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A GENERAL APPROACH TO DECOMPOSABLE BI-CAPACITIES

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We propose a concept of decomposable bi-capacities based on an analogous property of decomposable capacities, namely the valuation property. We will show that our approach extends the already existing concepts of decomposable bi-capacities. We briefly discuss additive and k-additive bi-capacities based on our definition of decomposability. Finally we provide examples of decomposable bi-capacities in our sense in order to show how they can be constructed.

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1. INTRODUCTION

Bipolar capacities were recently introduced by Grabisch and Labreuche [6, 7] and by Greco, Matarazzo and Slowinski [9] extending the concept of capacities acting most often on the unit interval to capacities on some bipolar scale, usually $[-1, 1]$. The main aim of this contribution is to present a general approach to decomposable bipolar capacities which were introduced in [16] and independently in [10] (however, in a slightly less general framework). The motivation to study decomposable bipolar capacities was stressed in [10] when discussing the axiomatic basis of non-compensatory preferences, and it can be extracted also from the letter of Benjamin Franklin to Joseph Prestly written in London Sept 19, 1772:

> [...], my way is to divide half a sheet of paper by a line into two columns; writing over the one Pro, and over the other Con. [...] When I have thus got them all together in one view, I endeavor to estimate their respective weights; and where I find two, one on each side, that seem equal, I strike them both out. If I find a reason pro equal to some two reasons con, I strike out the three. If I judge some two reasons con, equal to three reasons pro, I strike out the five; and thus proceeding I find at length where the balance lies; [...]

On the other hand and additionally, several properties have been introduced in the framework of capacities expressing a special structure of the capacity but also reducing the complexity necessary for the definition of the capacity itself. Capacities on some finite universe are special monotone set functions defined in the following way:

Definition 1. Consider some finite universe X. A set function $m : \mathcal{P}(X) \to [0,1]$ fulfilling $m(\emptyset) = 0$, $m(X) = 1$, and $m(A) \leq m(B)$ whenever $A \subseteq B$ (monotonicity) is called a *capacity* (or *fuzzy measure*) on X.

In order to determine the values of a capacity on some finite universe $X = \{1, \dots, n\}$ in fact $2^n - 2$ values have to be chosen. The complexity of such a procedure can be reduced if we know further properties of the capacity such as, e.g., S-decomposability [21], k-additivity [4], k-maxitivity [13] or p-symmetry [17].

Observe that the original notion of a capacity as introduced in [2], compare also [18], demanded the subadditivity of the set function w.r.t. the involved set operation, i.e., $m(A \cup B) \leq m(A) + m(B)$ for all $A, B \in \mathcal{P}(X)$. Since the nineties the subadditivity requirement is often omitted, and since it has no impact on the reduction of complexity we will continue in this spirit.

The present paper deals with decomposable bi-capacities which allow us to compute the bi-capacity of disjoint pairs by aggregating the values of the bi-capacity of basic elements. Note that a similar concept has already been introduced in [6] using the name S-decomposable bi-capacity (for some strict t-conorm S). We will show later that our approach covers this concept. Note further that our investigations for decomposable bi-capacities are done by taking into account relevant properties of S-decomposable capacities, also called pseudo-additive fuzzy measures, S-measures or just decomposable capacities [18, 20]. Other approaches for constructing bi-capacities from irreducible elements which are based on the underlying lattice structure of the set of disjoint subsets of the universe are possible (see, e.g., [8]) but will not be investigated within this contribution.

The paper is organized as follows: First we recall decomposable capacities and general bi-capacities. Then we introduce the concept of decomposable bi-capacities, followed by additive and k-additive bi-capacities. We show the relationship to already existing concepts and give some examples. Finally, we mention an approach to integration based on decomposable bi-capacities and provide further examples.

2. DECOMPOSABLE CAPACITIES AND BI-CAPACITIES

2.1. Decomposable capacity

Definition 2. [21] Consider some t-conorm S, i.e., a non-decreasing, associative, and commutative mapping $S : [0,1]^2 \to [0,1]$ with neutral element 0. A set function $m : \mathcal{P}(X) \to [0,1]$ is called an *S-decomposable capacity* (*pseudo-additive fuzzy measure, S-measure*) on X if $m(\emptyset) = 0$, $m(X) = 1$, and for all $A, B \in \mathcal{P}(X)$

$$A \cap B = \emptyset \quad \Rightarrow \quad m(A \cup B) = S(m(A), m(B)). \tag{1}$$

It is easy to see that for every t-conorm S the monotonicity of an S-decomposable capacity follows from the non-decreasingness of the t-conorm S. Furthermore, for every subset A of X the following property is fulfilled

$$m(A) = \underset{x \in A}{S}\, m(\{x\}),$$

expressing that the capacity of a set can be decomposed into capacities of singletons, i.e., can be computed by aggregating the values of the capacity applied to the singletons containing the elements of the corresponding set.

Moreover, an S-decomposable capacity m can be equivalently characterized by replacing (1) by the following valuation property

$$S(m(A \cap B), m(A \cup B)) = S(m(A), m(B)) \tag{2}$$

being fulfilled for all $A, B \in \mathcal{P}(X)$.

Note that since X is finite, the concept of S-decomposable capacities generalizes the classical concept of probabilities, i.e., additive capacities. Indeed, each probability p on X, i.e., a capacity fulfilling for all $A, B \in \mathcal{P}(X)$

$$A \cap B = \emptyset \quad \Rightarrow \quad p(A \cup B) = p(A) + p(B),$$

is an $S_{\mathbf{L}}$-decomposable capacity, where $S_{\mathbf{L}}$ denotes the Łukasiewicz t-conorm given by $S_{\mathbf{L}}(x, y) = \min(x + y, 1)$. It is clear that not all $S_{\mathbf{L}}$-decomposable capacities are also additive and therefore probabilities. Furthermore, each probability p on X with $|X| = n$ is uniquely determined by the probabilities of $n-1$ different singletons of X. On the other hand, in general all n singleton values are needed in order to describe an S-decomposable capacity.

Example 3. For $X = \{1, \ldots, n\}$, $n \geq 2$, let $k \in \mathbb{N}$, $k < n$ and define $m : \mathcal{P}(X) \to [0, 1]$ by $m(A) = \min(\frac{|A|}{k}, 1)$ for all $A \in \mathcal{P}(X)$. Then m is an $S_{\mathbf{L}}$-decomposable capacity which is not a probability. To describe m completely we need to know that $m(\{i\}) = \frac{1}{k}$ for all $i \in X$.

2.2. Bi-capacities

With the introduction of bi-capacities [6, 7], the concept of a capacity has been extended to mappings acting on pairs of disjoint sets taking values on a bipolar scale, most often on $[-1, 1]$. We consider a finite universe X. The set of all disjoint pairs of subsets of X will be denoted by

$$\mathcal{Q}(X) = \{(A, B) \in \mathcal{P}(X) \times \mathcal{P}(X) \mid A \cap B = \emptyset\}.$$

Definition 4. [6, 7] A function $v : \mathcal{Q}(X) \to \mathbb{R}$ is a *bi-capacity* if $v(\emptyset, \emptyset) = 0$, and $A \subseteq B$ implies that $v(A, C) \leq v(B, C)$ and $v(C, A) \geq v(C, B)$ for all $C \in \mathcal{P}(X \setminus B)$.

Furthermore, v is *normalized* if $v(X, \emptyset) = 1$ and $v(\emptyset, X) = -1$.

From the definition we get that $v(\emptyset, C) \leq 0$ and $v(C, \emptyset) \geq 0$ for all $C \in \mathcal{P}(X)$. From now on we will only deal with normalized bi-capacities.

Remark 5. Consider an arbitrary normalized bi-capacity $v : \mathcal{Q}(X) \to [-1,1]$. Then the set functions v^+, $v^- : \mathcal{P}(X) \to [0,1]$ defined by

$$v^+(A) = v(A, \emptyset) \quad \text{and} \quad v^-(A) = -v(\emptyset, A)$$

are standard capacities. Vice versa, for any two capacities $m_1, m_2 : \mathcal{P}(X) \to [0,1]$ the mapping $v^* : \mathcal{Q}(X) \to [-1,1]$ given by

$$v^*(A, B) = m_1(A) - m_2(B) \tag{3}$$

is a normalized bi-capacity. If a given bi-capacity can be described as the difference of two capacities as introduced in (3), then it is called a *bi-capacity of CPT Type* [7].

3. DECOMPOSABLE BI-CAPACITIES

3.1. General considerations

If we want to extend the concept of decomposable capacities to decomposable bi-capacities, the t-conorm as being applied in the case of capacities has to be replaced by a non-decreasing aggregation operator acting on the bipolar scale involved. Similarly as in the case of decomposable capacities (pseudo-additive fuzzy measures) the algebraic properties of the involved set operations will determine the properties of the aggregation operator we will deal with.

First of all, the "neutral" element of the bipolar scale should be preserved by the operator, i.e., 0 should be the neutral element of the operator. Therefore, we choose instead of a t-conorm an aggregation operator $U : I^2 \to I$ with $[-1,1] \subseteq I$ and with neutral element 0. We will show later that the aggregation operator U need not act on $[-1,1]$, but can take values from an arbitrary interval I with $[-1,1] \subseteq I$.

In analogy to the valuation property of decomposable measures, we expect from a decomposable bi-capacity to fulfill the following relationship for all $Q_1, Q_2 \in \mathcal{Q}(X)$

$$U(v(Q_1), v(Q_2)) = U(v(Q_1 \sqcup^* Q_2), v(Q_1 \sqcap^* Q_2)). \tag{4}$$

Note that since Q_1 and Q_2 are elements of $\mathcal{Q}(X)$ we have to clarify how the operations $\sqcup^*$ and $\sqcap^*$ are defined.

It has already been mentioned in [6, 7] that $\mathcal{Q}(X)$ equipped with the order $(A, B) \sqsubseteq (C, D)$ if and only if $A \subseteq C$ and $B \supseteq D$ is just the lattice $3^n = \{-1, 0, 1\}^n$, where the couple (A, B) corresponds to the vector $\tau \in \{-1, 0, 1\}^n$ containing 1 resp. -1 at each coordinate corresponding to an element of A resp. B and containing 0 at each remaining coordinate. Algebraically, $\tau = \mathbf{1}_A - \mathbf{1}_B$ where $\mathbf{1}_A : X \to \{0,1\}$ is the characteristic function (vector) of A.

Example 6. Consider $X = \{1,2,3\}$ and choose $A = \{1\}$ and $B = \{3\}$. Then the characteristic functions of these sets are given by $\mathbf{1}_A = (1,0,0)$ and $\mathbf{1}_B = (0,0,1)$. Consequently, $\tau \in \{-1,0,1\}^3$ corresponding to $(A, B) \in \mathcal{Q}(X)$ is described by

$$\tau = \mathbf{1}_A - \mathbf{1}_B = (1, 0, -1).$$

The supremum and infimum in the lattice $\mathcal{Q}(X)$ correspond to the standard supremum and infimum in $\{-1,0,1\}^n$ and thus they are defined by

$$\sup\{(A,B),(C,D)\} = (A\cup C, B\cap D),$$
$$\inf\{(A,B),(C,D)\} = (A\cap C, B\cup D).$$

Therefore, it is guaranteed that the necessary condition of disjointness is fulfilled. We demand the operations $\cup^*$ and $\cap^*$ to be defined as the supremum and the infimum in the lattice $(\mathcal{Q}(X),\sqsubseteq)$. Consequently, (4) can be rewritten as the following property for all $(A,B),(C,D)\in\mathcal{Q}(X)$

$$U(v(A,B),v(C,D)) = U(v(A\cup C,B\cap D),v(A\cap C,B\cup D)). \tag{5}$$

If we choose $(C,D)=(\emptyset,\emptyset)$ then we immediately get

$$v(A,B) = U(v(A,B),0) = U(v(A,B),v(\emptyset,\emptyset)) = U(v(A,\emptyset),v(\emptyset,B)). \tag{6}$$

Furthermore, for two arbitrary elements $i,j\in X$, $i\neq j$ the following relationship holds

$$U(v(\{i\},\emptyset),v(\{j\},\emptyset)) = U(v(\{i,j\},\emptyset),v(\emptyset,\emptyset)) = v(\{i,j\},\emptyset).$$

and similarly,

$$U(v(\emptyset,\{i\}),v(\emptyset,\{j\})) = v(\emptyset,\{i,j\}).$$

For a correct definition of the bi-capacity v by induction, we have to require the aggregation operator U to be additionally associative and commutative, i.e., U should be a uninorm on I with neutral element 0 (observe that the original definition of a uninorm in [3, 22] acts on $[0,1]$ only).

Definition 7. A non-decreasing mapping $U: I^2\to I$ is called a *uninorm* (on I) if it is associative, commutative and possesses a neutral element $e\in I$.

Consider a uninorm U on I with neutral element 0, then applying (5) we get by induction

$$v(A,\emptyset) = \underset{i\in A}{U} v(\{i\},\emptyset) \quad\text{and}\quad v(\emptyset,B) = \underset{j\in B}{U} v(\emptyset,\{j\}), \tag{7}$$

and also

$$v(A,B) = U(\underset{i\in A}{U} v(\{i\},\emptyset), \underset{j\in B}{U} v(\emptyset,\{j\})). \tag{8}$$

Therefore, a decomposable bi-capacity w.r.t. some uninorm U can be constructed by fixing the values of $v(\{i\},\emptyset)$ and $v(\emptyset,\{j\})$ for all $i,j\in X$. In order to guarantee that $v(X,\emptyset)=1$ and $v(\emptyset,X)=-1$ we additionally have to demand that

$$\underset{i\in X}{U} v(\{i\},\emptyset) = 1 \quad\text{and}\quad \underset{j\in X}{U} v(\emptyset,\{j\}) = -1.$$

The previous considerations lead us to the following definition of a decomposable bi-capacity.

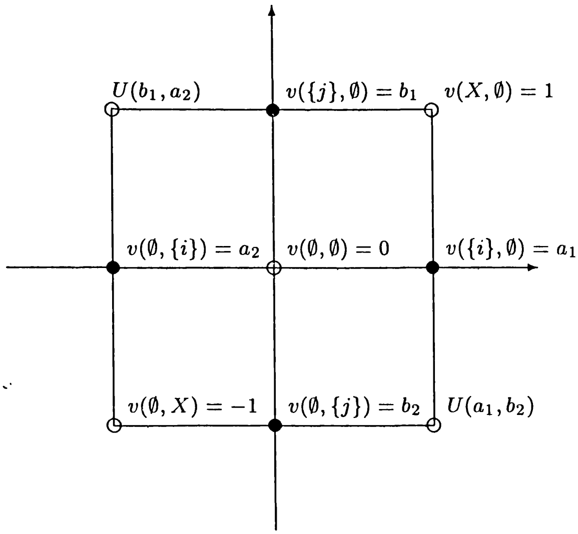

Fig. 1. Values of a normalized, U-decomposable bi-capacity $v : \mathcal{Q}(X) \to [-1, 1]$ with $X = \{i, j\}$.

Definition 8. For an arbitrary interval $I \supseteq [-1, 1]$ consider some uninorm $U : I^2 \to I$ with neutral element 0. A bi-capacity $v : \mathcal{Q}(X) \to [-1, 1]$ is called *decomposable* (*U-decomposable, U-bi-capacity*) if for all $(A, B), (C, D) \in \mathcal{Q}(X)$

$$U(v(A, B), v(C, D)) = U(v(A \cup C, B \cap D), v(A \cap C, B \cup D)).$$

Remark 9. Observe that for any normalized decomposable bi-capacity v, the set function $v^+ : \mathcal{P}(X) \to [0, 1]$ as defined in Remark 5 is an S^+-decomposable capacity, where the t-conorm S^+ is defined by

$$S^+(x, y) = \min(U(x, y), 1).$$

Similarly, the set function $v^- : \mathcal{P}(X) \to [0, 1]$ (see also Remark 5) is an S^--decomposable capacity with

$$S^-(x, y) = \min(-U(-x, -y), 1).$$

Moreover, for any $(A, B) \in \mathcal{Q}(X)$ it holds

$$v(A, B) = U(v(A, \emptyset), v(\emptyset, B) = U(v^+(A), -v^-(B)). \tag{9}$$

Remark 10. Further note that decomposable bi-capacities are constructed from the values of the bi-capacity applied to pairs composed from a singleton and the empty set (see also Figure 1), in contrast or additional to the alternative construction principle of bi-capacities based on the irreducible elements of the lattice $\mathcal{Q}(X)$ (see also [6, 8]). However, the approach linked to irreducible elements requires some pseudo-addition on $[-1,1]$ with neutral element -1, and thus it is a generalization of probability measures with values in $[-1,1]$, where 0 has no special role, while our approach will finally lead to some generalization of the CPT model.

Definition 11. Let $v : \mathcal{Q}(X) \to [-1,1]$ be a U-decomposable bi-capacity, where $U : \mathbb{R}^2 \to \mathbb{R}$ is given by $U(x,y) = x + y$. Then v is called an *additive* bi-capacity.

Proposition 12. A bi-capacity $v : \mathcal{Q}(X) \to [-1,1]$ is additive if and only if there exist two probability measures v_1, v_2 on X such that $v(A,B) = v^+(A) - v^-(B)$ for all $(A,B) \in \mathcal{Q}(X)$.

Proof. Consider an additive bi-capacity $v : \mathcal{Q}(X) \to [-1,1]$. Following Remark 9 (9) the bi-capacity can be rewritten by $v(A,B) = v^+(A) - v^-(B)$ where v^+ resp. v^- is also additive due to the additivity of v (and is therefore a probability).

In order to show sufficiency, we know because of (3) that v is a bi-capacity whenever it is given as a difference of two probability measures $v_1, v_2 : \mathcal{P}(X) \to [0,1]$, i.e., $v(A,B) = v_1(A) - v_2(B)$. Morover, for all $(A,B),(C,D) \in \mathcal{Q}(X)$

$$\begin{aligned} v(A,B) + v(C,D) &= (v_1(A) + v_1(C)) - (v_2(B) + v_2(D)) \\ &= (v_1(A \cup C) + v_1(A \cap C)) - (v_2(B \cap D) + v_2(B \cup D)) \\ &= (v_1(A \cup C) - v_2(B \cap D)) + (v_1(A \cap C) - v_2(B \cup D)) \\ &= v(A \cup C, B \cap D) + v(A \cap C, B \cup D), \end{aligned}$$

and thus v is an additive bi-capacity.

Observe that Proposition 12 can be generalized for any uninorm $U : I^2 \to I$, $[-1,1] \subseteq I$, with neutral element 0. Indeed, following the notation from Remark 9, $v : \mathcal{Q}(X) \to [-1,1]$ is a U-decomposable bi-capacity if and only if there exist a S^+-decomposable capacity $v_1 : \mathcal{P}(X) \to [0,1]$ and a S^--decomposable capacity $v_2 : \mathcal{P}(X) \to [0,1]$ such that $v(A,B) = U(v_1(A), -v_2(B))$.

Further note that our approach allows to introduce k-additive bi-capacities for $k \in \mathbb{N}$ following the approach from [14, 15], where k-additivity (compare also [4]) is linked to the additivity on the corresponding product space.

Definition 13. Fix $k \in \mathbb{N}$. Then a bi-capacity $v : \mathcal{Q}(X) \to [-1,1]$ is called *k-additive* whenever there exists an additive mapping $v_k : \mathcal{Q}(X^k) \to [-1,1]$ such that $v(A,B) = v_k(A^k, B^k)$ for all $(A,B) \in \mathcal{Q}(X)$.

Similarly as in Proposition 12, we have the following characterization of k-additive bi-capacities.

Proposition 14. A bi-capacity $v : \mathcal{Q}(X) \to [-1,1]$ is k-additive, for some $k \in \mathbb{N}$, if and only if there exist two k-additive capacities $v_1, v_2 : \mathcal{P}(X) \to [0,1]$ such that $v(A,B) = v_1(A) - v_2(B)$ for all $(A,B) \in \mathcal{Q}(X)$.

Proof. Recall that $v_1, v_2 : \mathcal{P}(X) \to [0,1]$ are k-additive capacities if and only if they can be represented as $v_1(A) = v_{1,k}(A^k)$ and $v_2(A) = v_{2,k}(A^k)$ for all $A \in \mathcal{P}(X)$, where $v_{1,k}, v_{2,k} : \mathcal{P}(X^k) \to [0,1]$ are additive set functions such that

$$v_{1,k}(A^k) \leq v_{1,k}(B^k), \quad v_{2,k}(A^k) \leq v_{2,k}(B^k), \quad \text{and} \quad v_{1,k}(X^k) = v_{2,k}(X^k) = 1$$

holds for all $A \subseteq B \subseteq X$ (observe that $v_{1,k}$ and $v_{2,k}$ need not be probability measures on X^k in general, see [14, 15]).

Consider an arbitrary k-additive bi-capacity $v : \mathcal{Q}(X) \to [-1,1]$. Then the capacities $v^+, v^- : \mathcal{P}(X) \to [0,1]$ as described in Remark 5 are evidently k-additive, since $v^+(A) = v_k(A^k, \emptyset)$ and $v^-(A) = -v_k(\emptyset, A^k)$. Moreover, from the additivity of v_k it follows that

$$v(A,B) = v_k(A^k, \emptyset) + v_k(\emptyset, B^k) = v^+(A) - v^-(B).$$

To see the sufficiency, we consider two k-additive capacities $v_1, v_2 : \mathcal{P}(X) \to [0,1]$ with $v_{1,k}, v_{2,k} : \mathcal{P}(X^k) \to \mathbb{R}$ being the additive set functions generating v_1, v_2, i.e., $v_1(A) = v_{1,k}(A^k)$ and $v_2(A) = v_{2,k}(A^k)$ for all $A \in \mathcal{P}(X)$. Then the mapping $v_k : \mathcal{Q}(X^k) \to \mathbb{R}$ given by $v_k(E,F) = v_{1,k}(E) - v_{2,k}(F)$ for all $(E,F) \in \mathcal{Q}(X^k)$ is additive. Moreover, the mapping $v : \mathcal{Q}(X) \to [-1,1]$ defined by

$$v(A,B) = v_1(A) - v_2(B)$$

is a bi-capacity because of (3), i.e., v is a k-additive bi-capacity. □

3.2. Relationship to existing concepts

In this section we recall alternative approaches to decomposable bi-capacities and investigate their relationship to our concept, especially the differences w.r.t. our approach.

So-called S-decomposable bi-capacities were introduced in [6] as bi-capacities satisfying

$$v(A,B) = (\underset{i \in A}{S}\, v(\{i\}, \emptyset)) \ominus_S (\underset{j \in B}{S} (-v(\emptyset, \{j\}))),$$

where S denotes a strict t-conorm and $\ominus_S$ the S-difference [5]. Note that if the function $s : [0,1] \to [0,\infty]$ is an additive generator of the strict t-conorm S, then the S difference $\ominus_S$ can be rewritten as

$$x \ominus_S y = g^{-1}(g(x) - g(y))$$

with

$$g(x) = \begin{cases} s(x), & \text{if } x \geq 0, \\ -s(-x), & \text{otherwise.} \end{cases}$$

It can be shown that

$$v(A,B) = g^{-1}\left(\sum_{i \in A} g(v(\{i\}, \emptyset)) + \sum_{j \in B} g(v(\emptyset, \{j\}))\right) = U(\underset{i \in A}{U}\, v(\{i\}, \emptyset), \underset{j \in B}{U}\, v(\emptyset, \{j\})),$$

where U is the uninorm on $I = [-1, 1]$ generated by g. Therefore S-decomposable bi-capacities are special cases of decomposable bi-capacities in our sense. However, in order to define an S-decomposable normalized bi-capacity v it is necessary to have $v(\{i\}, \emptyset) = 1$ and $v(\emptyset, \{j\}) = -1$ for some $i, j \in X$ due to the strictness of the involved t-conorm S. Note that this approach is related to generated and therefore Archimedean uninorms whereas in our approach an arbitrary uninorm on $I \supseteq [-1, 1]$ can be chosen.

Decomposable bi-capacities as introduced in [10] are based on a uninorm $U : [-1, 1]^2 \to [-1, 1]$ and evidently form a proper subclass of decomposable bi-capacities in the sense of Definition 8. Note that this concept excludes, e.g., additive bi-capacities from being decomposable.

3.3. Investigation of the uninorm involved

As mentioned previously, the uninorm involved can be defined on an arbitrary interval I with $[-1, 1] \subseteq I$. Since there is no uninorm which is continuous on the whole domain, but possibly on the whole domain up to the boundaries, we can avoid discontinuities on the bipolar scale of the normalized bi-capacities, i.e., on $= [-1, 1]$, by choosing a uninorm on $I \supset [-1, 1]$. In order to show how such a construction can be done, we provide the following simple example.

Example 15. Our aim is to construct a decomposable bi-capacity v on $\mathcal{Q}(X)$ of some finite universe X. We choose for the involved uninorm $U : [-2, 2]^2 \to [-2, 2]$ an extension of the 3-Π-operator [1] now acting on $[-2, 2]$

$$U(x_1, \dots, x_n) = 2 \cdot \frac{\prod_{i=1}^{n}(x_i + 2) - \prod_{i=1}^{n}(2 - x_i)}{\prod_{i=1}^{n}(x_i + 2) + \prod_{i=1}^{n}(2 - x_i)}.$$

We fix $v(\emptyset, \emptyset) = 0$, then we can choose $v(\{i\}, \emptyset) \geq 0$ and $v(\emptyset, \{i\}) \leq 0$ for all but one $i \in X$, since U is cancellative on $[-1, 1]^2$ and additionally the properties $v(X, \emptyset) = 1$ and $v(\emptyset, X) = -1$ have to be fulfilled. Table 1 shows the values of a decomposable bi-capacity $v : \mathcal{Q}(X) \to [-1, 1]$ on $X = \{1, 2\}$, with fixed values $v(\{1\}, \emptyset) = \frac{1}{2}$ and $v(\emptyset, \{1\}) = -\frac{1}{3}$. The values of a decomposable bi-capacity $v : \mathcal{Q}(X) \to [-1, 1]$ with underlying universe $X = \{1, 2, 3\}$ and $v(\{1\}, \emptyset) = v(\{2\}, \emptyset) = \frac{1}{2}$ and $v(\emptyset, \{1\}) = v(\emptyset, \{2\}) = -\frac{1}{4}$ are displayed in Table 2.

4. FINAL REMARKS AND CONCLUSION

We have mentioned already bi-capacities of CPT Type, which are characterized by the difference of two capacities, i.e., $v(A, B) = m_1(A) - m_2(B)$. In fact, this property expresses that the operations corresponding to the positive resp. negative part of the bipolar scale, i.e., $I^+ = I \cap [0, \infty]$ resp. $I^- = I \cap [-\infty, 0]$, are first treated independently of each other and are then combined for the determination of the final result.

Table 1. Decomposable bi-capacity with $|X| = 2$.

$v(A,B)$	$\emptyset$	$\{1\}$	$\{2\}$	X
$\emptyset$	0	$-\frac{1}{3}$	$-\frac{8}{11}$	-1
$\{1\}$	$\frac{1}{2}$	–	$-\frac{1}{4}$	–
$\{2\}$	$\frac{4}{7}$	$\frac{1}{4}$	–	–
X	1	–	–	–

Table 2. Decomposable bi-capacity with $|X| = 3$.

$v(A,B)$	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	X
$\emptyset$	0	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{11}{19}$	$-\frac{32}{65}$	$-\frac{4}{5}$	$-\frac{4}{5}$	-1
$\{1\}$	$\frac{1}{2}$	–	$\frac{8}{31}$	$-\frac{4}{47}$	–	–	$-\frac{1}{3}$	–
$\{2\}$	$\frac{1}{2}$	$\frac{8}{31}$	–	$-\frac{4}{47}$	–	$-\frac{1}{3}$	–	–
$\{3\}$	$\frac{1}{13}$	$-\frac{4}{23}$	$-\frac{4}{23}$	–	$-\frac{13}{31}$	–	–	–
$\{1,2\}$	$\frac{16}{17}$	–	–	$\frac{13}{31}$	–	–	–	–
$\{1,3\}$	$\frac{4}{7}$	–	$\frac{1}{3}$	–	–	–	–	–
$\{2,3\}$	$\frac{4}{7}$	$\frac{1}{3}$	–	–	–	–	–	–
X	1	–	–	–	–	–	–	–

Similarly, we expect that the introduced decomposable bi-capacities can be applied in decision making as a generalization of the CPT model using the so-called (S,T)-integral [12] or (S,U)-integral [11].

Further note that for our concept of decomposability of bi-capacities the uninorm U need not be symmetric on I^2 if the the order in (8) is fixed (for such operations we recommend [19] where associative, monotone operations on I with neutral element e are called pseudo-uninorms). Observe, however, that still S^+ and S^- should be symmetric. As an example of such an appropriate non-symmetric operation we mention $U : I^2 \to I$ given by

$$U(x,y) = \begin{cases} \min(x,y), & \text{if } x < 0, \\ \max(x,y), & \text{if } x > 0, \\ y, & \text{otherwise,} \end{cases}$$

where I is an arbitrary interval containing $[-1,1]$.

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