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1-LIPSCHITZ AGGREGATION OPERATORS AND QUASI-COPULAS

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In the paper, binary 1-Lipschitz aggregation operators and specially quasi-copulas are studied. The characterization of 1-Lipschitz aggregation operators as solutions to a functional equation similar to the Frank functional equation is recalled, and moreover, the importance of quasi-copulas and dual quasi-copulas for describing the structure of 1-Lipschitz aggregation operators with neutral element or annihilator is shown. Also a characterization of quasi-copulas as solutions to a certain functional equation is proved. Finally, the composition of 1-Lipschitz aggregation operators, and specially quasi-copulas, is studied.

Keywords: aggregation operator, 1-Lipschitz aggregation operator, copula, quasi-copula, kernel aggregation operator

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1. INTRODUCTION

The aim of this paper is to study 1-Lipschitz aggregation operators, and specially quasi-copulas. The study of these problems was motivated by several papers on fuzzy preference modeling [5, 6], or by papers concerning some problems in fuzzy probability calculus, e.g., by [10] and others. A distinguished example of 1-Lipschitz aggregation operators are copulas [17]. Well-known is the importance of copulas, as functions joining a multivariate distribution function to its one-dimensional distribution functions in statistical modeling and probability theory. The notion of a quasi-copula was introduced by Alsina, Nelsen and Schweizer in [1] and was used for characterizing operations on distribution functions that can be or cannot be derived from operations on random variables, cf. [17]. A simple characterization of quasi-copulas as special 1-Lipschitz functions has recently been given by Genest et al in [9], also see below. In [5] the construction of fuzzy preference structures by means of so-called generator triplets was studied. It was shown that a generator triplet (p, i, j) is monotone if and only if the indifference generator i is a commutative quasi-copula. Copulas and quasi-copulas also appear in applications of fuzzy logic where they are used for modeling conjunctors.

Let us start with recalling some basic notions that will be useful.

Let $n \in \mathbb{N}$, $n \geq 2$. n-ary aggregation operators are defined as non-decreasing functions $A : [0,1]^n \to [0,1]$ satisfying the boundary conditions $A(0,\dots,0) = 0$ and

$A(1, \ldots, 1) = 1$. In this paper we will deal with binary aggregation operators only, i.e. with $n = 2$, and therefore, if no confusion can appear, their name will often be shorten to aggregation operators only.

Aggregation operators satisfying the Lipschitz condition with constant 1, i.e., satisfying the property

$$|A(x_1, y_1) - A(x_2, y_2)| \leq |x_1 - x_2| + |y_1 - y_2|,$$

for all x_1, x_2, y_1, $y_2 \in [0, 1]$, will be called 1-Lipschitz aggregation operators.

From well-known types of binary aggregation operators, for example, the arithmetic mean M, the product operator Π, *Min* and *Max* operators, as well as weighted means, OWA operators, copulas, quasi-copulas, Choquet integral-based aggregation operators, Sugeno intergal-based aggregation operators are 1-Lipschitz aggregation operators. More details on these classes of aggregation operators can be found, e.g., in [2].

Distinguished classes of 1-Lipschitz aggregation operators are the classes of copulas and quasi-copulas.

A (two-dimensional) *copula* C is defined as a function $C : [0, 1]^2 \to [0, 1]$ with the properties

— $C(0, x) = C(x, 0) = 0$ and $C(x, 1) = C(1, x) = x$ for all $x \in [0, 1]$;

— $C(x_1, y_1) + C(x_2, y_2) \geq C(x_2, y_1) + C(x_1, y_2)$ for all x_1, x_2, y_1, $y_2 \in [0, 1]$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$.

The first property means that zero is annihilator and the element 1 neutral element of a copula. The second one is the moderate growth property or 2-monotonicity. From this property it follows that copulas are non-decreasing functions in each variable and also satisfy the Lipschitz condition mentioned above.

We omit the original definition of a (two dimensional) quasi-copula of Alsina et al in [1] and recall the more transparent one of Genest et al [9], who characterized *quasi-copulas* as functions $Q : [0, 1]^2 \to [0, 1]$ with the properties:

— $Q(0, x) = Q(x, 0) = 0$ and $Q(x, 1) = Q(1, x) = x$ for all $x \in [0, 1]$;

— Q is non-decreasing in each of its arguments;

— Q satisfies Lipschitz's condition (with constant 1).

Due to the 1-Lipschitz property, copulas as well as quasi-copulas are continuous functions on the unit square.

The relationship between copulas and quasi-copulas is given by the following characterization of quasi-copulas in terms of copulas [18]: the function $Q : [0, 1]^2 \to [0, 1]$ is a quasi-copula if and only if there exists a set $S \neq \emptyset$ of copulas such that for all $(x, y) \in [0, 1]^2$, $Q(x, y) = \sup\{C(x, y)\,;\, C \in S\}$.

Note that the conditions in the first item of the definition of a quasi-copula mean that quasi-copulas are aggregation operators with zero annihilator and neutral element equal to 1. One of these properties is superfluous because for 1-Lipschitz aggregation operators these properties are equivalent. Any aggregation operator A whose neutral element is $e_A = 1$, has the annihilator $a_A = 0$. However, in the case of 1-Lipschitz aggregation operators also the property $a_A = 0$ implies $e_A = 1$ (which is not true in general). This means that a 1-Lipschitz aggregation operator has neutral element $e_A = 1$ if and only if it has annihilator $a_A = 0$. Therefore quasi-copulas can be equivalently characterized as 1-Lipschitz aggregation operators with neutral element 1, or as 1-Lipschitz aggregation operators with zero annihilator. The set of all quasi-copulas will be denoted by $\mathcal{Q}$.

For any $Q \in \mathcal{Q}$, define the function $Q^* : [0,1]^2 \to [0,1]$ by $Q^*(x,y) = x + y - Q(x,y)$, which is called the dual of a quasi-copula Q. The dual of any quasi-copula is also a non-decreasing and 1-Lipschitz function, but with zero neutral element and annihilator equal to 1. Denote by $\mathcal{D}$ the set of all functions $f : [0,1]^2 \to [0,1]$ with mentioned properties. They will be called dual quasi-copulas. For each $f \in \mathcal{D}$ there is a quasi-copula Q such that $f = Q^*$, namely, $Q(x,y) = x + y - f(x,y)$.

The paper is further organized as follows. In the next section, the characterization of 1-Lipschitz aggregation operators as solutions to a functional equation similar to the Frank functional equation [8] is given, and moreover, it is shown that quasi-copulas and dual quasi-copulas play an important role in describing the structure of 1-Lipschitz aggregation operators with arbitrary annihilator or neutral element. Section 3 contains a characterization of quasi-copulas as solutions to a special type of functional equation, and also an additional necessary condition for being a quasi-copula. Section 4 is devoted to the study of composition of 1-Lipschitz aggregation operators, and again, a special attention is paid to quasi-copulas. The paper ends with several concluding remarks.

2. BINARY 1-LIPSCHITZ AGGREGATION OPERATORS

In the first subsection of this section we will characterize 1-Lipschitz aggregation operators in general. Then, in the second and third subsections, we describe the structure of 1-Lipschitz aggregation operators with any annihilator or neutral element from the unit interval.

2.1. Characterization of binary 1-Lipschitz aggregation operators

The following theorem shows that 1-Lipschitz aggregation operators can be characterized as solutions to a simple functional equation which is similar to the Frank functional equation [8].

Theorem 1. A binary aggregation operator A is 1-Lipschitz if and only if there is a binary aggregation operator B, such that for all x, $y \in [0,1]$ it holds

$$A(x,y) + B(x,y) = x + y\,. \tag{1}$$

P r o o f. (i) Let A be a 1-Lipschitz aggregation operator. We show that then the function B defined by $B(x,y) = x + y - A(x,y)$ is an aggregation operator. It is clear that B satisfies the boundary conditions $B(0,0) = 0$ and $B(1,1) = 1$. We prove that the 1-Lipschitz property of A implies the monotonicity of B.
Let $y, x_1, x_2 \in [0,1]$ are any points such that $x_1 < x_2$. Then

$$B(x_2,y) - B(x_1,y) = x_2 - x_1 + A(x_1,y) - A(x_2,y). \tag{2}$$

Due to the 1-Lipschitz property and monotonicity of A we have $A(x_2,y) - A(x_1,y) \leq x_2 - x_1$, which together with (2) gives $B(x_1,y) \leq B(x_2,y)$. Thus B is monotone in the first coordinate. An analogous claim is valid for the second coordinate and therefore B is monotone as an aggregation operator. It is clear that the pair (A,B) solves the equation (1).

(ii) Next, assume B is an aggregation operator. Mention that because of the inequality

$$|A(x_1,y_1) - A(x_2,y_2)| \leq |A(x_1,y_1) - A(x_2,y_1)| + |A(x_2,y_1) - A(x_2,y_2)|$$

which holds for all x_1, y_1, x_2, $y_2 \in [0,1]$, the 1-Lipschitz property of A follows from the 1-Lipschitz property of functions $A(.,y)$, $A(x,.)$, x, $y \in [0,1]$.
Let $x_1, x_2, y \in [0,1]$ be any points, and without loss of generality, let $x_1 \leq x_2$. Then due to monotonicity of B we have

$$0 \leq B(x_2,y) - B(x_1,y) = x_2 - x_1 - A(x_2,y) + A(x_1,y),$$

which leads to $A(x_2,y) - A(x_1,y) \leq x_2 - x_1$, that is, to the 1-Lipschitz property of the function $A(.,y)$. The proof for the 1-Lipschitz property of $A(x,.)$ is similar. □

In the sequel, for a given aggregation operator A denote $A^*(x,y) = x+y-A(x,y)$. By the previous theorem, A is a 1-Lipschitz aggregation operator if and only if the function A^* is an aggregation operator. Repeating this, we obtain that A^* is a 1-Lipschitz aggregation operator if and only if $(A^*)^*$ is an aggregation operator. Since $(A^*)^* = A$, we have that aggregation operator A is 1-Lipschitz if and only if A^* is a 1-Lipschitz aggregation operator.
In the framework of aggregation operators the standard dual to an aggregation operator A is defined by $A^d(x,y) = 1 - A(1-x, 1-y)$. However, the property $(A^*)^* = A$ also expresses certain type of duality between A and A^*.

If A is a 1-Lipschitz aggregation operator A then certainly

$$x + y - 1 \leq x + y - A^*(x,y) \leq x + y,$$

that is,

$$\max(x + y - 1, 0) \leq A(x,y) \leq \min(x + y, 1).$$

This means that the condition

$$T_L \leq A \leq S_L, \tag{3}$$

where $T_L(x,y) = \max(x+y-1,0)$ is the Łukasiewicz t-norm and $S_L(x,y) = \min(x+y,1)$ is the Łukasiewicz t-conorm, is a necessary condition for a binary aggregation operator to be 1-Lipschitz.

Finally, suppose that a 1-Lipschitz aggregation operator A has neutral element e_A. Then for $\forall x \in [0,1]$, $A^*(x,e_A) = A^*(e_A,x) = e_A$, which means that the element e_A is the annihilator of the operator A^*, i.e., $e_A = a_{A^*}$. Analogously, for the annihilator of A, if it exists, we have $a_A = e_{A^*}$.

2.2. The structure of binary 1-Lipschitz aggregation operators with annihilator

In this subsection we show that each 1-] pschitz aggregation operator with annihilator $a \in]0,1[$ is built up from a dual quasi-copula, a quasi-copula and the value a.

Let A be a 1-Lipschitz aggregation operator with annihilator $a_A \in [0,1]$. According to the previous discussions:

— if $a_A = 0$ then A is a quasi-copula;

— if $a_A = 1$ then $e_{A^*} = 1$, which means that the operator A^* is a quasi-copula, and thus A is a dual quasi-copula.

Now, let $a_A = a \in]0,1[$. Define the mappings φ_a, ψ_a by

$$\varphi_a(x) = \frac{x}{a}, \quad , \quad \psi_a(x) = \frac{x-a}{1-a}. \tag{4}$$

Then the function $Q_A : [0,1]^2 \to [0,1]$,

$$Q_A(x,y) = \psi_a\left(A\left(\psi_a^{-1}(x), \psi_a^{-1}(y)\right)\right) \tag{5}$$

is a quasi-copula, and the function $D_A : [0,1]^2 \to [0,1]$

$$D_A(x,y) = \varphi_a\left(A\left(\varphi_a^{-1}(x), \varphi_a^{-1}(y)\right)\right) \tag{6}$$

is a dual quasi-copula. We omit the details because the proofs go similarly as in the case of nullnorms, [3] .
Therefore

$$A(x,y) = \begin{cases} \varphi_a^{-1}\left(D_A\left(\varphi_a(x), \varphi_a(y)\right)\right) & \text{if } (x,y) \in [0,a] \times [0,a] \\ \psi_a^{-1}\left(Q_A\left(\psi_a(x), \psi_a(y)\right)\right) & \text{if } (x,y) \in [a,1] \times [a,1]. \end{cases}$$

If $(x,y) \in [0,a[\times]a,1]$, then

$$a = A(x,a) \leq A(x,y) \leq A(a,y) = a,$$

which means that $A(x,y) = a$, and the same is true for the rest of the unit square $]a,1] \times [0,a[$.

2.3. The structure of 1-Lipschitz aggregation operators with neutral element

A similar situation to the previous one is for 1-Lipschitz aggregation operators with neutral element.

Let A be a 1-Lipschitz aggregation operator with neutral element $e_A \in [0,1]$. Trivially,

— if $e_A = 1$ then A is a quasi-copula;

— if $e_A = 0$ then $a_{A^*} = 0$, and because A^* is a 1-Lipschitz aggregation operator, A^* is a quasi-copula, which implies that A is a dual quasi-copula.

Finally, assume that $e_A = e \in]0,1[$. Then the function $Q_A : [0,1]^2 \to [0,1]$,

$$Q_A(x,y) = \varphi_e \left(A \left(\varphi_e^{-1}(x), \varphi_e^{-1}(y) \right) \right) \tag{7}$$

is a quasi-copula, and the function $D_A : [0,1]^2 \to [0,1]$,

$$D_A(x,y) = \psi_e \left(A \left(\psi_e^{-1}(x), \psi_e^{-1}(y) \right) \right) \tag{8}$$

is a dual quasi-copula. Therefore

$$A(x,y) = \begin{cases} \varphi_e^{-1} \left(Q_A \left(\varphi_e(x), \varphi_e(y) \right) \right) & \text{if } (x,y) \in [0,e] \times [0,e] \\ \psi_e^{-1} \left(D_A \left(\psi_e(x), \psi_e(y) \right) \right) & \text{if } (x,y) \in [e,1] \times [e,1]. \end{cases}$$

In the case of uninorms [7] which is similar to this one, the values on the rest parts of the unit square are not determined uniquely, they are between the values of *Min* and *Max* operators, in general. In the case of 1-Lipschitz aggregation operators the values at the points $(x,y) \in [0,e[\times]e,1] \cup]e,1] \times [0,e[$ are determined uniquely. Indeed, if the operator A is 1-Lipschitz aggregation operator, the same is true for A^*, and moreover, $a_{A^*} = e$. Using the results of the previous subsection, the values of A^* at these points are $A^*(x,y) = e$, that is, $A(x,y) = x + y - e$ at all points $(x,y) \in [0,e[\times]e,1] \cup]e,1] \times [0,e[$.

3. CHARACTERIZATION OF QUASI-COPULAS

In the previous section we have shown that all 1-Lipschitz aggregation operators with annihilator or neutral element are fully characterized by quasi-copulas and dual quasi-copulas. In the case of commutative 1-Lipschitz aggregation operators also the corresponding quasi-copulas and dual quasi-copulas will be commutative. In this section we give a characterization of commutative quasi-copulas as solutions to a certain type of a functional equation.

Let us start with a slight modification of a given definition of a quasi-copula, showing that the boundary conditions characterizing quasi-copulas can be simplified.

Lemma 1. A function $Q : [0,1]^2 \to [0,1]$ is a quasi-copula if and only if it satisfies the following conditions:

(i) Q is non-decreasing;

(ii) Q is 1-Lipschitz;

(iii) $Q(0,1) = Q(1,0) = 0$ and $Q(1,1) = 1$.

Proof. It is clear that each quasi-copula fulfills the properties (i)–(iii). Conversely, from the 1-Lipschitz property and the conditions in (iii) we obtain the inequalities

$$\forall x \in [0,1] \, : \, Q(x,1) = Q(x,1) - Q(0,1) \leq x \quad \text{and} \quad Q(1,1) - Q(x,1) \leq 1 - x,$$

which give $x \leq Q(x,1) \leq x$, that is $Q(x,1) = x$. Analogously, for each $x \in [0,1]$, $Q(1,x) = x$, that is, 1 is the neutral element of Q. The fact that 0 is its annihilator follows from the monotonicity of Q and the properties in (iii) or from the discussion in Introduction. □

Remark 1. Since an aggregation operator A is always monotone and satisfies the property $A(1,1) = 1$, A is a quasi-copula if and only if it is 1-Lipschitz and $A(0,1) = A(1,0) = 0$.

As mentioned above, quasi-copulas can be characterized as solutions to a certain type of a functional equation. For simplicity, we prove the claim for commutative quasi-copulas.

Theorem 2. A commutative aggregation operator A is a commutative quasi-copula if and only if there exists an aggregation operator B such that for all $x, y \in [0,1]$ we have

$$A(x,y) + B(1-x,y) = y. \tag{9}$$

Proof. (i) Let A be a commutative quasi-copula. Define a function $B : [0,1]^2 \to [0,1]$ by

$$B(x,y) = y - A(1-x,y).$$

Then evidently $B(0,0) = 0$ and $B(1,1) = 1$. Next, let $x, y \in [0,1]$ be any elements and let $\epsilon \geq 0$ be an arbitrary number such that $x + \epsilon \in [0,1]$. Then

$$B(x+\epsilon,y) - B(x,y) = A(1-x,y) - A(1-x-\epsilon,y) \geq 0,$$

which follows from the monotonicity of A. Thus, B is monotone in the first coordinate.

For any $x, y \in [0,1]$ and $\epsilon \geq 0$ such that $y + \epsilon \in [0,1]$ we also have

$$B(x,y+\epsilon) - B(x,y) = \epsilon - (A(1-x,y+\epsilon) - A(1-x,y)) \geq 0,$$

because, due to the 1-Lipschitz property of A, it holds $A(1-x, y+\epsilon)-A(1-x, y) \leq \epsilon$. The function B is also monotone in the second coordinate. This means that B is an aggregation operator and moreover, the pair (A, B) solves the equation (9).

(ii) Let A be a commutative aggregation operator, which together with some aggregation operator B fulfills the equation (9). To show that A is a commutative quasi-copula, it is enough to show that A is 1-Lipschitz and $A(1,0) = 0$.
Put in the equation (9) $y = 0$. Then for each $x \in [0,1]$, it holds $A(x,0)+B(1-x,0) = 0$, which implies $A(x,0) = 0$ for each $x \in [0,1]$.
On the contrary, suppose that A is not a 1-Lipschitz operator. Then there is a $y \in [0,1[$ and an $\epsilon > 0$ such that $y + \epsilon \in [0,1]$ and

$$A(x, y+\epsilon) - A(x,y) > \epsilon\,.$$

Then

$$B(1-x, y+\epsilon) - B(1-x, y) = \epsilon - (A(x, y+\epsilon) - A(x,y)) < 0,$$

which contradicts the monotonicity of B. So, A is a 1-Lipschitz aggregation operator with the property $A(0,1) = A(1,0) = 0$, and by Lemma 1 it is a quasi-copula. □

Remark 2. The previous claim without the commutativity condition should have to be reformulated in the following way: An aggregation operator A is a quasi-copula if and only if there exist aggregation operators B and C such that for each x, $y \in [0,1]$ we have

$$A(x,y) + B(1-x, y) = y \quad \text{and} \quad A(x,y) + C(x, 1-y) = x.$$

In [10], the Bell inequalities were studied. It was shown that each commutative quasi-copula satisfies for each x, y, $z \in [0,1]$ the inequality

$$x - f(x,y) - f(x,z) + f(y,z) \geq 0. \tag{10}$$

However, this inequality, together with commutativity and monotonicity of f and neutral element equal to 1, does not fully characterize commutative quasi-copulas. Fulfilling the inequality (10) is only a necessary condition for functions to be commutative quasi-copulas, as is shown in the following example.

Example 1. The function $f : [0,1]^2 \to [0,1]$ defined by

$$f(x,y) = T_L(x,y).(2 - S_M(x,y)) \tag{11}$$

is non-decreasing, commutative, with neutral element $e = 1$ and fulfills the inequality (10), but it is not a quasi-copula.

To see this, consider the following subsets of the unit square:

$$U_0 = \{(x,y)\,;\, x+y \leq 1\}, \quad U_1 = \{(x,y)\,;\, x+y > 1 \wedge x \leq y\}$$

and

$$U_2 = \{(x, y)\,;\, x + y > 1 \wedge x \geq y\}.$$

Then $(x, y) \in U_0 \Rightarrow T_L(x, y) = 0 \Rightarrow f(x, y) = 0$.
Next, for all $(x, y) \in U_1$ we have

$$f(x, y) = (x + y - 1)(2 - y),$$

and for all $(x, y) \in U_2$, it is

$$f(x, y) = (x + y - 1)(2 - x).$$

It is clear that f is commutative and with neutral element $e = 1$. It is also continuous and partial derivatives at all inner points of U_1 are $\frac{\partial f}{\partial x}(x, y) = 2 - y \geq 0$, and $\frac{\partial f}{\partial y}(x, y) = 3 - 2y - x \geq 0$. The commutativity of f ensures similar inequalities for U_2, and therefore f is non-decreasing on $[0, 1]^2$.
However, the function f is not 1-Lipschitz. For example, for the point $(0.5, 0.9)$ the value of partial derivative is $\frac{\partial f}{\partial x}(0.5, 0.9) = 1.1 > 1$, which contradicts the 1-Lipschitz property of f.

Despite f is not a quasi-copula, it fulfills the inequality (10). To show this, consider only the case $x > \max(y, z)$, since in all other cases any commutative non-decreasing function f with neutral element 1 satisfies the inequality (10). Moreover, because of the commutativity of f, it is enough to pay attention to the case $y \leq z < x$ only.

► Consider first that $x + y \leq 1$. Then
$f(x, y) = 0$, $f(y, z) = 0$, and for the expression $E(x, y, z)$ on the left-hand side of (10) we obtain

$$E(x, y, z) = x - f(x, z) = f(x, 1) - f(x, z) \geq 0,$$

which follows from the monotonicity of f.
► Now, consider the case $x + y > 1$. Then because of $y \leq z < x$, also $x + z > 1$, and for the left-hand side expression $E(x, y, z)$ of (10) it holds

$$\begin{aligned} E(x, y, z) &= x - (x + y - 1).(2 - x) - (x + z - 1).(2 - x) \\ &+ \max(y + z - 1, 0)(2 - z) \\ &= x - (2 - x).(2x + y + z - 2) + \max(y + z - 1, 0)(2 - z). \end{aligned} \tag{12}$$

• If $y + z \leq 1$, then $T_L(y, z) = 0$ and $2x + y + z - 2 \leq 2x - 1$. Therefore

$$E(x, y, z) \geq x - (2 - x)(2x - 1) = 2(x - 1)^2 \geq 0.$$

• If $y + z > 1$, then $T_L(y, z) = y + z - 1$, and since $2 - z > 2 - x$, from (12) we obtain

$$E(x, y, z) \geq x - (2 - x)(2x - 1) = 2(x - 1)^2 \geq 0.$$

This ends the proof of the claim that f fulfills the inequality (10) despite it is not a quasi-copula.

4. ON COMPOSITION OF 1-LIPSCHITZ AGGREGATION OPERATORS

If A, B are n-ary aggregation operators and F is a binary aggregation operator then a function $F(A,B) : [0,1]^n \to [0,1]$ defined by

$$F(A,B)(x_1,\dots,x_n) = F(A(x_1,\dots,x_n), B(x_1,\dots,x_n)),$$

is also an n-ary aggregation operator and is called a composed aggregation operator. It is known, that although all three aggregation operators A, B, F are 1-Lipschitz, the composed aggregation operator $F(A,B)$ need not be of this property. For example, despite the Lukasiewicz t-conorm S_L is a 1-Lipschitz aggregation operator, the composed operator $S_L(S_L,S_L)$ does not possess this property [12]. However, if the outer operator F is a kernel aggregation operator, and A, B are 1-Lipschitz, then $F(A,B)$ is always 1-Lipschitz aggregation operator [4, 12].
Recall that a binary aggregation operator F has a *kernel property* if for all $u_1, u_2, v_1, v_2 \in [0,1]^2$ we have

$$|F(u_1,v_1) - F(u_2,v_2)| \leq \max\left(|u_1 - u_2|, |v_1 - v_2|\right).$$

It is clear that each kernel aggregation operator is also 1-Lipschitz. More details on kernel aggregation operators can be found in [13, 14, 15]. It can be shown that the kernel property of an outer operator is also a necessary condition for the 1-Lipschitz property of a composed aggregation operator. In the sequel, we will again deal with binary aggregation operators only.

Proposition 1. Let F be a binary aggregation operator. Then for any binary 1-Lipschitz aggregation operators A and B the composed aggregation operator $F(A,B)$ is 1-Lipschitz if and only if F is a kernel aggregation operator.

Proof. The sufficiency was proved in [12].
Necessity: Assume F is not a kernel aggregation operator. We show that then there exist 1-Lipschitz aggregation operators A, B, such that $F(A,B)$ is not 1-Lipschitz.

The kernel property of an aggregation operator is equivalent to its sub-shift invariantness [4]. Since F is not kernel, it is not sub-shift invariant, i.e., there exist such u, v, $a \in [0,1]$ that also $u+a$, $v+a \in [0,1]$ and

$$F(u+a, v+a) > a + F(u,v).$$

Suppose that $u \leq v$ and put

$$\begin{aligned} A(x,y) &= \min\left(1, \max(x+y-(v-u), 0)\right), \\ B(x,y) &= S_L(x,y) = \min(1, x+y). \end{aligned}$$

It can be easily shown that the operators A and B are 1-Lipschitz. If we choose the points $x = y = \frac{v}{2}$ and $x' = y' = \frac{v}{2} + \frac{a}{2}$, then

$$A\left(\frac{v}{2},\frac{v}{2}\right) = \min(1,u) = u, \qquad B\left(\frac{v}{2},\frac{v}{2}\right) = \min(1,v) = v,$$
$$A\left(\frac{v}{2}+\frac{a}{2},\frac{v}{2}+\frac{a}{2}\right) = \min(1,\max(a+u,0)) = a+u,$$
$$B\left(\frac{v}{2}+\frac{a}{2},\frac{v}{2}+\frac{a}{2}\right) = \min(1,v+a) = v+a,$$

and therefore

$$\begin{aligned}F(A,B)(x',y') - F(A,B)(x,y) &= F\left(A\left(\frac{v}{2}+\frac{a}{2},\frac{v}{2}+\frac{a}{2}\right),B\left(\frac{v}{2}+\frac{a}{2},\frac{v}{2}+\frac{a}{2}\right)\right) \\ &- F\left(A\left(\frac{v}{2},\frac{v}{2}\right),B\left(\frac{v}{2},\frac{v}{2}\right)\right) \\ &= F(u+a,v+a) - F(u,v) \\ &> a = |x'-x| + |y'-y|,\end{aligned}$$

which means that $F(A,B)$ is not a 1-Lipschitz aggregation operator. Note that aggregation operators of the type A were introduced in [16]. □

As a consequence of the previous theorem we obtain that if the outer operator F is kernel, then composition of two quasi-copulas is a quasi-copula. Observe that $F(0,0)=0$ ensures that zero is an annihilator of the composed operator whenever both inner operators have zero as their annihilator.
In the next part we show that for quasi-copulas, as a special type of 1-Lipschitz aggregation operators, the kernel property of F can be relaxed.

Lemma 2. Denote $K = \{(Q_1(x,y),Q_2(x,y))\,;\,(x,y)\in[0,1]^2, Q_1,\,Q_2\in\mathcal{Q}\}$. Then

$$K = \left\{(u,v)\,;\,u\in[0,1],\,v\in\left[\max(2u-1,0),\frac{u+1}{2}\right]\right\}.$$

Proof. The set K is built from all pairs $(Q_1(x,y),Q_2(x,y))$ of values of all quasi-copulas on $[0,1]$. It holds $K = \bigcup_{Q_1,Q_1\in\mathcal{Q}} K_{Q_1,Q_2}$, where K_{Q_1,Q_2} is an analogous set for a fixed pair of quasi-copulas Q_1, Q_2.
Recall that for each quasi-copula it holds

$$T_L(x,y) \le Q(x,y) \le T_M(x,y), \quad (x,y)\in[0,1]^2. \tag{13}$$

Let Q_1 be any quasi-copula.
• Assume first $Q_1(x,y)=0$. Then from the lower inequality in (13) we obtain $x+y-1\le 0$, which means that the point $(x,y)\in\{(x,y)\in[0,1]^2\,;\,y\le 1-x\}$. Because of the upper inequality in (13), for any quasi-copula $Q_2\in\mathcal{Q}$ at the points with $y\le 1-x$ we have

$$\max_{y\le 1-x}\min(x,y) = \frac{1}{2}.$$

So, if $Q_1(x,y)=0$, the values of each quasi-copula Q_2 will certainly be in the interval $[0,\frac{1}{2}]$.
• Further, assume $Q_1(x,y)=1$. From (13) we obtain $\min(x,y)=1$, i.e., $x=y=1$, which implies $Q_2(x,y)=1$.
If we denote $Q_1(x,y)=u$ and $Q_2(x,y)=v$, the previous results say that:

$$u=0\Rightarrow v\in[0,\frac{1}{2}] \quad\text{and}\quad u=1\Rightarrow v=1.$$

Finally, assume that $Q_1(x,y) = u \in]0,1[$. From (13) we have

$$\max(x+y-1,0) \le u \le \min(x,y),$$

i.e., $y \le 1-x+u$ and simultaneously, $x \ge u$ and $y \ge u$. This means that in the considered case, the points $(x,y) \in S_u$, where

$$S_u = \{(x,y) \in [0,1]^2\,;\, y \le 1-x+u,\, x \ge u,\, y \ge u\}.$$

Again, due to the upper inequality in (13), for each quasi-copula Q_2 at the points $(x,y) \in S_u$ it holds

$$Q_2(x,y) \le \max_{(x,y)\in S_u} \min(x,y) = \frac{u+1}{2}.$$

Monotonicity of quasi-copulas and the inequality $\max(x+y-1,0) \le Q_2(x,y)$ in (13) imply

$$Q_2(x,y) \ge Q_2(u,u) \ge \max(2u-1,0),$$

which is valid for all quasi-copulas $Q_2 \in \mathcal{Q}$ and all points $(x,y) \in S_u$.
We conclude that if $Q_1(x,y) = u \in]0,1[$, then

$$\max(2u-1,0) \le Q_2(x,y) \le \frac{u+1}{2}, \quad Q_2 \in \mathcal{Q}. \tag{14}$$

Note that the results for $u=0$ and $u=1$ can also be obtained from (14). □

We have shown that for any two quasi-copulas Q_1 and Q_2 the set of all points $(u,v) = (Q_1(x,y), Q_2(x,y))$ is the subset K of the unit square of the form

$$K = \left\{(u,v)\,;\, u \in [0,1],\, v \in \left[\max(2u-1,0), \frac{u+1}{2}\right]\right\}.$$

Theorem 3. Let F be an aggregation operator. For any quasi-copulas Q_1, Q_2, a composed aggregation operator $F(Q_1,Q_2)$ is a quasi-copula if and only if the operator F has the kernel property on the set K defined in Lemma 2.

Proof. *Sufficiency:* Let F be an aggregation operator with the kernel property on the set K, and let Q_1, Q_2 be any two quasi-copulas. The function $A = F(Q_1,Q_2)$ is an aggregation operator and therefore, A is a quasi-copula iff A is 1-Lipschitz and $A(1,0) = A(0,1) = 0$. The last property is evident,

$$A(1,0) = F(Q_1(1,0), Q_2(1,0)) = F(0,0) = 0,$$

and the same holds for $A(0,1)$.
To prove the 1-Lipschitz property of A, choose any (x_1,y_1), $(x_2,y_2) \in [0,1]^2$ and put $u = Q_1(x_1,y_1)$, $v = Q_2(x_1,y_1)$, $u' = Q_1(x_2,y_2)$, $v' = Q_2(x_2,y_2)$. Then

$$\begin{aligned} |A(x_1,y_1) - A(x_2,y_2)| &= |F(Q_1(x_1,y_1), Q_2(x_1,y_1)) - F(Q_1(x_2,y_2), Q_2(x_2,y_2))| \\ &= |F(u,v) - F(u',v')| \le \max(|u-u'|, |v-v'|), \end{aligned} \tag{15}$$

because (u,v), $(u',v') \in K$ and by the assumption the operator F is kernel on the set K. Further, after several trivial steps, using the 1-Lipschitz property of quasi-copulas Q_1, Q_2, (15) results in

$$|A(x_1,y_1) - A(x_2,y_2)| \le |x_1 - x_2| + |y_1 - y_2|,$$

which means that A is a 1-Lipschitz aggregation operator.

Necessity: We need to prove that if a composed aggregation operator $F(Q_1,Q_2) \in \mathcal{Q}$ for all Q_1, $Q_2 \in \mathcal{Q}$, then F is a kernel aggregation operator on the set K, or equivalently, if F is not a kernel aggregation operator on K, then there exist quasi-copulas Q_1, Q_2 such that $F(Q_1,Q_2) \notin \mathcal{Q}$.

Assume that F is not a kernel operator on K. Then F is not sub-shift invariant on K, i.e., there exist $(u,v) \in K$, $a \in [0,1]$, such that $(u+a, v+a) \in K$, and

$$F(u+a, v+a) > a + F(u,v). \tag{16}$$

Suppose that $u \le v$. Put $Q_1 = T_L$ and $Q_2 = (< 0, u-v+1 >, T_L)$, i.e., Q_2 is an ordinal sum [11]. If $u = v$, the operator Q_2 is also the Łukasiewicz t-norm, in other cases it is a non-trivial ordinal sum.

Let $x = v + \frac{a}{2}$, $y = u + 1 - v - \frac{a}{2}$. Then

$$Q_1(x,y) = \max(u,0) = u, \quad Q_2(x,y) = \max(v,0) = v,$$

$$Q_1(x+\frac{a}{2}, y+\frac{a}{2}) = \max(u+a,0) = u+a, \quad Q_2(x+\frac{a}{2}, y+\frac{a}{2}) = \max(v+a,0) = v+a$$

and therefore

$$F(Q_1,Q_2)(x+\frac{a}{2}, y+\frac{a}{2}) - F(Q_1,Q_2)(x,y) = F(u+a, v+a) - F(u,v) > a,$$

which means that $F(Q_1,Q_2)$ is not a 1-Lipschitz aggregation operator, thus not a quasi-copula. □

For composition of copulas the previous claim is not true. Despite the outer operator is kernel, the composition of two copulas need not to be a copula, as we can see in the following example.

Example 2. Let $F = \text{med}_k$, $k \in [0,1]$, i.e., $F(x,y) = \text{med}(x,y,k)$. Set $C_1 = T_L$ and $C_2 = T_P$, where T_P is the product t-norm. Then the composed operator is $A_k = \text{med}_k(T_L, T_P)$.

The operators C_1 and C_2 are copulas and each operator $F = \text{med}_k$ is a kernel aggregation operator on $[0,1]^2$. According to Theorem 4, the composed operator A_k is always 1-Lipschitz. For example, for $k = 0.5$ we obtain the operator

$$A_{0.5}(x,y) = \begin{cases} T_L(x,y) & \text{if } T_L(x,y) \ge 0.5 \\ T_P(x,y) & \text{if } T_P(x,y) \le 0.5 \\ 0.5 & \text{if } T_L \le 0.5 \le T_P(x,y). \end{cases}$$

The operator $A_{0.5}$ is not a copula because it is not 2-monotone. To show this, consider the points $x = \frac{2}{3}$, $y = \frac{3}{4}$, $x' = \frac{2}{3}$ and $y' = \frac{3}{4}$. Then we have

$$A_{0.5}\left(\frac{3}{4},\frac{3}{4}\right) + A_{0.5}\left(\frac{2}{3},\frac{2}{3}\right) - A_{0.5}\left(\frac{2}{3},\frac{3}{4}\right) - A_{0.5}\left(\frac{3}{4},\frac{2}{3}\right) =$$
$$0.5 + \frac{4}{9} - 0.5 - 0.5 = -\frac{1}{18} < 0,$$

which contradicts the 2-monotonicity of $A_{0.5}$.
Note that by the previous theorem, all operators A_k, $k \in [0,1]$, are quasi-copulas. The claim follows from the facts that T_L and T_P are quasi-copulas (each copula is also a quasi-copula) and the outer operator $\mathrm{med}(x,y,k)$ is kernel on $[0,1]^2$ and thus also on the set K.

Remark 3. Theorem 3 deals with the kernel property of an aggregation operator F on the set K from Lemma 2. However, for any aggregation operator F' such that $F|K = F'|K$, we have $F(Q_1,Q_2) = F'(Q_1,Q_2)$ for all pairs of quasi-copulas Q_1, $Q_2 \in \mathcal{Q}$. Moreover, if for any aggregation operator F which is kernel on the set K, we define a mapping $F' : [0,1]^2 \to [0,1]$ by

$$F'(x,y) = \begin{cases} F(x,y) & \text{if } (x,y) \in K \\ F\left(\frac{y+1}{2},y\right) & \text{if } y < 2x-1 \\ F(2y-1,y) & \text{if } y > \frac{x+1}{2}, \end{cases}$$

then F' is a kernel aggregation operator (on the unit square) and $F'|K = F|K$. Summarizing all above facts, for composition of quasi-copulas it is sufficient to deal with kernel aggregation operators as outer operators only, since no new composed operators can be obtained when kernel property on K is only required.

5. CONCLUSION

We have studied binary 1-Lipschitz aggregation operators. The main attention was paid to quasi-copulas, which were characterized as solutions to a certain functional equation. We have shown that quasi-copulas and dual quasi-copulas are also important for describing the structure of 1-Lipschitz aggregation operators with any neutral element or annihilator in the unit interval. We have also studied under which conditions the composition of 1-Lipschitz aggregation operators, and specially quasi-copulas, preserves these properties. We expect fruitful application of obtained results in preference modeling [5, 6] and statistics [18].

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