we have seen in Section 2, we have two different possibilities how to compute $q_X = E(X(t) - K)^+$. The Monte Carlo procedure supported by Theorem 2.1 provides the following approximations with $m = 1000, n = 100000$ and $x = 1, t = 2, \sigma_2 = 1$:

$\sigma_1 \backslash K$	1	2	3	4
1.5	0.6018±0.0031	0.5036±0.0031	0.4428±0.0070	0.4028±0.0084
2	0.6459±0.0029	0.5902±0.0044	0.5550±0.0057	0.5277±0.0068
2.5	0.6704±0.0027	0.6399±0.0037	0.6198±0.0046	0.6011±0.0055
3	0.6870±0.0026	0.6640±0.0033	0.6543±0.0040	0.6420±0.0046
0.8	0.4690±0.0036	0.2703±0.0065	0.1757±0.0084	0.1187±0.0098
0.6	0.4029±0.0037	0.1629±0.0066	0.0774±0.0082	0.0410±0.0040
0.4	0.3143±0.0036	0.0593±0.0027'	0.0141±0.0014'	0.0040±0.0008'

$\sigma_1 \backslash K$	0.8	0.6	0.4
1.5	0.6317±0.0026	0.6780±0.0020	0.7430±0.0013
2	0.6665±0.0037	0.6995±0.0020	0.7537±0.0013
2.5	0.6856±0.0024	0.7136±0.0019	0.7625±0.0013
3	0.6982±0.0023	0.7201±0.0019	0.7666±0.0013
0.8	0.5364±0.0029	0.6156±0.0021	0.7091±0.0013
0.6	0.4893±0.0029	0.5840±0.0021	0.6929±0.0013
0.4	0.4236±0.0028	0.5426±0.0020	0.6722±0.0012

The corresponding confidence intervals are constructed by the 3σ-method. This seems to be more appropriate than usual 0.05-level method in case when comparing so many values.

The second method is based on Theorem 2.2 and Formulas 2.6 and 2.7. The corresponding integrals were computed numerically with step $1/k$, where $k = 800$ or 1600. The rate of accuracy was estimated by the half-step method. The numbers that are not denoted by '*' are supposed to be almost accurate. The numbers denoted by '*' can differ from the exact value by $0.0001 - 0.0002$. The numbers denoted by '**' can differ from the exact value by $0.0002 - 0.0004$.

$\sigma_1 \backslash K$	1	2	3	4	0.8	0.6	0.4
1.5	0.6013	0.5058	0.4463	0.4037	0.6330*	0.6777*	0.7425
2	0.6457*	0.5903	0.5545	0.5278	0.6663*	0.7000*	0.7546*
2.5	0.6709*	0.6377	0.6162	0.5999	0.6857**	0.7134**	0.7620**
3	0.6860	0.6649	0.6514	0.6412*	0.6975**	0.7217**	0.7668**
0.8	0.4697*	0.2672	0.1722	0.1194	0.5366*	0.6146*	0.7093*
0.6	0.4206**	0.1640	0.0788	0.0423	0.4880*	0.5833*	0.6932*
0.4	0.3116**	0.0587	0.0133	0.0036	0.4226**	0.5414*	0.6716*

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Prof. RNDr. Josef Štěpán, DrSc. and Mgr. Petr Dostál, Department of Probability and Statistics, Faculty of Mathematics and Physics – Charles University of Prague, Sokolovská 83, CZ-186 75 Praha 8 Czech Republic.
e-mails: stepan@karlin.mff.cuni.cz, dostal@karlin.mff.cuni.cz