KYBERNETIKA — VOLUME 39 (2003), NUMBER 5, PAGES 643 – 650

CONSTRUCTION OF AGGREGATION OPERATORS: NEW COMPOSITION METHOD

TOMASA CALVO, ANDREA MESIAROVÁ AND ĽUBICA VALÁŠKOVÁ

A new construction method for aggregation operators based on composition of aggregation operators is proposed. Several general properties of this construction method are recalled. Further, several special cases are discussed. It will be also shown, that this construction generalize recently introduced twofold integral, which is exactly a composition of the Choquet and Sugeno integrals by means of min operator.

Keywords: aggregation operator, composition, two-fold integral
AMS Classification: 26B99, 26E60, 28E10

1. INTRODUCTION

Basic definitions, properties and constructions methods for aggregation operators are given in [2] and thus we will not repeat them. Recall only that given a binary aggregation operator A and two general aggregation operators (n-ary aggregation operators) B, C, the composite $D = A(B, C)$, $D(x_1, \dots, x_n) = A(B(x_1, \dots, x_n), C(x_1, \dots, x_n))$ is again a general aggregation operator (n-ary aggregation operator). Obviously, the continuity, idempotency, symmetry of all three A, B, C ensure the continuity, idempotency, symmetry of D, respectively. Among more sophisticated construction methods based on the composition of aggregation operators we recall hierarchical two-step (n-step) integrals discussed in [1, 7, 11], and double aggregation introduced and discussed in [4, 6].

In this contribution, we introduce and discuss a new construction method based on composition of aggregation operators. We restrict our considerations to n-ary aggregation operators, as the case of general aggregation operators can be then straightforwardly derived. We will discuss several special types of our construction and the properties of newly constructed operators.

2. $*$-COMPOSITION OF AGGREGATION OPERATORS

For a given t-norm T and its dual t-conorm S, their convex combination $C_\lambda = (1 - \lambda)T + \lambda S, \lambda \in [0, 1]$ can be understood as a composite $C_\lambda = W_\lambda(T, S)$, where

W_λ is the weighted arithmetic mean with weights $w_1 = 1-\lambda$ and $w_2 = \lambda$. Recall that the operator C_λ is called a linear convex compensatory operator [9, 16, 17]. In the theory of compensatory operators as studied and discussed in [8, 18], the high inputs are supposed to be aggregated in t-conorm style with upwards effect (i.e., output is even higher than any of inputs) while the low inputs are aggregated in t-norm style with downwards effect (similarly as by summation of positive inputs versus summation of negative inputs when acting on the real line). Yager and Filev [18] have proposed to "measure" the degree of highness of inputs $x_1, \dots, x_n$ by $T(x_1, \dots, x_n)$ and the degree of their lowness by $T(1-x_1, \dots, 1-x_n)$ and then to choose the parameter λ (which thus is dependent on the input vector $(x_1, \dots, x_n)$). For a given input vector $(x_1, \dots, x_n)$, the parameter λ is chosen to fulfil $(1-\lambda) : \lambda =$ "lowness":"highness"$= T(1-x_1, \dots, 1-x_n) : T(x_1, \dots, x_n)$. Then the aggregation $D = C_\lambda$ leads to

$$D(x_1, \dots, x_n) = \frac{T(1-x_1, \dots, 1-x_n)}{T(x_1, \dots, x_n) + T(1-x_1, \dots, 1-x_n)} T(x_1, \dots, x_n) + \frac{T(x_1, \dots, x_n)}{T(x_1, \dots, x_n) + T(1-x_1, \dots, 1-x_n)} S(x_1, \dots, x_n) = \frac{T(x_1, \dots, x_n)}{T(x_1, \dots, x_n) + T(1-x_1, \dots, 1-x_n)},$$

where T is supposed to have no zero divisors and the convention $\frac{0}{0} = \frac{1}{2}$ is applied. Observe that for $T = T_P$ we obtain the famous 3-Π-operator $D(x_1, \dots, x_n) = \frac{\Pi x_i}{\Pi x_i + \Pi(1-x_i)}$, see [18]. In what follows, we will propose a new method of construction aggregation operators generalizing the above discussed ideas.

Let $A : [0,1]^n \to [0,1]$ and $* : [0,1]^2 \to [0,1]$ be two aggregation operators and let $b \in [0,1]$ be a given constant. Define an operator $C_b : [0,1]^n \to [0,1]$ by

$$C_b(x_1, \dots, x_n) = A(x_1 * b, \dots, x_n * b) \tag{1}$$

Evidently, C_b is a non-decreasing operator; however $C_b(0, \dots, 0) = 0$ and $C_b(1, \dots, 1) = 1$ need not to be fulfilled, in general. Following the ideas of Yager and Filev [18] and the ideas of Calvo et al [2] with "flying parameter", we replace the constant b (independent of input $(x_1, \dots, x_n)$) by a constant $B(x_1, \dots, x_n)$ (dependent on input $(x_1, \dots, x_n)$), where B is an (arbitrary chosen) aggregation operator. Now, our operator $C : [0,1]^n \to [0,1]$ is given by

$$C(x_1, \dots, x_n) = A(x_1 * B(x_1, \dots, x_n), \dots, x_n * B(x_1, \dots, x_n)). \tag{2}$$

Proposition 1. Let $A, B : [0,1]^n \to [0,1]$ be two n-ary aggregation operators and let $* : [0,1]^2 \to [0,1]$ be a binary aggregation operator. Then the operator $C : [0,1]^n \to [0,1]$ given by (2) is an n-ary aggregation operator.

Proof. The monotonicity of all involved aggregation operators A, B and $*$ ensure the monotonicity of C. Moreover,

$C(0, \ldots, 0) = A(0 * B(0, \ldots, 0), \ldots, 0 * B(0, \ldots, 0)) = A(0 * 0, \ldots, 0 * 0) = A(0, \ldots, 0) = 0$, and similarly $C(1, \ldots, 1) = 1$. □

To illustrate the difference between the standard $A * B$ composition and the $*$-composition introduced in Proposition 1, let A be the product and B and $*$ the maximum operator. Then for any $n \in \mathbb{N}$, $\mathbf{x} \in [0,1]^n$,

$$(A * B)(\mathbf{x}) = A(\mathbf{x}) * B(\mathbf{x}) = (\prod_{i=1}^{n} x_i) \vee (\bigvee_{i=1}^{n} x_i) = \bigvee_{i=1}^{n} x_i,$$

while for the operator C introduced via (2) it holds

$$C(\mathbf{x}) = A(x_1 * B(\mathbf{x}), \ldots, x_n * B(\mathbf{x})) = \prod_{j=1}^{n} (x_j \vee (\bigvee_{i=1}^{n} x_i)) = (\bigvee_{i=1}^{n} x_i)^n.$$

Observe, however, that both composition-based construction methods coincide whenever A and $*$ commute (for deeper discussion on commuting aggregation operators we recommend [10]) and A is idempotent. For example, if A is the maximum operator and $*$ is the product, then in both cases we obtain as output $C(\mathbf{x}) = (\vee x_i) \cdot B(\mathbf{x})$ independently of B.

Several properties of $A, B, *$ are herited by C. Mostly the proofs are trivial and thus we omit them.

Proposition 2. Under requirements of Proposition 1 the following hold:

(i) if $A, B, *$ are continuous then C is continuous

(ii) if A, B are symmetric then C is symmetric

(iii) if $A, B, *$ are idempotent then C is idempotent

(iv) if $A, B, *$ are kernel operators then C is kernel operator

(v) if $A, B, *$ are shift invariant then C is shift invariant.

(vi) if $A, B, *$ are homogenous then C is homogeneous

(vii) if $A, B, *$ are stable under linear transformations then C is stable under linear transformations.

(viii) if $A, B, *$ are self-dual then C is self-dual

(ix) if $A, B, *$ have the annihilator a then C has the annihilator a.

(x) if $B, *$ have the annihilator a and a is idempotent of A then C has the annihilator a.

Recall only that an n-ary aggregation operator A is a kernel operator [3, 12] if for any $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in [0,1]^n$ it holds

$$|A(x_1, \ldots, x_n) - A(y_1, \ldots, y_n)| \le \max_{i \in \{1, \ldots, n\}} |x_i - y_i|.$$

By means of the following examples it can be shown that the next properties are not herited: bisymmetry, associativity, neutral element e, 1-Lipschitz property.

Example 1. Let $A, B, *$ be binary aggregation operators, where $A(x_1, x_2) = x_1 \cdot x_2$, $B(x_1, x_2) = x_1 * x_2 = \frac{x_1+x_2}{2}$. Then all three operators are bisymmetric, but $C(x_1, x_2) = A(x_1 * B(x_1, x_2), x_2 * B(x_1, x_2)) = \frac{(3x_1+x_2)\cdot(x_1+3x_2)}{16}$, and we can see that C is not bisymmetric.

Example 2. Let $A, B, *$ be binary aggregation operators, where $A(x_1, x_2) = \min(x_1, x_2)$, $B(x_1, x_2) = x_1 * x_2 = x_1 \cdot x_2$. Then all three operators are associative, but $C(x_1, x_2) = A(x_1 * B(x_1, x_2), x_2 * B(x_1, x_2)) = \min(x_1^2 \cdot x_2, x_1 \cdot x_2^2)$. This operator is not associative.

Example 3. Let $A, B, *$ be binary aggregation operators, where $A(x_1, x_2) = B(x_1, x_2) = x_1 * x_2 = x_1 \cdot x_2$. Then all three operators have 1 as a neutral element. On the other hand $C(x_1, x_2) = A(x_1 * B(x_1, x_2), x_2 * B(x_1, x_2)) = x_1^3 \cdot x_2^3$. We can easily see that 1 is not neutral element of operator C.

Example 4. Let $A, B, *$ be binary aggregation operators, where $A(x_1, x_2) = B(x_1, x_2) = x_1 * x_2 = S_L(x_1, x_2)$, where $S_L(x_1, x_2)$ is the Łukasiewicz t-conorm, $S_L(x_1, x_2) = \min(x_1 + x_2, 1)$. Since the Łukasiewicz t-conorm has the 1-Lipschitz property all three operators have it too. However, the composed operator

$$\begin{aligned}
&C(x_1, x_2) = A(x_1 * B(x_1, x_2), x_2 * B(x_1, x_2)) = \\
&\min(\min(x_1 + \min(x_1 + x_2, 1), 1) + \min(x_2 + \min(x_1 + x_2, 1), 1), 1) = \\
&\min(3x_1 + 3_2, 1) = S_L(3x_1, 3x_2)
\end{aligned}$$

does not have the 1-Lipschitz property. Recall that an n-ary aggregation operator A has the 1-Lipschitz property if for any $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in [0,1]^n$ it holds

$$|A(x_1, \ldots, x_n) - A(y_1, \ldots, y_n)| \le \sum_{i=1}^{n} |x_i - y_i|.$$

3. SPECIAL TYPES OF $*$-COMPOSITION

Evidently, if $x * y = x$ (i.e., $*$ is the projection to the first coordinate) then C constructed by means of A, B and $*$ via (2) is just A, independently of B. Similarly, if $* = P_2$ (i.e., $x * y = y$), then for any idempotent A we get $C = B$. Moreover, for idempotent aggregation operators $A, B, *$ we have the next observations:

α) $A = \vee, * = \wedge$ ensures $C = B$;

β) $A = \wedge, * = \wedge$ ensures $C = \wedge$;

γ) $B = \vee, * = \vee$ ensures $C = \vee$;

δ) $B = \wedge, * = \vee$ ensures $C = A$;

ϑ) $A = \wedge, * = \vee$ ensures $C = B$;

ξ) $B = \vee, * = \wedge$ ensures $C = A$;

η) $A = B = *$ binary bisymmetric ensures $C = A$.

For associative symmetric aggregation operators (which are determined by their binary form) we have the next observations

(i) $A = B = *$ is idempotent ensures $C = A$

(ii) $A = B = *$ symmetric ensures $C = A^{(n+1)}$, i.e.,

$$C(x_1, \dots, x_n) = A(\underbrace{x_1, \dots, x_1}_{(n+1)\text{-times}}, \dots, \underbrace{x_n, \dots, x_n}_{(n+1)\text{-times}})$$

Special types of idempotent, kernel, continuous aggregation operators are the Choquet and the Sugeno integrals [5, 14] and related general fuzzy integrals [11]. For a given fuzzy measure $m : \mathcal{P}(\{1, \dots, n\}) \to [0, 1]$, the corresponding aggregation operators derived by means of the Choquet and Sugeno integrals will be denoted as C_m and S_m, respectively.

Recall that

$$C_m(x_1, \dots, x_n) = \sum_{i=1}^{n} x_{(i)}(m(A_{(i)}) - m(A_{(i+1)}))$$

and

$$S_m(x_1, \dots, x_n) = \bigvee_{i=1}^{n} (x_{(i)} \wedge m(A_{(i)})),$$

where (.) is a permutation of $(1, \dots, n)$ such that $x_{(1)} \leq \dots \leq x_{(n)}$, and $A_{(i)} = \{(i), (i+1), \dots, (n)\}$ with convention $A_{(n+1)} = \emptyset$.

Let m_1, m_2 be two fuzzy measures on $\{1, \dots, n\}$. Due to Propositions 1, 2 and claims α), β), γ), δ) we have the following results:

— if $A = C_{m_1}$, $B = C_{m_2}$, $*$ is weighted mean with weights w and $(1-w)$ then $C = C_m$ with $m = wm_1 + (1-w)m_2 = m_1 * m_2$

— if $A = S_{m_1}$, $B = S_{m_2}$, $*$ is weighted maximum with weights w_1, w_2, $w_1 \vee w_2 = 1$, i.e., $x * y = (w_1 \wedge x) \vee (w_2 \wedge y)$, then $C = S_m$ with $m = m_1 * m_2$.

The next result illustrates the power of the newly proposed composition method.

Proposition 3. Let m_1 and m_2 be two fuzzy measures on $\{1, \dots, n\}$. If $A = C_{m_2}$, $B = S_{m_1}, * = \wedge$ then C is the two-fold integral recently introduced by Torra [15], see also [13].

Proof. We need to show that the formula of Torra expressing the twofold integral TI_{m_1,m_2}:

$$TI_{m_1,m_2} = \sum_{i=1}^{n} \left(\bigvee_{j=1}^{i} (x_{(j)} \wedge m_1(A_{(j)})) \right) (m_2(A_{(i)}) - m_2(A_{(i+1)}))$$

is equal to expression

$$\sum_{i=1}^{n} \left(x_{(i)} \wedge \left(\bigvee_{j=1}^{n} x_{(j)} \wedge m_1(A_{(j)}) \right) (m_2(A_{(i)}) - m_2(A_{(i+1)})) \right),$$

using the notation introduced above. It is clear that it is enough to prove that for all $k \in \{1, \dots, n\}$

$$\left(\bigvee_{j=1}^{k} (x_{(j)}) \wedge m_1(A_{(j)}) \right) = x_{(k)} \wedge \left(\bigvee_{j=1}^{n} x_{(j)} \wedge m_1(A_{(j)}) \right).$$

If

$$x_{(k)} \le \bigvee_{j=1}^{n} x_{(j)} \wedge m_1(A_{(j)})$$

then $x_{(k)} = x_{(k)} \wedge \left(\bigvee_{j=1}^{n} x_{(j)} \wedge m_1(A_{(j)}) \right)$. As far as it holds $x_{(k)} \ge x_{(p)}$ for all $p \in \{1, \dots, k\}$ and $m_1(A_{(k)}) \le m_1(A_{(p)})$ for all $p \in \{1, \dots, k\}$, it is obvious, that $x_{(p)} \le m_1(A_{(p)})$ for all $p \in \{1, \dots, k\}$. This means that we have $\bigvee_{j=1}^{k} x_{(j)} \wedge m_1(A_{(j)})) = x_{(k)} = x_{(k)} \wedge \left(\bigvee_{j=1}^{n} x_{(j)} \wedge m_1(A_{(j)}) \right)$.

On the other hand, if

$$x_{(k)} \ge \bigvee_{j=1}^{n} x_{(j)} \wedge m_1(A_{(j)})$$

then $\bigvee_{j=1}^{n} x_{(j)} \wedge m_1(A_{(j)}) = x_{(k)} \wedge \left(\bigvee_{j=1}^{n} x_{(j)} \wedge m_1(A_{(j)}) \right)$ and since $x_{(k)} \le x_{(p)}$ for all $p \in \{k+1, \dots, n\}$ and $m_1(A_{(k)}) \ge m_1(A_{(p)})$ for all $p \in \{k+1, \dots, n\}$, necessarily $x_{(p)} \ge m_1(A_{(p)})$ for all $p \in \{k+1, \dots, n\}$. Consequently we get that $\bigvee_{j=1}^{k} (x_{(j)}) \wedge m_1(A_{(j)}) = \bigvee_{j=1}^{n} x_{(j)} \wedge m_1(A_{(j)}) = x_{(k)} \wedge \left(\bigvee_{j=1}^{n} x_{(j)} \wedge m_1(A_{(j)}) \right)$. □

Observe that all properties of the two-fold integral shown in [15] can be obtained easily looking on it as a special $*$-composition. For example, for any subset $E \subsetneq \{1, 2, \dots, n\}$ we can straightforwardly show

$$TI_{m_1,m_2}(\mathbf{1}_E) = C_{m_2}(S_{m_1}(\mathbf{1}_E) \wedge \mathbf{1}_E) = m_1(E) C_{m_2}(\mathbf{1}_E) = m_1(E) m_2(E).$$

Similarly, because of Proposition, evidently $TI_{m_1,m_2} \le C_{m_2}$ and $TI_{m_1.m_2} \le S_{m_2}$. Note that if $A = C_{m_1}$, $B = S_{m_2}$, $* = \vee$ then C can be understood as a dual of two-fold integral given by

$$\sum_{i=1}^{n}\left(x_{(i)} \vee \left(\bigwedge_{j=i+1}^{n} x_{(j)} \vee m_2(A_{(j)})\right)(m_1(A_{(i)}) - m_1(A_{(i+1)}))\right).$$

Indeed, denoting C constructed via (2) from $A, B, *$ by C_{A*B} we have

$$C_{C_{m_1} \vee S_{m_2}} = (C_{C_{m_1^d} \wedge S_{m_2^d}})^d,$$

where $m^d(A) = 1 - m(A^c)$, i.e., m^d is the dual fuzzy measure to m and $(C(\mathbf{x}))^d = 1 - C(\mathbf{1} - \mathbf{x})$. Moreover,

$$C_{C_{m_1} \vee S_{m_2}} + C_{C_{m_1} \wedge S_{m_2}} = C_{m_1} + S_{m_2}.$$

We list some properties of the above introduced aggregation operators. Let m^* and m_* be the strongest and the weakest fuzzy measure on $\{1, 2, \dots, n\}$, respectively. Then $C_{m^*} = S_{m^*} = \vee$ and $C_{m_*} = S_{m_*} = \wedge$. Consequently, from α), β), γ), δ), ϑ), ξ) we get:

$$\begin{aligned}
&C_{C_m \wedge S_{m^*}} = C_m\\
&C_{C_{m^*} \wedge S_m} = S_m\\
&C_{C_m \wedge S_{m_*}} = C_{C_{m_*} \wedge S_m} = \wedge\\
&C_{C_m \vee S_{m_*}} = C_m\\
&C_{C_{m_*} \vee S_m} = S_m\\
&C_{C_m \vee S_{m^*}} = C_{C_{m^*} \vee S_m} = \vee.
\end{aligned}$$

4. CONCLUSION

Newly introduced $*$-composition allows to build more complex aggregation operators from simpler operators, and thus it allows a flexible modelling of several aspects and expected attitudes of constructed operators.

ACKNOWLEDGEMENT

The work on this contribution was supported by the European projects COST 274 TARSKI, CEEPUS SK-42, by grants VEGA 1/8331/01, VEGA 2/3163/23, APVT-20-023402 and TIC 2000-1368-C03-01, BFM 2003-05308, and by the Spanish Research Group LOBFI.

(Received August 15, 2003.)

REFERENCES

[1] P. Benvenuti and R. Mesiar: A note on Sugeno and Choquet integrals. In: Proc. 7th Internat. Conference on Information Processing and Management of Uncertainty in Knowledge-based Systems (IPMU'2000), Madrid, pp. 582–585.

[2] T. Calvo, A. Kolesárová, M. Komorníková, and R. Mesiar: Aggregation operators: properties, classes and construction methods. In: Aggregation Operators (T. Calvo, G. Mayor, and R. Mesiar, eds.), Physica–Verlag, Heidelberg 2002, pp. 3–107.

[3] T. Calvo and R. Mesiar: Stability of aggregation operators. In: Proc. 1st Internat. Conference in Fuzzy Logic and Technology (EUSFLAT 2001), Leicester, pp. 475–478.

[4] T. Calvo and A. Pradera: Double aggregation operators. In: Proc. Summer School on Aggregation Operators 2001 (AGOP 2001), Oviedo, pp. 19–22.

[5] G. Choquet: Theory of capacities. Ann. Inst. Fourier *5* (1953–54), 131–295.

[6] T. Calvo and A. Pradera: Some characterizations based on double integrals. In: Proc. 1st Internat. Conference in Fuzzy Logic and Technology, Leicester, pp. 470–474.

[7] K. Fujimoto and T. Murofushi: Hierarchical decomposition of the Choquet integral. In: Fuzzy Measures and Integrals. Theory and Applications (M. Grabisch, T. Murofushi, and M. Sugeno, eds.), Physica–Verlag, Heidelberg 2000, pp. 94–103.

[8] E. P. Klement, R. Mesiar, and E. Pap: On the relationship of associative compensatory operators to triangular norms and conorms. Internat. J. Uncertain. Fuzziness Knowledge-based Systems *4* (1996), 129–144.

[9] E. P. Klement, R. Mesiar, and E. Pap: Triangular Norms (Trends in Logic, Volume 8 – Studia Logica Library). Kluwer, Dordrecht 2000.

[10] R. Mesiar and S. Saminger: Commuting aggregation operators. In: Proc. 3rd Internat. Conference in Fuzzy Logic and Technology (EUSFLAT'2003), Zittau, pp. 308–311.

[11] R. Mesiar and D. Vivona: Two-step integral with respect to fuzzy measure. Tatra Mount. Math. Publ. *16* (1999), 359–368.

[12] J. Mordelová and E. Muel: Kernel aggregation operators. In: Proc. Summer School on Aggregation Operators 2001 (AGOP'2001), Oviedo, pp. 95–98.

[13] Y. Narukawa and V. Torra: Twofold integral and multi-step Choquet integral. In: Proc. Summer School on Aggregation Operators 2003, Alcala, pp. 135–140.

[14] M. Sugeno: Theory of Fuzzy Integrals and Applications. Ph.D. Dissertation, Tokyo Institute of Technology, Tokyo 1974.

[15] V. Torra: Twofold integral: A Choquet integral and Sugeno integral generalization. Submitted for publication.

[16] I. B. Türksen: Inter-valued fuzzy sets and 'compensatory AND'. Fuzzy Sets and Systems *51* (1992), 295–307.

[17] H.-J. Zimmermann and P. Zysno: Latent connectives in human decision making. Fuzzy Sets and Systems *4* (1980), 37–51.

[18] R. R. Yager and D. P. Filev: Essentials of Fuzzy Modelling and Control. Wiley, New York 1994.

Prof. Dr. Tomasa Calvo, Department of Computer Science, University of Alcalá, E-28871 Alcalá de Henares, Madrid. Spain.

e-mail: tomasa.calvo@uah.es

Dr. Andrea Mesiarová, Mathematical Institute of the Slovak Academy of Sciences, Štefánikova 49, 814 73 Bratislava. Slovakia.

e-mail: mesiarova@mat.savba.sk

Dr. Ľubica Valášková, Slovak University of Technology, Radlinského 11, 813 68 Bratislava. Slovakia.

e-mail: luba@vox.svf.stuba.sk