KYBERNETIKA — VOLUME 39 (2003), NUMBER 5, PAGES 569 – 582

COMPUTING COMPLEXITY DISTANCES BETWEEN ALGORITHMS

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We introduce a new (extended) quasi-metric on the so-called dual p-complexity space, which is suitable to give a quantitative measure of the improvement in complexity obtained when a complexity function is replaced by a more efficient complexity function on all inputs, and show that this distance function has the advantage of possessing rich topological and quasi-metric properties. In particular, its induced topology is Hausdorff and completely regular.

Our approach is applied to the measurement of distances between infinite words over the decimal alphabet and some advantages of our computations with respect to the ones that provide the classical Baire metric are discussed.

Finally, we show that the application of fixed point methods to the complexity analysis of Divide & Conquer algorithms, presented by M. Schellekens (Electronic Notes in Theoret. Comput. Sci. *1* (1995)), can be also given from our approach.

Keywords: invariant extended quasi-metric, complexity function, balanced quasi-metric, infinite word, Baire metric, contraction mapping, Divide & Conquer algorithm

AMS Subject Classification: 54E50, 5H25, 54H99, 68Q25

1. INTRODUCTION AND PRELIMINARIES

In the sequel the letters $\mathbb{R}^+$, ω and $\mathbb{N}$ will denote the set of nonnegative real numbers, the set of nonnegative integer numbers and the set of positive integers numbers, respectively.

M. Schellekens introduced in [23] the theory of complexity spaces as a part of the development of a topological foundation for the complexity analysis of programs and algorithms. Later on, S. Romaguera and M. Schellekens [21] introduced the so-called dual complexity space and obtained several quasi-metric properties of the complexity space which are interesting from a computational point of view, via the analysis of its dual. Recently, it was shown in [22] that the dual complexity space admits a structure of a (quasi-)normed semilinear space in the sense of [20] (see Section 2).

In [7], the notion of dual complexity has been extended to the "p-dual" case, where $p > 1$, for including in this theoretical approach to computational complexity,

[1]The authors acknowledge the support of the Spanish Ministry of Science and Technology, Plan Nacional I+D+I, Grant BFM2003-02302 and FEDER.

algorithms with running time $\mathcal{O}(2^n/n^r)$, $0 < r \leq 1$ (see Section 2). However, for all $p \in [1, +\infty)$, the quasi-pseudo-metric generated in a natural way induces a T_0 topology that is not even T_1.

In this paper we construct a new distance function on the dual p-complexity space, namely, an invariant extended quasi-metric, which is suitable to measure progress made in lowering the complexity by replacing a given program by another program which is more efficient on all inputs. In particular, it permits us to give a numerical quantification of progress made in lowering the complexity by replacing a given program Q by a program P which is more efficient on all inputs. Moreover, this distance function possesses rich topological and quasi-metric properties as Hausdorffness and complete regularity, among others. We also apply this extended quasi-metric to the measurement of distances between infinite words over the decimal alphabet and analyze some advantages of our methods with respect to the ones that use the classical Baire metric. In this way, we partially reconcile the theory of computational complexity with the corresponding to denotational semantics. Finally, we show that, similarly to the approach made by Schellekens in [23], Divide & Conquer algorithms induce contraction mappings for our extended quasi-metric, and then a Banach-type fixed point theorem is applicable to our context.

Our main reference for general topology is [4] and for quasi-pseudo-metric spaces they are [6] and [14].

Let us recall that a quasi-pseudo-metric on a set X is a nonnegative real-valued function d on $X \times X$ such that for all $x, y, z \in X$: (i) $d(x, x) = 0$, and (ii) $d(x, z) \leq d(x, y) + d(y, z)$.

If, in addition, d satisfies: (iii) $d(x, y) = 0$ if and only if $x = y$, then d is called a quasi-metric on X.

We will also consider extended quasi-(pseudo-)metrics. They satisfy the three above axioms, except that we allow $d(x, y) = +\infty$.

If d is a quasi-(pseudo-)metric on X, then the function d^{-1} defined on $X \times X$ by $d^{-1}(x, y) = d(y, x)$, is also a quasi-(pseudo-)metric on X, and d^s defined on $X \times X$ by $d^s(x, y) = \max\{d(x, y), d^{-1}(x, y)\}$, is a (pseudo-)metric on X. If d is an extended quasi-(pseudo-)metric on X, then d^{-1} and d^s are an extended quasi-(pseudo-)metric and an extended (pseudo-)metric on X, respectively.

Each extended quasi-pseudo-metric d on a set X induces a topology $\mathcal{T}(d)$ on X which has as a base the family of open d-balls $\{B_d(x, r) : x \in X,\ r > 0\}$, where $B_d(x, r) = \{y \in X : d(x, y) < r\}$ for all $x \in X$ and $r > 0$.

If d is an extended quasi-metric on X, then the topology $\mathcal{T}(d)$ induced by d is T_1.

A quasi-metric space is a pair (X, d) such that X is a (nonempty) set and d is a quasi-metric on X. The notion of an extended quasi-metric space is defined in the obvious manner.

A semilinear space on $\mathbb{R}^+$ (a cone in the sense of [12]) is a triple $(X, +, \cdot)$ such that $(X, +)$ is an Abelian monoid, and $\cdot$ is a function from $\mathbb{R}^+ \times \mathbb{X}$ to X such that for all $x, y \in X$ and $r, s \in \mathbb{R}^+$:

(i) $r \cdot (s \cdot x) = (rs) \cdot x$;
(ii) $r \cdot (x + y) = (r \cdot x) + (r \cdot y)$;
(iii) $(r + s) \cdot x = (r \cdot x) + (s \cdot x)$;

(iv) $1 \cdot x = x$.

A norm on a semilinear space $(X, +, \cdot)$ is a function $p : X \to \mathbb{R}^+$ such that for all $x, y \in X$ and $r \in \mathbb{R}^+$:

(i) $p(x) = 0$ if and only if $x = 0$;

(ii) $p(r \cdot x) = rp(x)$;

(iii) $p(x + y) \leq p(x) + p(y)$.

A normed semilinear space is a pair (X, p) such that X is a semilinear space and p is a norm on X.

The following is a classical and useful example of a normed semilinear space.

For each $p \in [1, +\infty)$ denote by l_p the set of infinite sequences $\mathbf{x} := (x_n)_{n\in\omega}$ of real numbers such that $\sum_{n=0}^{\infty} \mid x_n \mid^p < +\infty$.

It is well known that $(l_p, \|.\|_p)$ is a Banach space, where $\| \ . \ \|_p$ is the norm on l_p defined by $\| \mathbf{x} \|_p = (\sum_{n=0}^{\infty} \mid x_n \mid^p)^{1/p}$ for all $\mathbf{x} \in l_p$ (see, for instance, [8]).

Let $l_p^+ = \{\mathbf{x} \in l_p : x_n \geq 0 \text{ for all } n \in \omega\}$, and let $\|.\|_{+p}$ be the restriction of $\|.\|_p$ to l_p^+. It is well known, and easy to see, that $(l_p^+, \|.\|_{+p})$ is a normed semilinear space, which is called the positive cone of $(l_p, \|.\|_p)$.

2. A NEW COMPLEXITY DISTANCE

Let us recall [23] that the complexity (quasi-pseudo-metric) space consists of the pair $(\mathcal{C}, d_\mathcal{C})$, where

$$\mathcal{C} = \left\{ f \in (0, +\infty]^\omega : \sum_{n=0}^{\infty} 2^{-n} \frac{1}{f(n)} < +\infty \right\},$$

and $d_\mathcal{C}$ is the quasi-pseudo-metric on $\mathcal{C}$ given by

$$d_\mathcal{C}(f, g) = \sum_{n=0}^{\infty} 2^{-n} \max\left\{ \left(\frac{1}{g(n)} - \frac{1}{f(n)}, 0 \right) \right\},$$

for all $f, g \in \mathcal{C}$. (We adopt the convention that $\frac{1}{+\infty} = 0$.)

The dual complexity space, introduced in [21], can be directly used for the complexity analysis of algorithms in the case that the running time of computing is the complexity measure ([21], p. 313). Contrarily to the complexity space $(\mathcal{C}, d_\mathcal{C})$, it can be endowed with a structure of normed semilinear space ([22]). Furthermore, the dual has a definite appeal, since in this context, it has a minimum which corresponds to the minimum of semantic domains.

Recall that the dual complexity space consists of the pair $(\mathcal{C}^*, d_{\mathcal{C}^*})$, where $\mathcal{C}^* = \{f \in (\mathbb{R}^+)^\omega : \sum_{n=0}^{\infty} 2^{-n} f(n) < +\infty\}$, and $d_{\mathcal{C}^*}$ is the quasi-pseudo-metric on $\mathcal{C}^*$ given by $d_{\mathcal{C}^*}(f, g) = \sum_{n=0}^{\infty} 2^{-n} \max\{g(n) - f(n), 0\}$, for all $f, g \in \mathcal{C}^*$.

The quasi-pseudo-metric spaces $(\mathcal{C}^*, d_{\mathcal{C}^*})$ and $(\mathcal{C}, d_\mathcal{C})$ are isometric via the inversion mapping $\Psi : \mathcal{C}^* \to \mathcal{C}$, i.e., $\Psi(f) = 1/f$ for all $f \in \mathcal{C}^*$ (see [21]).

For $p \in [1, +\infty)$, the dual p-complexity space, introduced in [7], is the normed semilinear space $(\mathcal{C}_p^*, q_p)$, where

$$\mathcal{C}_p^* = \left\{ f \in (\mathbb{R}^+)^\omega : \sum_{n=0}^{\infty} (2^{-n} f(n))^p < +\infty \right\},$$

and q_p is the norm on $\mathcal{C}_p^*$ given by

$$q_p(f) = \left(\sum_{n=0}^{\infty} (2^{-n} f(n))^p \right)^{1/p}.$$

It was proved in [7] that the normed semilinear spaces $(\mathcal{C}_p^*, q_p)$ and $(l_p^+, \|.\|_{+p})$ are isometrically isomorphic in the following sense: There is a bijective linear mapping $\phi : \mathcal{C}_p^* \to l_p^+$ such that $\|\phi(f)\|_{+p} = q_p(f)$ for all $f \in \mathcal{C}_p^*$. In fact, the mapping ϕ is given by the rule $\phi(f)(n) = 2^{-n} f(n)$, $f \in \mathcal{C}_p^*$, $n \in \omega$.

Now, for each $f, g \in \mathcal{C}_p^*$ put

$$d_{q_p}(f, g) = \left(\sum_{n=0}^{\infty} (2^{-n} \max\{(g(n) - f(n), 0\})^p \right)^{1/p}.$$

Then d_{q_p} is a T_0 quasi-pseudo-metric on $\mathcal{C}_p^*$, which, clearly, is not a quasi-metric ([7]). Note, in particular, that the quasi-pseudo-metric space $(\mathcal{C}_1^*, d_{q_1})$ is exactly the dual complexity space, as defined above.

According to Section 4 of [23], the intuition behind the complexity distance between two functions $f, g \in \mathcal{C}_p^*$ is that $d_{q_p}(f, g)$ measures relative progress made in lowering the complexity by replacing any program Q with complexity function g by any program P with complexity function f. Therefore, if $f \neq g$, condition $d_{q_p}(f, g) = 0$ can be interpreted as g is "more efficient" than f. In particular $q_p(f) = d_{q_p}(0, f)$ measures relative progress made in lowering the complexity by replacing f by the "optimal" complexity function 0, assuming that the complexity measure is the running time of computing. Thus, if $q_p(g) < q_p(f)$, there is an increasing in complexity when g is replaced by f, i.e., g is "more efficient" than f.

We want to show that these computational interpretations are also provided by the extended quasi-metric e_{q_p} which will be constructed below. We also give a numerical quantification of the improvement in complexity obtained when a complexity function g is replaced by a more efficient complexity function f, via the properties of e_{q_p}.

Similarly to [13], an extended quasi-metric d on a semilinear space $(X, +, \cdot)$ is said to be invariant if for each $x, y, z \in X$ and $r > 0$, $d(x + z, y + z) = d(x, y)$ and $d(r \cdot x, r \cdot y) = rd(x, y)$, where we use the natural convention that $r(+\infty) = +\infty$ for all $r > 0$.

Theorem 1. For each $p \in [1, +\infty)$ let $e_{q_p} : \mathcal{C}_p^* \times \mathcal{C}_p^* \to [0, +\infty]$ be given by

$e_{q_p}(f, g) = q_p(g - f)$ if $f \leq g$, and

$e_{q_p}(f,g) = +\infty,$ otherwise.

Then e_{q_p} is an invariant extended quasi-metric on $\mathcal{C}_p^*$.

Proof. Fix $p \in [1,+\infty)$. Let $f,g \in \mathcal{C}_p^*$ such that $e_{q_p}(f,g) = 0$. Then $f \leq g$ and $q_p(g-f) = 0$ (note that, indeed, $g - f \in \mathcal{C}_p^*$ because $f \leq g$). It immediately follows that $f = g$.

Now let $g,h \in \mathcal{C}_p^*$ be such that $e_{q_p}(f,h) < +\infty$ and $e_{q_p}(h,g) < +\infty$. Then $f \leq h$ and $h \leq g$. So $e_{q_p}(f,g) = q_p(g-f) \leq q_p(g-h) + q_p(h-f) = e_{q_p}(h,g) + e_{q_p}(f,h)$.

Therefore e_{q_p} is an extended quasi-metric on $\mathcal{C}_p^*$.

It remains to show that e_{q_p} is invariant. To this end, let $f,g,h \in \mathcal{C}_p^*$ and let $r > 0$. If $e_{q_p}(f,g) = +\infty$, it clearly follows that $e_{q_p}(f+h, g+h) = +\infty$ and $e_{q_p}(rf,rg) = +\infty$. Otherwise, we have $f \leq g$, and thus $f + h \leq g + h$. So

$$e_{q_p}(f+h,g+h) = q_p((g+h)-(f+h)) = q_p(g-f) = e_{q_p}(f,g),$$

and

$$e_{q_p}(rf,rg) = q_p(fg - rf) = rq_p(g-f) = re_{q_p}(f,g).$$

The proof is complete. □

Remark 1. Given the dual p-complexity space $(\mathcal{C}_p^*, q_p)$, we show that $(d_{q_p})^s \leq e_{q_p}$. Indeed, let $f,g \in \mathcal{C}_p^*$. If $e_{q_p}(f,g) = +\infty$, the conclusion is obvious; otherwise, we have $f \leq g$, and thus $d_{q_p}(g,f) = 0$. Therefore $e_{q_p}(f,g) = q_p(g-f) = d_{q_p}(f,g) = (d_{q_p})^s(f,g)$.

Consequently $(\mathcal{C}_p^*, \mathcal{T}(e_{q_p}))$ is a submetrizable topological space. (Let us recall that a topological space $(X,\mathcal{T})$ is said to be submetrizable if there is a metric topology on X weaker than $\mathcal{T}$).

Balanced (extended) quasi-metric spaces were introduced by D. Doitchinov [3] in order to obtain a satisfactory theory of quasi-metric completion that preserves complete regularity.

Recall that an extended quasi-metric space (X,d) is said to be balanced if given $r,s > 0$, $(x_k)_{k\in\mathbb{N}}$, $(y_k)_{k\in\mathbb{N}}$ sequences in X with $\lim_{m,k\to+\infty} d(y_m,x_k) = 0$, and points $x,y \in X$ with $d(x,x_k) \leq r$ and $d(y_k,y) \leq s$ for all $k \in \mathbb{N}$, then $d(x,y) \leq r+s$.

It is well known that each balanced extended quasi-metric space is Hausdorff and completely regular (see [3]).

Theorem 2. For each $p \in [1,+\infty)$ the extended quasi-metric space $(\mathcal{C}_p^*, e_{q_p})$ is balanced.

Proof. Let $r,s > 0$, $(f_k)_{k\in\mathbb{N}}$, $(g_k)_{k\in\mathbb{N}}$ be sequences in $\mathcal{C}_p^*$ with $\lim_{m,k\to+\infty} e_{q_p}(g_m,f_k) = 0$, and $f,g \in \mathcal{C}_p^*$ with $e_{q_p}(f,f_k) \leq r$ and $e_{q_p}(g_k,g) \leq s$ for all $k \in \mathbb{N}$. Thus $f \leq f_k$ and $g_k \leq g$ for all $k \in \mathbb{N}$. Moreover $g_m \leq f_k$ eventually.

We first note that $f \leq g$. Indeed, let $n_0 \in \mathbb{N}$. For an arbitrary $\varepsilon > 0$ there is $k \in \mathbb{N}$ such that

$$\left(\sum_{n=0}^{\infty}(2^{-n}(f_k(n) - g_k(n))^p\right)^{1/p} < \varepsilon.$$

Thus $f_k(n_0) - g_k(n_0) < 2^{n_0}\varepsilon$. Hence

$$f(n_0) \leq f_k(n_0) < 2^{n_0}\varepsilon + g_k(n_0) \leq 2^{n_0}\varepsilon + g(n_0).$$

We deduce that $f(n_0) \leq g(n_0)$ for all $n_0 \in \mathbb{N}$, i.e. $f \leq g$.

Finally, choose $k \in \mathbb{N}$ such that $g_k \leq f_k$. Then for any $m \in \mathbb{N}$, we obtain

$$\left(\sum_{n=0}^{m}(2^{-n}(g(n) - f(n)))^p\right)^{1/p} \leq \left(\sum_{n=0}^{m}(2^{-n}(g(n) - g_k(n) + f_k(n) - f(n)))^p\right)^{1/p}$$

$$\leq \left(\sum_{n=0}^{m}(2^{-n}(g(n) - g_k(n)))^p\right)^{1/p} + \left(\sum_{n=0}^{m}(2^{-n}(f_k(n) - f(n)))^p\right)^{1/p}.$$

We immediately deduce that

$e_{q_p}(f,g) \leq e_{q_p}(g_k,g) + e_{q_p}(f,f_k) \leq s + r.$

Therefore $(\mathcal{C}_p^*, e_{q_p})$ is a balanced extended quasi-metric space. □

Corollary. For each $p \in [1,+\infty)$ the extended quasi-metric space $(\mathcal{C}_p^*, e_{q_p})$ is Hausdorff and completely regular.

Note that Hausdorffness of $(\mathcal{C}_p^*, e_{q_p})$ also follows immediately from the fact noted in Remark 1 that $(\mathcal{C}_p^*, \mathcal{T}(e_{q_p}))$ is a submetrizable topological space.

Theorem 3. Let $p \in [1,+\infty)$, let $(f_k)_{k\in\mathbb{N}}$ be a decreasing sequence in the dual p-complexity space $(\mathcal{C}_p^*, q_p)$ and let $f : \omega \to \mathbb{R}^+$ given by

$$f(n) = \inf_{k\in\mathbb{N}} f_k(n) \quad \text{for all } n \in \omega.$$

Then the following statements hold.

(1) $f \in \mathcal{C}_p^*$ and $\lim_{k\to+\infty} e_{q_p}(f, f_k) = 0$. So $\lim_{k\to+\infty} q_p(f_k - f) = 0$.

(2) $q_p(f) = \inf_{k\in\mathbb{N}} q_p(f_k)$.

Proof. We first show that $f \in \mathcal{C}_p^*$ and that $(f_k)_{k\in\mathbb{N}}$ converges to f in $(\mathcal{C}_p^*, e_{q_p})$. Indeed, let $\varepsilon > 0$. Since $f_1 \in \mathcal{C}_p^*$ there is $n_\varepsilon \in \omega$ such that

$$\sum_{n=n_\varepsilon+1}^{\infty}(2^{-n}f_1(n))^p < \varepsilon/3.$$

So $\sum_{n=n_\varepsilon+1}^{\infty}(2^{-n}f(n))^p < \varepsilon/3$, and, hence, $f \in \mathcal{C}_p^*$.

Furthermore, since $f_k \leq f_1$ for all $k > 1$, it follows that

$$\sum_{n=n_\varepsilon+1}^{\infty}(2^{-n}f_k(n))^p < \varepsilon/3$$

for all $k \in \mathbb{N}$.

By definition of f and the fact that $f_{k+1} \leq f_k$ for all $k \in \mathbb{N}$, there is $k_1 \in \mathbb{N}$ such that for each $k \geq k_1$, $(2^{-n}(f_k(n) - f(n)))^p < \varepsilon/3$, $n = 0, 1, ..., n_\varepsilon$. Hence for $k \geq k_1$,

$$\begin{aligned} e_{q_p}(f, f_k) &\leq \sum_{n=0}^{\infty}(2^{-n}(f_k(n) - f(n)))^p \\ &\leq \sum_{n=0}^{n_\varepsilon}(2^{-n}(f_k(n) - f(n)))^p + \sum_{n=n_\varepsilon+1}^{\infty}(2^{-n} f_k(n))^p \\ &< \frac{\varepsilon}{3}\left(\sum_{n=0}^{\infty} 2^{-n}\right) + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

We conclude that $(f_k)_{k\in\mathbb{N}}$ converges to f in $(\mathcal{C}_p^*, e_{q_p})$. So, by Remark 1, $\lim_{k\to+\infty} q_p(f_k - f) = 0$. Therefore, statement (1) holds.

Finally, for each $\varepsilon > 0$ there is $k_\varepsilon \in \mathbb{N}$ such that for $k \geq k_\varepsilon$, $q_p(f_k - f) < \varepsilon$, so $q_p(f_k) < q_p(f) + \varepsilon$. Thus, statement (2) is satisfied. □

As we have noted above, if $f, g \in \mathcal{C}_p^*$, $f \neq g$, satisfy $f \leq g$, then there is an improvement in the running time of computing when g is replaced by f. In this case the positive number $q_p(g - f)$ provides a numerical quantification of such an improvement.

Furthermore, if $(f_k)_{k\in\mathbb{N}}$ is a decreasing sequence in $\mathcal{C}_p^*$, Theorem 3 shows that f represents the infimum of all running time of computing corresponding to the complexity functions f_k, $k \in \mathbb{N}$, where $f = \inf_{k\in\mathbb{N}} f_k$.

This interesting computational fact can be formulated in the framework of the so-called right K-sequentially complete quasi-metric spaces.

Similarly to [19], a sequence $(x_n)_{n\in\mathbb{N}}$ in an extended quasi-metric space (X, d) is called right K-Cauchy if for each $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $d(x_k, x_n) < \varepsilon$ whenever $k \geq n \geq n_0$. The extended quasi-metric space (X, d) is called right K-sequentially complete quasi-metric if every right K-Cauchy sequence is convergent.

Right K-sequential completeness plays a crucial role in the study of completeness of hyperspaces and function spaces on quasi-metric spaces (see, for instance, [15] and Section 9 of [14]).

Theorem 4. For each $p \in [1, +\infty)$ the extended quasi-metric space $(\mathcal{C}_p^*, e_{q_p})$ is right K-sequentially complete.

Proof. Fix $p \in [1, +\infty)$. Let $(f_k)_{k\in\mathbb{N}}$ be a right K-Cauchy sequence in $(\mathcal{C}_p^*, e_{q_p})$. For $\varepsilon = 1$, there is $n_0 \in \mathbb{N}$ such that $e_{q_p}(f_k, f_n) < 1$ whenever $k \geq n \geq n_0$. Consequently, $f_k \leq f_n$ whenever $k \geq n \geq n_0$, and thus, the sequence $(f_k)_{k\geq n_0}$ is decreasing. By Theorem 3, $(f_k)_{k\in\mathbb{N}}$ converges in $(\mathcal{C}_p^*, e_{q_p})$ to the function $f \in \mathcal{C}_p^*$ defined by $f(n) = \inf_{k\geq n_0} f_k(n)$ for all $n \in \omega$. We conclude that $(\mathcal{C}_p^*, e_{q_p})$ is right K-sequentially complete. □

As we indicated above the quasi-pseudo-metric space $(\mathcal{C}_p^*, d_{q_p})$ is a T_0 non T_1 space, and hence, d_{q_p} is not balanced. Next, we shall show that completeness properties of $(\mathcal{C}_p^*, d_{q_p})$ are quite different to completeness properties of $(\mathcal{C}_p^*, e_{q_p})$. In fact, it was proved in [7] (see also [21]), that $(\mathcal{C}_p^*, d_{q_p})$ is Smyth complete. (Let us recall that a T_0 (extended) quasi-pseudo-metric space (X, d) is Smyth complete provided that every rigth K-Cauchy sequence in (X, d^{-1}) converges in the (extended) metric space (X, d^s)). The following example shows that $(\mathcal{C}_p^*, d_{q_p})$ is not right K-sequentially complete.

Example 1. Let $(f_k)_{k\in\mathbb{N}}$ be the sequence of functions defined on ω by $f_k(n) = k$ whenever $n \in \omega$. Clearly $(f_k)_{k\in\mathbb{N}}$ is a right K-Cauchy sequence in $(\mathcal{C}_p^*, d_{q_p})$, for each $p \in [1, +\infty)$, because $d_{q_p}(f_k, f_j) = 0$ whenever $k \geq j$. Since for each $f \in \mathcal{C}_p^*$ and each $k \in \mathbb{N}$, $f_k \leq f + \max\{(f_k - f), 0\}$, it follows that

$$k\left(\frac{2^p}{2^p-1}\right)^{1/p} = q_p(f_k) \leq q_p(f) + q_p(\max\{(f_k - f), 0\}) = q_p(f) + d_{q_p}(f, f_k).$$

So, $(f_k)_{k\in\mathbb{N}}$ does not converges in $(\mathcal{C}_p^*, d_{q_p})$.

Next we show that $(\mathcal{C}_p^*, e_{q_p})$ is not Smyth complete.

Example 2. Let $(f_k)_{k\in\mathbb{N}}$ be the sequence in $\mathcal{C}_p^*$ given by $f_k(n) = 1 - 2^{-k}$, whenever $n \in \omega$. Then, for $k \leq j$, we have

$$e_{q_p}(f_k, f_j) \leq 2^{-k}\left(\frac{2^p}{2^p-1}\right)^{1/p},$$

and, therefore, $(f_k)_{k\in\mathbb{N}}$ is right K-Cauchy in $(\mathcal{C}_p^*, (e_{q_p})^{-1})$ for each $p \in [1, +\infty)$. Clearly $(f_k)_{k\in\mathbb{N}}$ is not convergent in $(\mathcal{C}_p^*, (e_{q_p})^s)$, because $(e_{q_p})^s(f, g) = +\infty$ for all $f, g \in \mathcal{C}_p^*$ with $f \neq g$.

As an application of the theory developed above we shall measure distances between some infinite words over the decimal alphabet via the complexity extended quasi-metric e_{q_p} and we shall compare our computations with the ones that provide the classical Baire metric.

Let $\Sigma = \{0, 1, 2, ..., 9\}$ and let Σ^ω be the set of all infinite words over Σ. Each $w \in \Sigma^\omega$ will be denoted by $w_0 w_1 w_2 \ldots$.

Let us recall ([11], [17], [1], [16]) that the Baire metric is given by

$$D(v, w) = 2^{-\ell(v,w)} \quad \text{if } v \neq w \quad \text{and} \quad D(w, w) = 0,$$

for all $v, w \in \Sigma^\omega$, where $\ell(v, w)$ is defined as the length of the nonempty common prefix of v and w if it exists, and $\ell(v, w) = 0$ otherwise.

It is clear that we may identify Σ^ω with the subset $\mathcal{C}_{p,\Sigma}^*$ of $\mathcal{C}_p^*$ defined by $\mathcal{C}_{p,\Sigma}^* = \{f \in \mathcal{C}_p^* : f(n) \in \Sigma \text{ for all } n \in \omega\}$.

The following result illustrates the relationship between the complexity extended quasi-metric e_{q_p} and the Baire metric.

Proposition 1. Let $f, g \in \mathcal{C}^*_{p,\Sigma}$ be such that $f \leq g$. If $e_{q_p}(f,g) < 2^{-k}$ for some $k \in \omega$, then $D(f,g) \leq 2^{-(k+1)}$.

Proof. Since

$$e_{q_p}(f,g) = \left(\sum_{n=0}^{\infty}(2^{-n}(g(n) - f(n)))^p\right)^{1/p},$$

it follows from our assumption that $g(n) = f(n)$ for $n = 0, 1, ..., k$. Therefore $\ell(f,g) \geq k+1$. So $D(f,g) \leq 2^{-(k+1)}$. □

Corollary. Let $(f_j)_{j \in \mathbb{N}}$ be a sequence in $\mathcal{C}^*_{p,\Sigma}$ such that $\lim_{j \to +\infty} e_{q_p}(f, f_j) = 0$ for some $f \in \mathcal{C}^*_{p,\Sigma}$. Then $\lim_{j \to +\infty} D(f, f_j) = 0$.

Remark 2. It follows from the preceding corollary that the topology induced by the Baire metric is weaker than $\mathcal{T}(e_{q_p})$ on $\mathcal{C}^*_{p,\Sigma}$.

Now let $\Sigma_0^\omega = \{w \in \Sigma^\omega : w_0 = 0\}$. Then, we may identify Σ_0^ω with the subset $\mathcal{C}^*_{p,\Sigma_0}$ of $\mathcal{C}^*_{p,\Sigma}$, defined by $\mathcal{C}^*_{p,\Sigma_0} = \{f \in \mathcal{C}^*_{p,\Sigma} : f(0) = 0\}$.

Next we compute a paradigmatic particular case in the realm of $\mathcal{C}^*_{p,\Sigma_0}$. Let $f :=$ 0000... and let us consider the sequence $(f_j)_{j \in \mathbb{N}}$ given by

$f_1 := 01111...$
$f_2 := 00111...$
....
$f_j := \overbrace{0000}^{j\text{-times}}\ 111...$

Obviously $(f_j)_{j \in \mathbb{N}}$ is a decreasing sequence in $\mathcal{C}^*_{p,\Sigma_0}$. Since $f(n) = \inf_{j \in \mathbb{N}} f_j(n)$ for all $n \in \omega$, it follows from Theorem 3 that $(f_j)_{j \in \omega}$ converges to f with respect to $\mathcal{T}(e_{q_p})$, and thus it converges to f with respect to the Baire metric D by the corollary of Proposition 1. In particular, we have $D(f, f_j) = 2^{-j}$ and

$$e_{q_p}(f, f_j) = \left(\sum_{n=j}^{\infty}(2^{-n})^p\right)^{1/p} = \left(\frac{2^{-(j-1)p}}{2^p - 1}\right)^{1/p} = \frac{2}{(2^p - 1)^{1/p}} D(f, f_j).$$

for all $j \in \mathbb{N}$.

In general, the sequence $(g_j)_{j \in \mathbb{N}}$ given by

$g_1 := 0kkkk...$
$g_2 := 00kkk...$
....
$g_j := \overbrace{0000}^{j\text{-times}}\ kkk...,$

with $1 < k \leq 9$, converges to f with respect to $\mathcal{T}(e_{q_p})$ and hence with respect to the Baire metric. However, a direct computation shows that $D(f, g_j) = D(f, f_j)$ but $e_{q_p}(f, g_j) = ke_{q_p}(f, f_j)$ for all $j \in \mathbb{N}$. This fact makes evident an interesting computational advantage of our complexity distance e_{q_p} with respect to the Baire metric. Indeed, while the Baire metric does not distinguish between the distances from g_j to f and from f_j to f, the (extended) quasi-metric e_{q_p} is sensitive to such differences in a satisfactory way. For instance, if we consider the sequence $(h_j)_{j\in\omega}$ where

$$h_1 := 0mmmm...$$
$$h_2 := 00mmm...$$
$$\dots\dots\dots\dots\dots$$
$$h_j := \overbrace{0000}^{j\text{-times}}\ mmm...,$$

with $1 \leq m < k \leq 9$, we obtain $e_{q_p}(f, h_j) = \frac{m}{k} e_{q_p}(f, g_j) < e_{q_p}(f, g_j)$ for all $j \in \omega$, as it was desirable.

Next, we extend this approach to discuss, in our context, the problem of the approximation for any real number $\omega \in (0, 1)$, which admits a rational decimal expansion, i.e. $\omega := 0\omega_1\omega_2...\omega_i 0000....$, where $\omega_i > 0$.

In this case, we consider the sequence $(g_j)_{j\in\mathbb{N}}$ given by

$$g_1 := 0\omega_1\omega_2...\omega_i 0kkkk...$$
$$g_2 := 0\omega_1\omega_2...\omega_i 00kkk...$$
$$\dots\dots\dots\dots\dots$$
$$g_j := 0\omega_1\omega_2...\omega_i\ \overbrace{0000}^{j\text{-times}}\ kkk...,$$

with $1 < k \leq 9$. Then $D(\omega, g_j) = 2^{-(i+j+1)}$ and

$$e_{q_p}(\omega, g_j) = \left(\sum_{n=i+j+1}^{\infty} (2^{-n}k)^p \right)^{1/p} = \frac{2k}{(2^p - 1)^{1/p}} D(\omega, g_j),$$

for all $j \in \mathbb{N}$.

Finally, let $(h_j)_{j\in\omega}$ be the sequence given by

$$h_1 := 0\omega_1\omega_2...\omega_i 0mmmm...$$
$$h_2 := 0\omega_1\omega_2...\omega_i 00mmm...$$
$$\dots\dots\dots\dots\dots$$
$$h_j := 0\omega_1\omega_2...\omega_i\ \overbrace{0000}^{j\text{-times}}\ mmm...,$$

with $1 \leq m < k \leq 9$. Therefore $D(\omega, h_j) = 2^{-(i+j+1)} = D(\omega, g_j)$; however $e_{q_p}(\omega, h_j) = \frac{m}{k} e_{q_p}(\omega, g_j)$ for all $j \in \omega$, as it was desirable.

3. A BANACH FIXED POINT THEOREM: APPLICATION TO DIVIDE & CONQUER ALGORITHMS

Motivated in part for applications of quasi-metric structures to computer science, there exist several generalizations of the classical Banach fixed point theorem to the setting of quasi-metric spaces in the literature ([2], [5], [9], [10], [17], [18], [23], etc.).

Similarly to the metric case, by a contraction mapping from an extended quasi-metric spaces (X, d) to an extended quasi-metric space (Y, d') we mean a mapping $T : X \to Y$ such that there is $0 < \alpha < 1$ with $d'(Tx, Ty) \leq \alpha d(x, y)$ for all $x, y \in X$. In this case, we say that α is a contraction constant of T (see, for instance, [23]).

Our next result, that generalizes Banach's fixed point theorem to extended right K-sequentially complete quasi-metric spaces, will be useful later on.

Theorem 5. Let (X, d) be a Hausdorff right K-sequentially complete extended quasi-metric space and let T be a contractive mapping from X into itself. If there is $x_0 \in X$ such that $d(Tx_0, x_0) < +\infty$, then T has a fixed point.

Proof. (We sketch the proof since it follows from standard arguments). For each $n \in \mathbb{N}$ let $T^n x_0 = x_n$. Clearly $d(x_{n+1}, x_n) \leq \alpha^n d(Tx_0, x_0)$ for all $n \in \omega$, where α is a contraction constant for T. By the triangle inequality it easily follows that for each $k, n \in \mathbb{N}$,

$$d(x_{n+k}, x_n) < \frac{\alpha^n}{1-\alpha} d(Tx_0, x_0),$$

and, consequently, $(x_n)_{n\in\mathbb{N}}$ is a right K-Cauchy sequence in (X, d). Let $y \in X$ such that $\lim_{n\to+\infty} d(y, x_n) = 0$. By continuity of T, $\lim_{n\to+\infty} d(Ty, Tx_n) = 0$. Since for each n, $Tx_n = x_{n+1}$, it follows from Hausdorffness of (X, d), that $y = Ty$. Hence y is a fixed point of T. □

The following easy example deals with some natural questions that one can consider in the light of Theorem 5.

Example 3. Let $X = \{x, y\}$. Define $d(x, y) = d(y, x) = +\infty$ and $d(x, x) = d(y, y) = 0$. Then (X, d) is a Hausdorff right K-sequentially complete extended quasi-metric space. (Note that actually d is an extended metric on X.)

Let $Tx = x$ and $Ty = y$. Clearly T satisfies all conditions of Theorem 5 and x and y are fixed points of T, so it does not have a unique fixed point.

Now let $Tx = y$ and $Ty = x$. Then T is a contractive mapping without fixed point and $d(Tx, x) = d(Ty, y) = +\infty$. This shows that condition "$d(Tx_0, x_0) < +\infty$, for some $x_0 \in X$", cannot be omitted in the statement of Theorem 5.

In Section 6 of [23], Schellekens applied his theory of complexity spaces to the complexity analysis of Divide & Conquer algorithms. In particular, he proved that Divide & Conquer algorithms induce contraction mappings from the complexity space $(\mathcal{C}, d_{\mathcal{C}})$ into itself. Via the inversion mapping Ψ defined in Section 2, it was shown in [21] that these applications can be also obtained based on the dual complexity space $(\mathcal{C}_1^*, d_{q_1})$. Here we shall prove that Divide & Conquer algorithms also induce contraction mappings for any extended quasi-metric space $(\mathcal{C}_p^*, e_{q_p})$, $p \in [1, +\infty)$.

Let $a, b, c \in \mathbb{N}$ with $a, b \geq 2$, let n range over the set $\{b^k : k \in \omega\}$ and let $h \in \mathcal{C}$. A functional Φ corresponding to a Divide & Conquer algorithm in the sense of [23], is typically defined by $\Phi(f)(1) = c$, and $\Phi(f)(n) = af(n/b) + h(n)$ if $n \in \{b^k : k \in \mathbb{N}\}$.

This functional intuitively corresponds to a Divide & Conquer algorithm which recursively splits a given problem in a subproblems of size n/b and which takes $h(n)$ time to recombine the separately solved problems into the solution of the original problem (see Section 6 of [23]).

Extending Theorem 6.1 of [23], it was shown in Section 4 of [21], that for the dual complexity space $(\mathcal{C}_1^*, d_{q_1})$, the functional Φ^* given by

$$\Phi^*(f)(n) = \begin{cases} 1/c & \text{if } n = 1 \\ \frac{f(n/b)}{a + f(n/b)h(n)} & \text{if } n \in \{b^k : k \in \mathbb{N}\}, \end{cases}$$

is a contraction mapping with contraction constant $1/a$.

In our context, for each $p \in [1, +\infty)$, define

$\mathcal{C}_p^* \mid_{b,c} = \{f : f$ is the restriction to arguments n of the form b^k, $k \in \omega$, of $f' \in \mathcal{C}_p^*$ such that $f'(1) = 1/c\}$.

Observe that each $f \in \mathcal{C}_p^* \mid_{b,c}$ can be considered as an element of $\mathcal{C}_p^*$, by defining $f(n) = 0$ whenever $n \notin \{b^k : k \in \omega\}$. Thus, if for each $f \in \mathcal{C}_p^* \mid_{b,c}$, $\Phi^*(f)$ is defined as above, we obtain the following contraction mapping theorem.

Theorem 6. Let $f, g \in \mathcal{C}_p^* \mid_{b,c}$. Then the following statements hold.
(1) $\Phi^*(f), \Phi^*(g) \in \mathcal{C}_p^* \mid_{b,c}$;
(2) $\Phi^*(f) \leq \Phi^*(g)$ whenever $f \leq g$;
(3) $e_{q_p}(\Phi^*(f), \Phi^*(g)) \leq \frac{1}{a} e_{q_p}(f, g)$.

Proof. Since statements (1) and (2) follow directly from the definitions we only show (3). If $e_{q_p}(f, g) = +\infty$, the conclusion is obvious. If $e_{q_p}(f, g) < +\infty$, we have $f \leq g$, and by (2), $\Phi^*(f) \leq \Phi^*(g)$. Therefore

$$\begin{aligned} (e_{q_p}(\Phi^*(f), \Phi^*(g)))^p &= (q_p(\Phi^*(g) - \Phi^*(f)))^p \\ &= \sum_{n=0}^{\infty} (2^{-n}((\Phi^*(g) - \Phi^*(f))(n)))^p \\ &= \sum_{n \in \{b^k : k \in \mathbb{N}\}} \left(2^{-n}\left(\frac{g(n/b)}{a + g(n/b)h(n)} - \frac{f(n/b)}{a + f(n/b)h(n)}\right)\right)^p \\ &\leq \sum_{n \in \{b^k : k \in \mathbb{N}\}} \left(2^{-n}\left(\frac{a(g(n/b) - f(n/b))}{a^2}\right)\right)^p \\ &\leq \frac{1}{a^p} \sum_{n=0}^{\infty} (2^{-n}(g(n) - f(n))^p = \frac{1}{a^p}(e_{q_p}(f, g))^p. \end{aligned}$$

Hence $e_{q_p}(\Phi^*(f), \Phi^*(g)) \leq \frac{1}{a} e_{q_p}(f, g)$. This completes the proof. □

Theorem 7. Let $p \in [1, +\infty)$. Then the mapping $\Phi^* : \mathcal{C}_p^* |_{b,c} \to \mathcal{C}_p^* |_{b,c}$ has a unique fixed point.

Proof. We shall apply Theorem 5. To this end, first note that $(\mathcal{C}_p^* |_{b,c}, e_{q_p})$ is a Hausdorff extended quasi-metric space because $\mathcal{C}_p^* |_{b,c}$ is a subset of $\mathcal{C}_p^*$.

Next we show that $(\mathcal{C}_p^* |_{b,c}, e_{q_p})$ is right K-sequentially complete. Indeed, let $(f_j)_{j \in \mathbb{N}}$ be a right K-Cauchy sequence in $(\mathcal{C}_p^* |_{b,c}, e_{q_p})$. By Theorem 4, $(f_j)_{j \in \mathbb{N}}$ converges to a function $f \in \mathcal{C}_p^*$. Moreover, there is $n_0 \in \mathbb{N}$ such that $f(n) = \inf_{j \geq n_0} f_j(n)$ for all $n \in \omega$ (see the proof of Theorem 4). Since each f_j is in $\mathcal{C}_p^* |_{b,c}$, it follows that $f(n) = 0$ for all $n \notin \{b^k : k \in \omega\}$, and thus $f \in \mathcal{C}_p^* |_{b,c}$.

Now let $f_0 \in \mathcal{C}_p^* |_{b,c}$ defined by $f_0(b^k) = 1/c$ for all $k \in \omega$ and $f_0(n) = 0$ otherwise. We show that $\Phi^*(f_0) \leq f_0$. Indeed, for $n \notin \{b^k : k \in \omega\}$, we have $\Phi^*(f_0)(n) = 0$. On the other hand $\Phi^*(f_0)(1) = 1/c = f_0(1)$. Finally, for $n = b^k$, $k \in \mathbb{N}$, we have

$$\Phi^*(f_0)(n) = \frac{f_0(b^{k-1})}{a + f_0(b^{k-1})h(b^k)} \leq f_0(b^{k-1}) = \frac{1}{c} = f_0(n).$$

Therefore $\Phi^*(f_0) \leq f_0$, so $e_{q_p}(\Phi^*(f_0), f_0) < +\infty$. Since by Theorem 6, Φ^* is a contraction mapping from $(\mathcal{C}_p^* |_{b,c}, e_{q_p})$ into itself, we may apply Theorem 5, and thus Φ^* has a fixed point, namely, g. Suppose that $g_1 \in \mathcal{C}_p^* |_{b,c}$ satisfies $\Phi^*(g_1) = g_1$. We shall prove, by induction over k, that $g = g_1$. In fact, $g(1) = \Phi^*(g)(1) = 1/c$ and $g_1(1) = \Phi^*(g_1)(1) = 1/c$. Moreover

$$g(b) = \Phi^*(g)(b) = \frac{g(1)}{a + g(1)h(b)} = \frac{1}{ac + h(b)},$$

and, similarly,

$$g_1(b) = \Phi^*(g_1)(b) = \frac{1}{ac + h(b)}.$$

So $g(b) = g_1(b)$.

Now let $g(b^{k-1}) = g_1(b^{k-1})$, where $k \geq 2$. Then

$$g(b^k) = \Phi^*(g)(b^k) = \frac{g(b^{k-1})}{a + g(b^{k-1})h(b^k)} = \Phi^*(g_1)(b^k) = g_1(b^k).$$

Hence $g = g_1$, and consequently Φ^* has a unique fixed point. □

(Received February 3, 2003.)

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