RESTRICTED IDEALS AND THE GROUPABILITY PROPERTY. TOOLS FOR TEMPORAL REASONING[1]

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In the field of automatic proving, the study of the sets of prime implicants or implicates of a formula has proven to be very important [17, 19, 20, 21]. If we focus on non-classical logics and, in particular, on temporal logics, such study is useful even if it is restricted to the set of unitary implicants/implicates [3, 4, 5, 6]. In this paper, a new concept we call *restricted ideal/filter* is introduced, it is proved that the set of restricted ideals/filters with the relation of inclusion has lattice structure and its utility for the efficient manipulation of the set of unitary implicants/implicates of formulas in propositional temporal logics is shown. We introduce a new property for subsets of lattices, which we call *groupability*, and we prove that the existence of *groupable* subsets in a lattice allows us to express restricted ideals/filters as the inductive closure for a binary non-deterministic operator and, consequently, the presence of this property guarantees a proper computational behavior of the set of unitary implicants/implicates.

Keywords: lattice, ideal, induction, temporal reasoning, prime implicants/implicates.

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1. INTRODUCTION

The future of computation points to the incorporation of the non-determinism. In the literature, the concept of non-deterministic automata as a formal model of computation has been widely developed. The necessity of the incorporation of non-determinism has been also discussed in the literature. So, for example, in [25] the author presents a discussion about how the study of non-determinism is useful for natural language processing, in [9] the author shows how formal non-deterministic models are useful in describing interactive systems. Another example is designing a circuit or a network: non-determinism characterizes the flexibility allowed in the design [24].

The importance of expressing non-determinism in a logic program is well-known; for instance Prolog contains a spurious constructor, called the *cut* which is widely used to improve execution speed. The need of non-determinism is also emerging in

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deductive databases [10]. So, two main classes of logic-based languages have been extensively studied in the literature as the theoretical basis for relational database languages and their extensions. One is the class of first-order languages and another one is the class of Datalog languages. Different works [1, 2] have brought into focus the need for having non-deterministic operators in such languages in addition to recursion and fixpoint.

Most works about non-determinism are based on simulation by means of algorithms and deterministic automata. Nonetheless, in the future it will be necessary to develop a formal theory that regards this aspect as inherent to it. In the development of a theory for non-determinism the classical operations need to be modified to account for the presence of non-determinism, that is, it is necessary to use non-deterministic operators both of fixed and flexible arity (i.e., applications of A^n in 2^A or of A^* in 2^A, respectively, where A^* is the universal language over A)

On the other hand, most objects used in Mathematica, Logic or Computer Science are *defined inductively*. By this we mean that we frequently define a set S of objects as: "the smallest set of objects containing a given set X of atoms, and closed under a given set $\mathcal{F}$ of constructors". In this definition, the constructors are deterministic operators. Therefore, we need to extend this notion for non-deterministic operators and, since the associative property is very restrictive for non-deterministic operators, we need a non-trivial extension of such property, that allows us to work with either binary or flexible arity non-deterministic operators. The new property that we have introduced is called *groupability*.

Particularly, in the applied aspects, our interest is focussed in prime implicants/implicates. The calculation of prime implicants/implicates of a formulae is useful in situations where satisfying models are desired as in circuits design, in error analysis during hardware verification and, more recently, in Artificial Intelligence applications (diagnosis, abductive reasoning, automated theorem proving, compilation of knowledge bases and, more generally, in casual and hypothetical reasoning [17, 19, 20, 21]). So, the design of efficient methods for the computation of prime implicants/implicates from a logical expression has been topic of research over decades and numerous algorithms have been proposed in the literature [16, 18].

If we focus in the field of automatic proving, such study is useful even if it is restricted to the set of unitary implicants/implicates [3, 4, 5, 6]. So, the results obtained by our research group [14, 15] are based on the efficient manipulation of these sets and they allow us a very important improvement in the efficiency of any prover.

The greatest obstacle we have found when trying to apply the obtained results for Classical Logic and multivalued logics [8, 12, 14, 15] to non-classical logics and, in particular, to temporal logics, is the higher complexity of the set of unitary implicants/implicates with the relation of *"logic implication"*.

From this starting point, a theoretical study within the framework of lattice theory is carried out in this paper[2]. In order to make this work as self-contained as

[2]We will assume that the concept of lattice and of ideal and filter of a lattice are known. Concretely, we will use the concepts of ideal and filter generated by a set X, which we will denote by $(X]$ and by $[X)$, respectively [11].

possible, in Section 2, we introduce propositional temporal logics which are extension of the Classical Propositional Logic, and we formalize them as abstract algebras. In particular, we will focus on FNext and FNext±, which are propositional temporal logics with discrete and linear flow of time. In Section 3, we introduce the concept of unitary formula, in order to define the concept of temporal literal. In particular we describe the set of temporal literals for the logics FNext and FNext±. In Section 4, concepts of implicant and implicate of a formula are introduced and we study, within a general theoretical framework, the ideals and filters of a lattice when they are restricted to a subset. In Section 5, we study the structure of the set of ideals of a lattice restricted to a subset. In Section 6, we introduce the operators we have called *"non-deterministic operators"* and we define a property, called *"groupability"*, which is fundamental in order to manipulate efficiently restricted ideals. This property allows us to characterize restricted ideals in terms of inductive closure for a binary non-deterministic operator. In Section 7, we prove that the set of literals in both FNext and FNext± are groupable, what, all in all, guarantees that in these logics we can manipulate efficiently the sets of implicant/implicate literals of a formula.

2. TEMPORAL LOGICS THAT ARE EXTENSIONS OF CLASSICAL LOGIC

Now we introduce the formalization of temporal propositional logic as abstract algebras and characterize those which are extensions of classical propositional logic. Given that a *logical system* is formed by a *formal language* and a *model theory* regarding such a language, *temporal propositional logic* is defined as $(\mathcal{L}, \mathcal{I})$ where $\mathcal{L}$ is a propositional language and $\mathcal{I}$ a set of interpretations for such a language.

A propositional language can be defined as a Ω-algebra freely generated by a set of atoms, $\mathcal{V}$, in the Ω category, in the following way :

Definition 2.1. Let $\mathcal{V} = \{p, q, \dots, p_1, q_1, \dots, p_n, q_n, \dots\}$ be a numerable and infinite set and $\Omega = \{op_1, \dots, op_r\}$ a finite domain of operators such that $\Omega(0) \subseteq \mathcal{V}$ where $\Omega(0)$ denotes the 0 arity operator set. A *propositional language* is the Ω-algebra of the words $\mathcal{L} = \langle\mathcal{V}\rangle = (L, op_1, \dots, op_r)$. *Atoms or atomic formulae* are the elements of $\mathcal{V}$, *logical constants* are the $\Omega(0)$ elements, *propositional connectives* are the elements of $\Omega \setminus \Omega(0)$, and *propositional formulae* are the elements of $\mathcal{L}$.

The model theory is given by the interpretation set, such interpretations being defined in terms of Ω-algebras as follows:

Definition 2.2. Let $\mathcal{L} = \langle\mathcal{V}\rangle = (L, op_1, \dots, op_r)$ be a propositional language. We call *temporal interpretation* of $\mathcal{L}$ any pair $I = (\mathcal{M}_t, h)$ where:

- $\mathcal{M}_t = ((T, \leq), op_1, \dots, op_r)$ where $(T, \leq)$ is a flow of time and $(2^T, op_1, \dots, op_r)$ is a algebra similar [3] to $\mathcal{L}$.

[3]Two Ω-algebras are similar if they have the same similarity, that is, if the lists made by the arities of their operators are coincident. On the other hand, although we use the same symbols for operators in $\mathcal{L}$ and in $\mathcal{M}_t$, there is not ambiguity because it is clear in the context we use them.

- h is a homomorphism of $\mathcal{L}$ in $\mathcal{M}_t$.

Definition 2.3. Let $(\mathcal{L}, \mathcal{I})$ be a temporal propositional logic with $\mathcal{L} = (L, op_1, \dots, op_r)$ and $\mathcal{I}$ a set of temporal interpretations for $\mathcal{L}$.

A formula ϕ is *true* in a temporal interpretation $I = (\mathcal{M}_t, h)$, and denoted by $\models_I \phi$, if $h(\phi) = T$. A formula ϕ is *valid*, and denoted by $\models \phi$, if it is true for every $I \in \mathcal{I}$. A propositional formula, $\phi \in L$, is *satisfiable* if for some temporal interpretation $I = (\mathcal{M}_t, h)$ there exists a $t \in h(\phi)$; in this case, we say that h is a *model of* ϕ *in* t.

Two formulae ϕ and ψ are *semantically equivalent*, and denoted by $\phi \equiv \psi$, if, for each temporal interpretation $I = (\mathcal{M}_t, h)$, it is verified that $h(\phi) = h(\psi)$.

2.1. Classical Logic Extensions

We are interested in temporal propositional logics preserving all the laws of classical propositional logic. These are known as *extensions of classical logic*. In an informal way a temporal propositional logic is an extension of classical propositional logics if, for each instant of time, its restriction is a classical logic. Formally:

Definition 2.4. Let $\mathbf{L} = (\mathcal{L}, \mathcal{I})$ be a temporal propositional logic, where $\mathcal{L} = \langle \mathcal{V} \rangle = (L, \mathcal{G})$. $\mathbf{L}$ is an *extension of classical logic* if there exist $\mathcal{F} \subseteq \mathcal{G}$ and a matrix $\mathcal{M} = (\{0,1\}, \{1\}, \mathcal{F})$, such that, for each $t \in T$, $\mathbf{L}(t) = (\mathcal{L}(t), \mathcal{M}, \mathcal{I}(t))$, where

1. $\mathcal{L}(t) = \langle L \rangle = (L, \mathcal{F})$
2. $\mathcal{I}(t) = \{I_{(t)} \mid I \in \mathcal{I}\}$ where, if $I = (\mathcal{M}_t, h) \in \mathcal{I}$

$$I_{(t)} : \mathbf{L} \longrightarrow \{0,1\}; \qquad I_{(t)}(\phi) = 1 \text{ if and only if } t \in h(\phi)$$

is a classical propositional logic.

Our aim is to deepen, in a formal way, the study of implicate and implicant sets in temporal logics. We will begin by defining the semantic implication relation.

Definition 2.5. Let $\mathbf{L} = (\mathcal{L}, \mathcal{I})$ be a temporal propositional logic that is an extension of classical logic. We define a binary relation, $\trianglelefteq$, in its language, $\mathcal{L}$, as follows: let $\phi, \psi \in \mathcal{L}$ then $\phi \trianglelefteq \psi$ if and only if $\models \phi \to \psi$

The relation $\trianglelefteq$ is a preorder. Indeed, there are different $\phi, \psi \in \mathcal{L}$ such that $\phi \trianglelefteq \psi$ and $\psi \trianglelefteq \phi$. For example, $\phi = (p \vee q) \wedge (p \vee \neg q)$ and $\psi = p$. Therefore, in the quotient set $\mathcal{L}/_{\equiv}$, the relation $\trianglelefteq$ defined as: $[\phi] \trianglelefteq [\psi]$ if and only if $\phi \trianglelefteq \psi$ is a partial-order relation, with a bounded lattice structure with element zero $[\bot]$ and element one $[\top]$. Furthermore, $(\mathcal{L}/_{\equiv}, \trianglelefteq)$ is a Boolean algebra.

2.2. FNext and FNext± Logics

We illustrate previous definitions by using FNext and FNext± logics. In these, the flow of time is $(\mathbb{Z}, \leq)$, and is infinite, discrete and linear and the propositional variable set is $\mathcal{V} = \{p, q, \dots, p_1, q_1, \dots, p_n, q_n, \dots\}$. FNext is the extension of classical propositional logic with the connectives: $\oplus$ ("tomorrow"), F ("sometime in the future"), and G ("always in the future").

Definition 2.6. FNext is a temporal propositional logic $(\mathcal{L}_{FNext}, \mathcal{I})$ given by :

- $\mathcal{L}_{FNext} = \langle \mathcal{V} \rangle = (L_{FNext}, \bot, \top, \neg, \oplus, \mathtt{F}, \mathtt{G}, \rightarrow, \vee, \wedge)$ where the arity list of these operators (their similarity) is $(0, 0, 1, 1, 1, 1, 2, *, *)$.[4]
- $\mathcal{I}$, the interpretation set of the type $(\mathcal{M}_t, h)$ where:
  - $\mathcal{M}_t = ((\mathbb{Z}, \leq), \emptyset, \mathbb{Z}, {}^c, \oplus, \mathtt{F}, \mathtt{G}, \rightarrow, \cup, \cap)$ where: ^{c} is the complementary operator; $\oplus, \mathtt{F}, \mathtt{G} : 2^{\mathbb{Z}} \rightarrow 2^{\mathbb{Z}}$ are given by $\oplus\Gamma = \{t \in \mathbb{Z} \mid t+1 \in \Gamma\}$ $\mathtt{F}\Gamma = \{t \in \mathbb{Z} \mid [t+1) \cap \Gamma \neq \emptyset\}$,[5] $\mathtt{G}\Gamma = \{t \in \mathbb{Z} \mid [t+1) \subseteq \Gamma\}$ and $\rightarrow: 2^{\mathbb{Z}} \times 2^{\mathbb{Z}} \rightarrow 2^{\mathbb{Z}}$ given by $\rightarrow(\Gamma_1, \Gamma_2) = \Gamma_1^c \cup \Gamma_2$.
  - h is any homomorphism of $\mathcal{L}_{FNext}$ in $\mathcal{M}_t$.

FNext± is an extension of FNext with past temporal connectives: $\ominus$ ("yesterday"), P ("sometime in the past"), and H ("always in the past").

Definition 2.7. FNext± is the temporal propositional logic $(\mathcal{L}_{FNext\pm}, \mathcal{I})$ given by:

- $\mathcal{L}_{FNext\pm} = \langle \mathcal{V} \rangle = (L_{FNext\pm}, \bot, \top, \neg, \oplus, \mathtt{F}, \mathtt{G}, \ominus, \mathtt{P}, \mathtt{H}, \rightarrow, \vee, \wedge)$ where its similarity is $(0, 0, 1, 1, 1, 1, 1, 1, 1, 2, *, *)$.
- $\mathcal{I}$, the set of interpretations of the type $(\mathcal{M}_t, h)$ where:
  - $\mathcal{M}_t = ((\mathbb{Z}, \leq), \emptyset, \mathbb{Z}, {}^c, \oplus, \mathtt{F}, \mathtt{G}, \ominus, \mathtt{P}, \mathtt{H}, \rightarrow, \cup, \cap)$ where ^{c}, $\oplus$, F, G and $\rightarrow$ are defined in the same way as in FNext, and $\ominus, \mathtt{P}, \mathtt{H} : 2^{\mathbb{Z}} \rightarrow 2^{\mathbb{Z}}$ are given by $\ominus\Gamma = \{t \in \mathbb{Z} \mid t-1 \in \Gamma\}$ $\mathtt{P}\Gamma = \{t \in \mathbb{Z} \mid (t-1] \cap \Gamma \neq \emptyset\}$ $\mathtt{H}\Gamma = \{t \in \mathbb{Z} \mid (t-1] \subseteq \Gamma\}$
  - h is any homomorphism of $\mathcal{L}_{FNext\pm}$ in $\mathcal{M}_t$.

3. UNITARY FORMULAE AND LITERALS

In this section we introduce the concepts of unitary formula and literal.

[4] The symbol $*$ indicates that the operator has flexible arity.

[5] $[t+1)$ denotes the filter generated by $\{t+1\}$ in $(\mathbb{Z}, \leq)$. That is, $[t+1) = \{x \in \mathbb{Z} \mid t < x\}$.

Definition 3.1. Let $(\mathcal{L},\mathcal{I})$ be a temporal propositional logic, an extension of classical logic, where $\mathcal{L} = \langle\mathcal{V}\rangle$ in the (Ω) category. We define the set of *unitary formulae*, $\mathcal{L}^{mon}$, as the word algebra freely generated by $\mathcal{V}$ in the $(\Omega(1))$ category.

Therefore, unitary formulae are the constants and the language formulae which only have monary connectives. For example, in FNext and in FNext± these are

$$\mathcal{L}^{mon}_{FNext} = \{\top,\bot\} \cup \{\gamma_1 \dots \gamma_k \ell_p \mid \ell_p \in \mathcal{V}^{\pm},\ \gamma_i \in \{\neg,\oplus,\mathtt{F},\mathtt{G}\},\ 1 \le i \le k\}$$

$$\mathcal{L}^{mon}_{FNext\pm} = \{\top,\bot\} \cup \{\gamma_1 \dots \gamma_k \ell_p \mid \ell_p \in \mathcal{V}^{\pm}, \gamma_i \in \{\neg,\oplus,\mathtt{F},\mathtt{G},\ominus,\mathtt{P},\mathtt{H}\}, 1 \le i \le k\}$$

where $\mathcal{V}^{\pm}$ is the *set of classical literals*, i.e., $\{p, \neg p \mid p \in \mathcal{V}\}$. From now on, ℓ_p will denote a classical literal in p and $\overline{\ell_p}$ will denote the opposite literal.

Definition 3.2. Let $(\mathcal{L},\mathcal{I})$ be a temporal propositional logic, an extension of the classical logic, where $\mathcal{L} = \langle\mathcal{V}\rangle$ in the (Ω) category. We define the set of *temporal literals*[6] as follows:

$$Lit = \{[\phi] \in \mathcal{L}/_{\equiv} \mid [\phi] \cap \mathcal{L}^{mon} \neq \varnothing\}$$

Therefore, a literal is a class of $\mathcal{L}/_{\equiv}$ containing some unitary formula.

The equivalence laws in FNext allow us to choose a canonical representative for each literal[7] and define the FNext literals set, up to equivalence, as $Lit^{+} = \bigcup_{\ell_p \in \mathcal{V}^{\pm}} Lit^{+}(\ell_p)$ where $Lit^{+}(\ell_p) = \{\top,\bot\} \cup \{\mathtt{FG}\ell_p, \mathtt{GF}\ell_p\} \cup \{\oplus^n \ell_p, \mathtt{F}\oplus^n \ell_p, \mathtt{G}\oplus^n \ell_p \mid n \in \mathbb{N}\}$.

FNext± equivalence laws and the algorithm described in [5] allows us to define the literal set of FNext±, up to equivalence, as : $Lit^{\pm} = \bigcup_{\ell_p \in \mathcal{V}^{\pm}} Lit^{\pm}(\ell_p)$ where

$$\begin{aligned} Lit^{\pm}(\ell_p) = \ \{\top,\bot\}\ &\cup \{\mathtt{FG}\ell_p, \mathtt{GF}\ell_p, \mathtt{PH}\ell_p, \mathtt{HP}\ell_p, \mathtt{FP}\ell_p, \mathtt{GH}\ell_p\} \\ &\cup \{\odot^n \ell_p, \mathtt{F}\odot^n \ell_p, \mathtt{G}\odot^n \ell_p, \mathtt{P}\odot^n \ell_p, \mathtt{H}\odot^n \ell_p \mid\ n \in \mathbb{Z}\} \end{aligned}$$

and $\odot^n \ell_p$ denotes: $\oplus \overset{n}{\dots} \oplus \ell_p$ if $n > 0$; $\ominus \overset{-n}{\dots} \ominus \ell_p$ if $n < 0$, and $\odot^0 \ell_p = \ell_p$.

4. UNITARY IMPLICATES AND IMPLICANTS

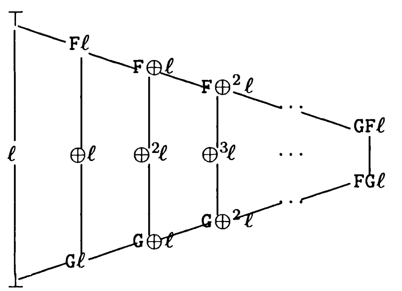
Implicates and implicants are widely used in several areas of artificial intelligence. For example, they are used to formally model truth maintenance systems (TMSs) and assumption-based truth maintenance systems (ATMSs), for circumscription, model-based diagnosis, abduction, and relational databases [17, 19, 20, 21].

Our research focuses on the field of automated deduction where it is possible to obtain good results using the concept of unitary implicants and implicates. The advantage of using unitary implicants and implicates resides in the large amount of information they provide with a lower cost.

The difficulty of calculating unitary implicates and implicants sets in temporal logic is due to the fact that the sets can be infinite. In order to overcome this problem we must deepen the study of the structure of these sets. In this section we provide some glimpses into a study of this kind to justify the introduction of the new algebraic structures.

[6] As we will only work with temporal literals, from now on we will omit the adjective *temporal*.

[7] In [4] is shown an algorithm with linear cost to obtain the canonical representative



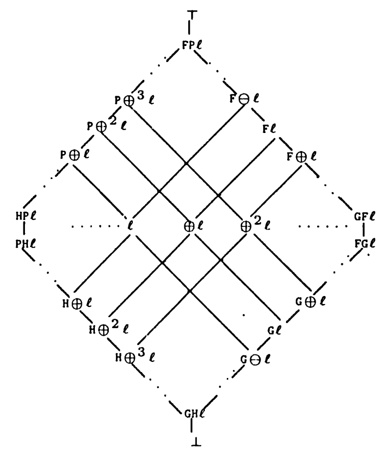


Fig. 1. The lattice $(Lit^+(\ell_p), \trianglelefteq)$ in FNext. **Fig. 2.** The poset $(Lit^\pm(\ell_p), \trianglelefteq)$ in FNext±.

Definition 4.1. Let $(\mathcal{L}, \mathcal{I})$ be a temporal logic, an extension of classical logic, let us consider $(\mathcal{L}/_\equiv, \trianglelefteq)$ and $\phi, \psi \in \mathcal{L}/_\equiv$. If $\phi \trianglelefteq \psi$, we say that ϕ is an *implicant* of ψ, and ψ is an *implicate* of ϕ.

We take as a starting point the following result, which is a direct consequence of the definitions of ideal and filter in lattices [11].

Proposition 4.2. Let $(\mathcal{L}, \mathcal{I})$ be a temporal logic, an extension of classical logic, and let us consider the lattice $(\mathcal{L}/_\equiv, \trianglelefteq)$ and $\varphi \in \mathcal{L}/_\equiv$. The implicant set of φ is the ideal $(\varphi]$, and the set of implicates is the filter $[\varphi)$ in the lattice $(\mathcal{L}/_\equiv, \trianglelefteq)$. In this way, the sets of implicant literals and implicates are, respectively: $(\varphi] \cap Lit$ and $[\varphi) \cap Lit$.

Given that the concepts of implicate and filter are dual concepts of implicant and ideal, from now on we will only refer to the latter.

Our aim is to design efficient algorithms for calculating the sets of implicant literals of a formula. To solve this, we need to make an algebraic study of the behavior of intersections of the type $I \cap Lit$ where I is an ideal of $(\mathcal{L}/\equiv, \trianglelefteq)$.

Let $(A, \leq)$ be a lattice, $\varnothing \neq B \subseteq A$ and $X \subseteq A$, we denote by:

- $X{\downarrow}_B$, the restriction of $X{\downarrow}$ to B, i.e., $X{\downarrow}_B = X{\downarrow} \cap B$. [8]

[8] $X{\downarrow}$ denotes the lower closure of X, that is, $X{\downarrow} = \bigcup_{x \in X} (x] = \bigcup_{x \in X} \{y \in A \mid y \leq x\}$

- If I is an ideal of A, I_B denotes the restriction of I to B. In more specific terms $(X]_B$ denotes the restriction on B of the ideal generated by X, $(X]$; that is, $(X]_B = (X] \cap B$ and this is called *ideal generated by X in B*.

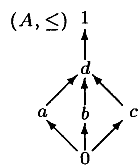
From now on, we will use indistinctly $(A, \leq)$ or $(A, \vee, \wedge)$ to denote a lattice considered as an ordered structure or as an algebraic structure, respectively.

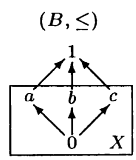
4.1. Restrictions of ideals

We introduce here the concept of ideal restricted to the subset B of a lattice A and the basic results that call for the introduction of new structures. In a dual way, the corresponding results are obtained for filters.

Definition 4.3. Let $(A, \leq)$ be a lattice and $\varnothing \neq B \subseteq A$. If $X \subseteq B$ satisfies that $(X]_B = X$, then X is an *ideal restricted to B*.

Example 4.1. Let us consider the lattice $(A, \leq)$, $B = \{0, a, b, c, 1\}$ and $X = \{0, a, b, c\}$. X is an ideal restricted to B, because $(X] = \{0, a, b, c, d\}$ and $(X]_B = X$. However, X is not a filter restricted to B because $[X) = A$ and $[X)_B = B \neq X$.





The following proposition justifies the above definition.

Proposition 4.4. Given a lattice $(A, \leq)$, $\varnothing \neq B \subseteq A$ and $X \subseteq B$. Then:

1. $(X]_B = \{b \in B \mid \text{there exists a finite } X_0 \subseteq X \text{ such that } b \leq \bigvee_{x \in X_0} x\}$.
2. X is an ideal restricted to B if and only if there is an ideal I of A such that $X = I_B$.
3. $(X]_B$ is the intersection of all the ideals restricted to B including X.

Proof. These results are a direct consequence of the well-known results about lattices [11].

Let see now some properties related with restricted ideals.

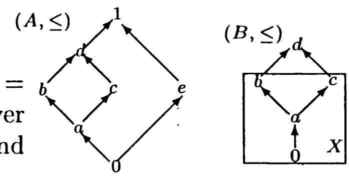
Proposition 4.5. Let $(A, \leq)$ be a lattice, $\varnothing \neq B \subseteq A$ and $X \subseteq B$.

i) If X is an ideal restricted to B, then $X = X{\downarrow}_B$.

ii) If $(A, \leq)$ is bounded and $0 \in B$, then 0 belongs to every ideal restricted to B.

Proof. i) Obviously, $X \subseteq X{\downarrow_B}$. On the other hand, $(X]_B^{\cdot} = X$, and so $X{\downarrow_B} \subseteq (X]_B = X$. Item ii) is an immediate consequence of item i).

The next example ensures that the opposite of item 1 in the previous proposition is not true.

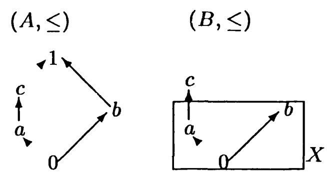


Example 4.2. Let us consider the lattice $(A, \leq)$, $B = \{0, a, b, c, d\}$ and $X = \{0, a, b, c\}$. $X{\downarrow_B} = X$, however X is not an ideal restricted to B because $(X] = B$ and $(X]_B = B$.

Theorem 4.6. Let $(A, \vee, \wedge)$ be a lattice and $X \subseteq B \subseteq A$. If X is an ideal restricted to B, then X is an ideal of the partial lattice $(B, \vee_B, \wedge_B)$. [9]

Proof. From Proposition 4.5, we have that $X{\downarrow_B} = X$. Moreover, if $a, b \in X$ and $a \vee b$ exists in B, we have that $a \vee b \in (X]_B = X$.

The following examples ensure that the opposite of the previous theorem is not true:



Example 4.3. Let us consider the lattice $(A, \leq)$, $B = \{0, a, b, c\}$ and $X = \{0, a, b\}$.
$(X] = A$, $(X]_B = B \neq X$, and so X is not an ideal restricted to B, however, X is an ideal of the partial lattice $(B, \vee_B, \wedge_B)$, because X is closed for $\vee_B$ and $X{\downarrow_B} = X$.

Now we give a new and useful characterization of restricted ideals.

Theorem 4.7. Let $(A, \leq)$ be a lattice, $\varnothing \neq B \subseteq A$ and $X \subseteq B$. Then, X is an ideal restricted to B iff for every finite subset $X_0 \subseteq X$, we have that $(X_0]_B \subseteq X$.

Proof. If X is an ideal restricted to B and X_0 is a finite subset of X, we have that $(X_0]_B \subseteq (X]_B \subseteq X$. Conversely, let us suppose that for every finite subset $X_0 \subseteq X$, we have that $(X_0]_B \subseteq X$. We have to prove that $(X]_B \subseteq X$. By Proposition 4.4, if $b \in (X]_B$, there exists a finite $X_0 \subseteq X$ such that $b \in (X_0]_B$ and by hypothesis, we have that $b \in X$.

Example 4.4. Let us consider the lattice $(2^{\mathbb{R}}, \subseteq)$. We have that $2^{\mathbb{Z}}$ is an ideal restricted to $2^{\mathbb{Q}}$ because, for all $\{C_1, \ldots, C_n\} \subseteq 2^{\mathbb{Z}}$, we have that

$$(\{C_1, \ldots, C_n\}]_{2^{\mathbb{Q}}} = 2^{C_1 \cup \cdots \cup C_n} \subseteq 2^{\mathbb{Z}}$$

[9] $(B, \vee_B, \wedge_B)$ is a partial lattice of $(A, \vee, \wedge)$ if $B \subseteq A$ and $\vee_B$ and $\wedge_B$ are the restriction to B of $\vee$ and $\wedge$, respectively. Moreover, $\varnothing \neq I \subseteq B$ is an ideal of $(B, \vee_B, \wedge_B)$ if it satisfies the two following conditions: (1) for all $a, b \in I$, if $a \vee_B b$ exist then $a \vee_B b \in I$; (2) if $x \leq a \in I$ then $x \in I$. See pages 52-54 in [11].

Remark. From now on, when we say FNext (or FNext±), we mean the lattice $(\mathcal{L}_{FNext/\equiv}, \trianglelefteq)$ (or $(\mathcal{L}_{FNext\pm/\equiv}, \trianglelefteq)$) and the subset $B = Lit^+$ (or $B = Lit^{\pm}$).

Example 4.5. In the lattice $(\mathcal{L}_{FNext/\equiv}, \trianglelefteq)$, we have that:

1. $X = \{\mathtt{G} \oplus^m \ell_p \mid m \in \mathbb{N}\} \cup \{\bot\}$ is an ideal restricted to Lit^+.
2. $Y = Lit^+ - \{\top, \mathtt{F}\ell_p\}$ is not an ideal restricted to Lit^+ because $\{\mathtt{F}\oplus\ell_p, \oplus\ell_p\} \subseteq Y$ and $(\{\mathtt{F} \oplus \ell_p, \oplus\ell_p\}]_{Lit^+} = (\mathtt{F} \oplus \ell_p \vee \oplus\ell_p)\downarrow_{Lit^+} = \mathtt{F}\ell_p\downarrow_{Lit^+} \not\subseteq Y$

Proposition 4.8. Let $(A, \leq)$ be a bounded lattice and $\{0,1\} \subseteq B \subseteq A$. The unique subset of B that is, simultaneously, an ideal and a filter restricted to B, is B.

Proof. Let us suppose that $X \subseteq B$ is an ideal restricted to B and a filter restricted to B. Because X is a filter restricted to B, we have that $1 \in [X)_B = X$ and, so $X\downarrow = A$. On the other hand, X is an ideal restricted to B, and Proposition 4.5 ensures that $X = X\downarrow_B$. Then, we have that $X = X\downarrow_B = X\downarrow \cap B = A \cap B = B$.

5. OPERATIONS WITH RESTRICTIONS OF IDEALS

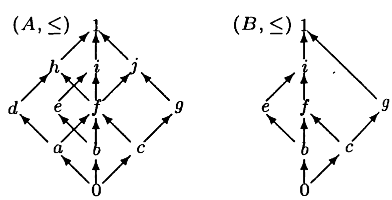
From now on, if $(A, \leq)$ is a lattice and $\varnothing \neq B \subseteq A$, $\mathcal{I}deals_B(A)$ denotes the set of ideals restricted to B in A. We are interested in the structure of $(\mathcal{I}deals_B(A), \subseteq)$ and so, we begin by analyzing the behavior of restricted ideals when they intersect.

Lemma 5.1. Let $(A, \leq)$ be a lattice and $\varnothing \neq B \subseteq A$. Let X_1 and X_2 be two ideals restricted to B. Then, $X_1 \cap X_2$ is an ideal restricted to B.

Proof. Indeed, if $X_1 = (X_1]_B$ and $X_2 = (X_2]_B$ we obviously obtain that $X_1 \cap X_2 \subseteq (X_1 \cap X_2]_B$. On the other hand, $(X_1 \cap X_2]_B \subseteq (X_1]_B \cap (X_2]_B = X_1 \cap X_2$.

The following example shows that the union of restricted ideals is not always a restricted ideal.

Example 5.1. Let us consider the lattice $(A, \leq)$ and the subset B in the following diagrams. The subsets $X_1 = \{0, b, c, f, g\}$ and $X_2 = \{0, b, c, e, f, i\}$ of B are ideals restricted to B. For these subsets we have that:



- $X_1 \cap X_2$ is an ideal restricted to B, and $X_1 \cap X_2 = \{0, b, c, f\} = (\{0, b, c, f\}]_B$
- However, $X_1 \cup X_2$ is not an ideal restricted to B, because $X_1 \cup X_2 = \{0, b, c, e, f, g, i\} \subsetneq (\{0, b, c, e, f, g, i\}]_B = B$

Using the theorem below, we study the structure of $(\mathcal{I}deals_B(A), \subseteq)$.

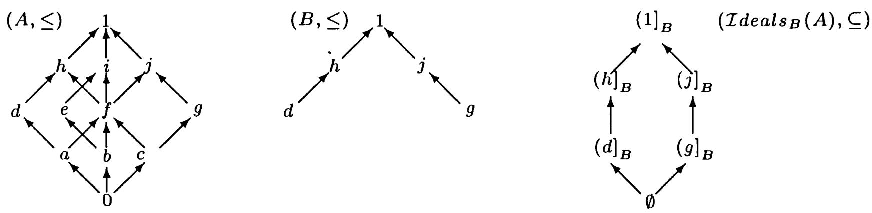
Theorem 5.2. Let $(A, \leq)$ be a lattice and $\varnothing \neq B \subseteq A$. $(\mathcal{I}deals_B(A), \subseteq)$ is a lattice; also

$$\inf(X_1, X_2) = X_1 \cap X_2; \qquad \sup(X_1, X_2) = (X_1 \cup X_2]_B$$

Proof. The equality $\inf(X_1, X_2) = X_1 \cap X_2$ is a direct consequence of Lemma 5.1. On the other hand, to prove that $\sup(X_1, X_2) = (X_1 \cup X_2]_B$, it is enough to verify that $(X_1 \cup X_2]_B$ exists, which is the case, because from item 2 in Proposition 4.4, $(X_1 \cup X_2]_B$ is the intersection of all the ideals restricted to B having $X_1 \cup X_2$, among them, we find at least $B = A_B$.

The following example shows that the empty set can be an element of $\mathcal{I}deals_B(A)$.

Example 5.2. The diagrams show a lattice $(A, \leq)$, the subset $B = \{d, g, h, j, 1, \}$ and the lattice $(\mathcal{I}deals_B(A), \subseteq)$:



We have that $(\mathcal{I}deals_B(A), \subseteq)$ and $(\mathcal{I}deals(A), \subseteq)$ have both the lattice structure.

Lemma 5.3. Let $(A, \leq)$ be a lattice and $X_1, X_2 \subseteq A$. Then, we have that

$$(X_1 \cup X_2] = ((X_1] \cup (X_2]]$$

Proof. Since $X_1 \subseteq (X_1]$ and $X_2 \subseteq (X_2]$, we have that $X_1 \cup X_2 \subseteq (X_1] \cup (X_2]$ and so, $(X_1 \cup X_2] \subseteq ((X_1] \cup (X_2]]$. Conversely, if $y \in ((X_1] \cup (X_2]]$ there exists a finite subset $X_0 \subseteq (X_1] \cup (X_2] \subseteq (X_1 \cup X_2]$ such that $y \leq \bigvee_{x \in X_0} x$, and therefore $y \in (X_1 \cup X_2]$

Theorem 5.4. Let $(A, \leq)$ be a lattice and $\varnothing \neq B \subseteq A$. $(\mathcal{I}deals_B(A), \subseteq)$ is isomorphic to a sublattice of $(\mathcal{I}deals(A), \subseteq)$.

Proof. Let us consider $f : \mathcal{I}deals_B(A) \to \mathcal{I}deals(A)$ defined by $f(X) = (X]$. Obviously f is injective. Let us prove that is an homomorphism. If $X_1, X_2 \in \mathcal{I}deals_B(A)$, then $f(X_1 \wedge X_2) = f(X_1) \wedge f(X_2)$ and Lemma 5.3 ensures that

$$f(X_1 \vee X_2) = (X_1 \cup X_2] = ((X_1] \cup (X_2]] = f(X_1) \vee f(X_2)$$

6. INDUCTIVE CLOSURE AND GROUPABILITY

Once we have characterized the ideals and filters restricted to a subset, our aim is to obtain an efficient manipulation of them. For this reason, we introduce the operators we call *non-deterministic operators* [22, 23] for a set A, and we give a property for the subsets of a lattice, called *groupability*.

Definition 6.1. Let A be a non-empty set. If $F : A^n \to 2^A$ is a total application, then we say that F is a *non-deterministic operator* (ndo) of arity n in A. If $F : A^* \to 2^A$ is a total application, where A^* is the universal language defined in A, we say that F is a *non-deterministic operator of flexible arity* in A. In this case, we define the *n-particularization of F* as the ndo with arity n in A given by $F_n(x_1, \ldots, x_n) = F(x_1 \ldots x_n)$, for all $x_1, \ldots, x_n \in A$.

For all $X \subseteq A$, $F(a_1 \ldots, a_{i-1}, X, a_{i+1}, \ldots, a_n) = \bigcup_{x \in X} F(a_1 \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n)$. Therefore, $F(a_1 \ldots, a_{i-1}, \varnothing, a_{i+1}, \ldots, a_n) = \varnothing$.

Definition 6.2. Let $(A, \leq)$ be a lattice and $\varnothing \neq B \subseteq A$. We denote by Φ_*^B the ndo of flexible arity:

$$\Phi_*^B : B^* \to 2^B \qquad \Phi_*^B(b_1 b_2 \ldots b_n) = (b_1 \vee b_2 \vee \cdots \vee b_n){\downarrow_B}$$

and by Φ_2^B its 2-particularization, i.e., $\Phi_2^B : B^2 \to 2^B$ with $\Phi_2^B(b_1, b_2) = (b_1 \vee b_2){\downarrow_B}$.

Obviously, Φ_2^B and Φ_*^B are commutative. We can now to generalize the concept of inductive closure by using non-deterministic operators.

Definition 6.3. Let A be a set, $X \subseteq A$ and $\mathcal{F}$ a set of nd-operators in A. We define the *nd-inductive closure*[10] *of X under $\mathcal{F}$* as $\mathcal{C}\ell_{\mathcal{F}}(X) = \bigcup_{i \in \mathbb{N}} X_i$ where, if $\mathrm{ar}(F)$ denotes the arity of F, $X_0 = X$ and $X_{i+1} = X_i \cup \bigcup_{F \in \mathcal{F}} F(X_i^{\mathrm{ar}(F)})$
We say that $\mathcal{C}\ell_{\mathcal{F}}$ is an *inductive closure operator* and, if $\mathcal{C}\ell_{\mathcal{F}}(X) = X$, we say that X is *closed under $\mathcal{F}$*.

Proposition 6.4. Let $(A, \leq)$ be a lattice, $\varnothing \neq B \subseteq A$. For all $X \subseteq B$, we have that $(X]_B = \mathcal{C}\ell_{\Phi_*^B}(X)$. That is, $(X]_B$ is the inductive closure of X under Φ_*^B.

Proof. It is a direct consequence of Proposition 4.4.

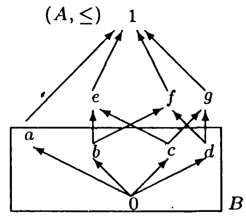
The following example shows that this result is not true for Φ_2^B, and therefore we can not characterize $(X]_B$ as the inductive closure of X under Φ_2^B.

[10] Or *inductive closure* if no confusion arises.

Example 6.1. Let us consider the lattice $(A, \leq)$ and $B = \{0, a, b, c, d\} \subseteq A$. Let $X = \{b, c, d\} \subseteq B$. We have that $(X]_B = B$. However, the inductive closure of X under Φ_2^B is $\{0, b, c, d\} \neq (X]_B$ because:

$$\Phi_2^B(b, c) = \{0, b, c\};\ \Phi_2^B(b, d) = \{0, b, d\}$$

$$\text{and}\quad \Phi_2^B(c, d) = \{0, c, d\}$$



We are interested in establishing conditions for B that allow us to characterize $(X]_B$ as the inductive closure of X under Φ_2^B. In our analysis, we begin with the following result relating the actions of Φ_2^B and Φ_*^B.

From now on, $[n]$ denotes the set of natural numbers $\{1, 2, \ldots, n\}$ and, if X is a non-empty finite set of natural numbers, $S(X)$ is the set of permutations of X.

Proposition 6.5. Let $(A, \leq)$ be a lattice, $\varnothing \neq B \subseteq A$ and $\omega = b_1 b_2 \cdots b_n \in B^*$ with $n \geq 2$. For all permutation $\sigma \in S([n])$ we have that

$$\Phi_2^B(\Phi_2^B(\ldots \Phi_2^B(\Phi_2^B(b_{\sigma(1)}, b_{\sigma(2)}),\ b_{\sigma(3)})\ \ldots, b_{\sigma(n-1)}), b_{\sigma(n)}) \subseteq \Phi_*^B(\omega) \quad (1)$$

Therefore,

$$\bigcup_{\sigma \in S([n])} \Phi_2^B(\ldots \Phi_2^B(\Phi_2^B(b_{\sigma(1)}, b_{\sigma(2)}),\ b_{\sigma(3)})\ \ldots, b_{\sigma(n)}) \subseteq \Phi_*^B(\omega) \quad (2)$$

Proof. We will prove it by induction over the length n of ω.

- If $n = 2$ the result is obvious, because $\Phi_2^B(b_1, b_2) = \Phi_2^B(b_2, b_1) = \Phi_*^B(b_1, b_2)$.
- Let us now assume that the result is true for length n. Let $\omega = b_1 \ldots b_{n+1}$ and $\sigma \in S_{n+1}$, then,

$$\begin{aligned}\Phi_2^B(\Phi_2^B(\ldots \Phi_2^B(b_{\sigma(1)}, b_{\sigma(2)}), \ldots b_{\sigma(n)}),\ b_{\sigma(n+1)})&\\ &\overset{\dagger}{\subseteq} \Phi_2^B(\Phi_*^B(b_{\sigma(1)} b_{\sigma(2)} \ldots b_{\sigma(n)}), b_{\sigma(n+1)})\\ &\overset{\dagger\dagger}{\subseteq} \Phi_*^B(b_{\sigma(1)} b_{\sigma(2)} \ldots b_{\sigma(n)} b_{\sigma(n+1)}) \overset{\dagger\dagger\dagger}{=} \Phi_*^B(\omega)\end{aligned}$$

  were we use the induction hypothesis in †; and †† is also true because if $b \in \Phi_2^B(b', b_{\sigma(n+1)})$ with $b' \in \Phi_*^B(b_{\sigma(1)} b_{\sigma(2)} \ldots b_{\sigma(n)})$, then it satisfies that $b \leq b' \vee b_{\sigma(n+1)} \leq b_{\sigma(1)} \vee b_{\sigma(2)} \vee \cdots \vee b_{\sigma(n)} \vee b_{\sigma(n+1)}$ and, consequently, $b \in \Phi_*^B(b_{\sigma(1)} b_{\sigma(2)} \ldots b_{\sigma(n)} b_{\sigma(n+1)})$. Furthermore, ††† is true by the commutativity of Φ_*^B

In the following example we show the case in which all the inclusions (1) of Proposition 6.5 are strict, and however the inclusion (2) is an equality.

Example 6.2. Let us consider the set $U = \{1,2,3,4,5,6\}$, the lattice $(2^U, \subseteq)$, the subset $B = \{\{1,2\}, \{3,4\}, \{5,6\}, \{1,3\}, \{4,6\}, \{1,3,5\}, \{2,4,6\}\}$, $X = \{1,2\}$, $Y = \{3,4\}$ and $Z = \{5,6\}$. We have that

$$\Phi_*^B(XYZ) = B$$
$$\Phi_2^B(\Phi_2^B(X,Y),Z) = \{\{1,2\}, \{3,4\}, \{5,6\}, \{1,3\}, \{4,6\}, \{1,3,5\}\}$$
$$\Phi_2^B(\Phi_2^B(Y,Z),X) = \{\{1,2\}, \{3,4\}, \{5,6\}, \{1,3\}, \{4,6\}, \{2,4,6\}\}$$
$$\Phi_2^B(\Phi_2^B(X,Z),Y) = \{\{1,2\}, \{3,4\}, \{5,6\}, \{1,3\}, \{4,6\}\}$$

and we have that:

$$\Phi_*^B(XYZ) = \Phi_2^B(\Phi_2^B(X,Y),Z) \cup \Phi_2^B(\Phi_2^B(Y,Z),X) \cup \Phi_2^B(\Phi_2^B(X,Z),Y) = B$$

In the following example we show that the inclusion (2), and therefore all the inclusions (1) are strict.

Example 6.3. Let us consider the lattice A and the subset B of the example 6.1. If $\omega = bcd$, we have that $\Phi_*^B(bcd) = B$. However,

$$\Phi_2^B(\Phi_2^B(b,c),d) \cup \Phi_2^B(\Phi_2^B(b,d),c) \cup \Phi_2^B(\Phi_2^B(c,d),b) = \{0,b,c,d\} \neq B$$

In order to simplify the study of the ideals restricted to B, it is advantageous that inclusion (2) becomes an equality. So, we introduce the following property:

Definition 6.6. Let F be a ndo with flexible arity in A and F_2 its 2-particularization. We say that F has the property of *groupability* if, for all $\omega = b_1 b_2 \ldots b_n \in A$ with $\texttt{Length}(\omega) = \texttt{n} > 2$, we have that

$$F(\omega) = \bigcup_{\sigma \in S([n])} F_2(F_2(F_2(\ldots F_2(F_2(b_{\sigma(1)}, b_{\sigma(2)}), b_{\sigma(3)}) \ldots), b_{\sigma(n-1)}), b_{\sigma(n)})$$

Let $(A, \leq)$ be a lattice and $\varnothing \neq B \subseteq A$. We say that B is *groupable* if Φ_*^B has the groupability property.

Example 6.4. In the lattice $(A, \leq)$ of the example 6.1, $B = \{0,a,b,c,d\}$ is not groupable.

We can already ratify that we have obtained our objective, that is, a property that allows us to substitute a nd-operator with flexible arity by its 1-particularization and 2-particularization in the inductive closures when we cannot ensure the associativity. Therefore we have the following theorem whose proof is trivial.

Theorem 6.7. Let A be a set, ε the empty chain in A^*, $\mathcal{F}$ a family of ndos in A, $F \in \mathcal{F}$ with flexible arity, F_1 its 1-particularization, F_2 its 2-particularization and $\mathcal{G} = (\mathcal{F} \setminus \{F\}) \cup \{F_1, F_2\}$. If F is groupable, then, for all $X \subseteq A$:

$$\mathcal{C}\ell_{\mathcal{F}}(X) = \mathcal{C}\ell_{\mathcal{G}}(X \cup F(\varepsilon))$$

Moreover, if $F(\varepsilon) \subseteq F(x)$ for every $x \in A$, then $\mathcal{C}\ell_{\mathcal{F}} = \mathcal{C}\ell_{\mathcal{G}}$.

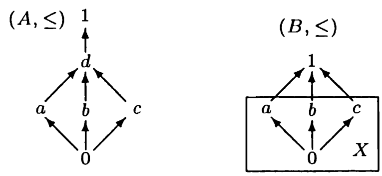
The following corollary allows us to characterize $(X]_B$ as the inductive closure of X under Φ_2^B, if B is a groupable set.

Corollary 6.8. Let $(A, \leq)$ be a lattice and $\varnothing \neq B \subseteq A$ a groupable set. Then, for every $X \subseteq B$, we have that $(X]_B = \mathcal{C}\ell_{\Phi_2^B}(X)$.

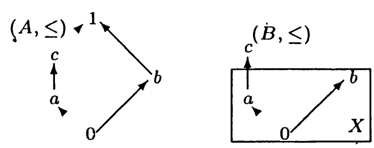
The example 6.1 shows that we can not eliminate in Corollary 6.8 the condition that B is groupable. As a consequence of Corollary 6.8, under the groupability hypothesis we can improve Theorem 4.7 that characterizes the restricted ideals:

Corollary 6.9. Let $(A, \leq)$ be a lattice, $\varnothing \neq B \subseteq A$ a groupable set and $X \subseteq B$. Then, X is an ideal restricted to B if and only if for all $x_1, x_2 \in X$ we have that $(\{x_1, x_2\}]_B \subseteq X$.

Example 6.5. Let us consider the lattice $(A, \leq)$, $B = \{0, a, b, c, 1\} \subseteq A$ and $X = \{0, a, b, c\}$. It is easy to see that B is groupable and Corollary 6.9 allows us to conclude that X is an ideal restricted to B.



Example 6.6. Let us consider the lattice $(A, \leq)$, $B = \{0, a, b, c\}$ and $X = \{0, a, b\}$. We can see that B is groupable and by Corollary 6.9, we have that X is not an ideal restricted to B, because $c \leq a \vee b$ (in A), and however, $c \notin X$.



Now, we give an useful characterization of groupability which justifies its name.

Theorem 6.10. Let $(A, \leq)$ be a lattice and $\varnothing \neq B \subseteq A$. B is groupable if and only if for every $\omega \in B^*$ with $\mathtt{Length}(\omega) > 2$ it satisfies that:

$$\Phi_*^B(\omega) = \bigcup_{\substack{\omega_1 \subseteq \omega \\ \mathtt{Length}(\omega_1) > 1 \\ \omega_2 = \omega \setminus \omega_1}} \Phi_*^B(\Phi_*^B(\omega_1)\omega_2)$$

Proof. It is equivalent to prove that, for all $\omega = b_1 b_2 \dots b_n \in B^*$, the following equality holds:

$$\bigcup_{\substack{\omega_1 \subseteq \omega \\ \mathrm{Length}(\omega_1)>1 \\ \omega_2=\omega\setminus\omega_1}} \Phi_*^B(\Phi_*^B(\omega_1)\omega_2) = \bigcup_{\sigma\in S([n])} \Phi_2^B(\dots(\Phi_2^B(b_{\sigma(1)}, b_{\sigma(2)})\dots, b_{\sigma(n)})$$

Firstly, if $b \in \Phi_2^B(\Phi_2^B(\dots\Phi_2^B(\Phi_2^B(b_{\sigma(1)}, b_{\sigma(2)}), b_{\sigma(3)})\dots, b_{\sigma(n-1)}), b_{\sigma(n)})$ we have that for any $1 < r < n$ exists x such that

$$\begin{aligned} x &\in \Phi_2^B(\Phi_2^B(\dots\Phi_2^B(\Phi_2^B(b_{\sigma(1)}, b_{\sigma(2)}), b_{\sigma(3)})\dots b_{\sigma(r-1)}), b_{\sigma(r)}) \subseteq \\ &\overset{\dagger}{\subseteq} \Phi_*^B(b_{\sigma(1)}\cdots b_{\sigma(r)}) \end{aligned}$$

and it satisfies

$$\begin{aligned} b &\in \Phi_2^B(\Phi_2^B(\dots\Phi_2^B(\Phi_2^B(x, b_{\sigma(r+1)}), b_{\sigma(r+2)})\dots, b_{\sigma(n-1)}), b_{\sigma(n)}) \subseteq \\ &\overset{\dagger}{\subseteq} \Phi_*^B(x, b_{\sigma(r+1)}\cdots b_{\sigma(n)}) \end{aligned}$$

where, in (†) we have used Proposition 6.5. So that, $b \in \Phi_*^B(\Phi_*^B(\omega_1)\omega_2)$ where $\omega_1 = b_{\sigma(1)}\cdots b_{\sigma(r)}$ and $\omega_2 = b_{\sigma(r+1)}\cdots b_{\sigma(n)}$.

Conversely, we prove by induction that there exists $\sigma \in S([n])$ such that

$$\bigcup_{\substack{\omega_1 \subseteq \omega \\ \mathrm{Length}(\omega_1)>1 \\ \omega_2=\omega\setminus\omega_1}} \Phi_*^B(\Phi_*^B(\omega_1)\omega_2) \subseteq \Phi_2^B(\Phi_2^B(\dots\Phi_2^B(\Phi_2^B(b_{\sigma(1)}, b_{\sigma(2)}), b_{\sigma(3)})\dots, b_{\sigma(n-1)}), b_{\sigma(n)})$$

If $n = 3$ the result is obvious. Let us now assume that the result is true for any chain with length lower than n (being $n > 3$), and we will prove it for chains with length n.

If $\omega = b_1 \dots b_n$ and $b \in \bigcup_{\substack{\omega_1 \subseteq \omega \\ \mathrm{Length}(\omega_1)>1 \\ \omega_2=\omega\setminus\omega_1}} \Phi_*^B(\Phi_*^B(\omega_1)\omega_2)$ then there exist $\sigma' \in S([n])$ such that $b \in \Phi_*^B(\Phi_*^B(\omega_1)\omega_2)$ with $\omega_1 = b_{\sigma'(1)}\cdots b_{\sigma'(r)}$ and $\omega_2 = b_{\sigma'(r+1)}\cdots b_{\sigma'(n)}$. That is, there exists x such that $x \in \Phi_*^B(\omega_1)$ and $b \in \Phi_*^B(x\omega_2)$.

By induction hypothesis there exists $\sigma'' \in S(\{\sigma'(1), \cdots, \sigma'(r)\})$ such that

$$x \in \Phi_2^B(\dots(\Phi_2^B(b_{\sigma''(\sigma'(1))}, b_{\sigma''(\sigma'(2))})\dots, b_{\sigma''(\sigma'(r))})$$

and there exists $\sigma''' \in S(\{\sigma'(r+1), \cdots, \sigma'(n)\})$ such that

$$b \in \Phi_2^B(x\dots(\Phi_2^B(b_{\sigma'''(\sigma'(r+1))}, b_{\sigma'''(\sigma'(r+2))})\dots, b_{\sigma'''(\sigma'(n))})$$

The permutation $\sigma \in S([n])$ defined by:

$$\sigma(i) = \begin{cases} \sigma''(\sigma'(i)), & \text{if } 1 \le i \le r \\ \sigma'''(\sigma'(i)), & \text{if } r < i \le n \end{cases}$$

verifies that $b \in \Phi_2^B(\Phi_2^B(\dots(\Phi_2^B(b_{\sigma(1)}b_{\sigma(2)})\, b_{\sigma(3)})\dots)\, b_{\sigma(n-1)})\, b_{\sigma(n)})$.

Definition 6.11. Let $(A, \leq)$ be a lattice and $\varnothing \neq B \subseteq A$. We say that Φ_*^B has the property of *strong groupability* if for every chain $\omega = b_1 \dots b_n \in B^*$ with $n \geq 2$, there exists a permutation $\sigma \in S([n])$ such that

$$\Phi_*^B(\omega) = \Phi_*^B(\Phi_2^B(b_{\sigma(1)}, b_{\sigma(2)})\, b_{\sigma(3)} \dots\, b_{\sigma(n)})$$

The following result is immediate.

Theorem 6.12. Let $(A, \leq)$ be a lattice and $\varnothing \neq B \subseteq A$. Then

1. If B is strong groupable, then B is groupable.
2. If for any chain $\omega = b_1 b_2 \dots b_n \in B^*$, with $\texttt{Length}(\omega) = \texttt{n} > 2$ we have that there exists a permutation $\sigma \in S([n])$ such that

$$\Phi_*^B(\omega) = \Phi_2^B(\Phi_2^B(\dots \Phi_2^B(\Phi_2^B(b_{\sigma(1)}, b_{\sigma(2)}), b_{\sigma(3)}) \dots), b_{\sigma(n-1)}), b_{\sigma(n)})$$

then we have that B is strong groupable.

7. GROUPABILITY IN Lit^+ AND $Lit^\pm$

In the theoretical aspects, we have completed the study about ideals/filters of a lattice, when they are restricted to a subset, and we have characterized them as the inductive closure of a set using an ndo of flexible arity. The property called groupability allows us to characterize the ideals/filters restricted as the inductive closure of a set using a binary ndo.

Taking up again the applied aspects that justified the theoretical study, we remember that the utility of the unitary implicants/implicates sets to improve the efficiency of any prover has been widely proved [3, 12, 13, 14, 15, 23]. The greatest obstacle we have found in temporal logic is that we will must manage flexible arity operators and its storage is not possible. However, the presence of the groupability property in the set of literals allows us to replace its management by using binary operators. This property also allows us to extend the results given in [3, 4, 5, 6] to these temporal logics and to other in which the literals set satisfy this property.

In this section we prove that the set of literals for the logics FNext (Lit^+) and FNext$\pm$ ($Lit^\pm$) are strong groupable. Firstly, we begin with some previous results.

Lemma 7.1. Let $\Gamma \subseteq Lit^\pm$ and $\Gamma_p = \Gamma \cap (Lit^\pm(p) \cup Lit^\pm(\overline{p}))$ for each $p \in \mathcal{V}$.

1. $(\Gamma]_{Lit^\pm} = Lit^\pm$ if and only if there exists $p \in \mathcal{V}$ such that $(\Gamma_p]_{Lit^\pm} = Lit^\pm$. [11]
2. $[\Gamma)_{Lit^\pm} = Lit^\pm$ if and only if there exists $p \in \mathcal{V}$ such that $[\Gamma_p)_{Lit^\pm} = Lit^\pm$. [12]

[11] Notice that the condition $(\Gamma]_{Lit^\pm} = Lit^\pm$ ensures that $\Gamma \neq \varnothing$ and that, if it is unitary, then $\Gamma = \{\top\}$.

[12] Notice that the condition $[\Gamma)_{Lit^\pm} = Lit^\pm$ ensures that $\Gamma \neq \varnothing$ and that, if it is unitary, then $\Gamma = \{\bot\}$.

Let us consider a finite subset $\Gamma \subseteq Lit^+$. Obviously, if $\top \in \Gamma$ or there exist $\ell_1, \ell_2 \in \Gamma$ such that $\overline{\ell_1} \trianglelefteq \ell_2$, we can ensure that $(\Gamma]_{Lit^+} = Lit^+$. However, there exist an infinite number of cases. Indeed, for all $n \in \mathbb{N}$ we have that $\mathtt{F} \oplus^{n+1} \ell_p \vee \oplus^{n+1} \ell_p = \mathtt{F} \oplus^n \ell_p$ and so, for all $n \in \mathbb{N}$:

$$\left(\mathtt{G} \oplus^n \overline{\ell_p}, \oplus^{n+1} \ell_p, \dots, \oplus^{n+k} \ell_p, \mathtt{F} \oplus^{n+k} \ell_p\right]_{Lit^+} = Lit^+$$

The following theorem allows us to determinate the finite sets $\Gamma \subseteq Lit^\pm$ such that $(\Gamma]_{Lit^\pm} = Lit^\pm$.

Theorem 7.2. Let $\varnothing \neq \Gamma \subseteq Lit^\pm$ a non-unitary set. Then, $(\Gamma]_{Lit^\pm} = Lit^\pm$ if and only if one of the following conditions is satisfied:

(i) There exist $\ell_1, \ell_2 \in \Gamma$ such that $\overline{\ell_1} \trianglelefteq \ell_2$.

(ii) There exist $\ell_p \in \mathcal{V}^\pm$, and $m_1, m_2 \in \mathbb{Z}$ with $m_1 < m_2$, such that:

$$\{\mathtt{G} \odot^{m_1} \overline{\ell_p}, \mathtt{F} \odot^{m_2} \ell_p\} \cup \{\odot^m \ell_p \mid m_1 < m \leq m_2\} \subseteq \Gamma$$

(iii) There exist $\ell_p \in \mathcal{V}^\pm$, and $m_1, m_2 \in \mathbb{Z}$ with $m_1 > m_2$, such that:

$$\{\mathtt{H} \odot^{m_1} \overline{\ell_p}, \mathtt{P} \odot^{m_2} \ell_p\} \cup \{\odot^m \ell_p \mid m_1 > m \geq m_2\} \subseteq \Gamma$$

(iv) There exist $\ell_p \in \mathcal{V}^\pm$ such that $\varnothing \neq \Gamma \cap Lit^\pm(\overline{\ell_p}) \neq \{\bot\}$, and $m_1, m_2 \in \mathbb{Z}$ with $m_1 \leq m_2$, such that:

$$\{\mathtt{P} \odot^{m_1} \ell_p, \mathtt{F} \odot^{m_2} \ell_p\} \cup \{\odot^m \ell_p \mid m_1 \leq m \leq m_2\} \subseteq \Gamma$$

Proof. As a consequence of the previous lemma, it is enough to prove that the result is true for any non-empty, non-unitary and finite $\Gamma \subseteq Lit^\pm(p) \cup Lit^\pm(\overline{p})$.

Its sufficiency is obvious. We show its necessity by proving that if no condition is satisfied then $(\Gamma]_{Lit^\pm} \neq Lit^\pm$.

If condition (i) is not satisfied, for all $\ell_p \in \mathcal{V}^\pm$ we have that $\mathtt{FP}\ell_p \notin \Gamma$, because $\overline{\mathtt{FP}\ell_p} = \mathtt{GH}\overline{\ell_p}$, and for the interpretation h such that $h(\overline{\ell_p}) = \mathbb{Z}$, we have that $0 \notin h(\bigvee_{\ell \in \Gamma} \ell)$ and so, $\bigvee_{\ell \in \Gamma} \ell \neq \top$, that is $(\Gamma]_{Lit^\pm} \neq Lit^\pm$.

Now, we consider the following cases:

1. $\Gamma \cap \{\mathtt{F} \odot^m p, \mathtt{F} \odot^m \overline{p} \mid m \in \mathbb{Z}\} = \varnothing$ and $\Gamma \cap \{\mathtt{P} \odot^m p, \mathtt{P} \odot^m \overline{p} \mid m \in \mathbb{Z}\} = \varnothing$.

2. $\Gamma \cap \{\mathtt{F} \odot^m p, \mathtt{F} \odot^m \overline{p} \mid m \in \mathbb{Z}\} \neq \varnothing$ and $\Gamma \cap \{\mathtt{P} \odot^m p, \mathtt{P} \odot^m \overline{p} \mid m \in \mathbb{Z}\} = \varnothing$.

3. $\Gamma \cap \{\mathtt{F} \odot^m p, \mathtt{F} \odot^m \overline{p} \mid m \in \mathbb{Z}\} = \varnothing$ and $\Gamma \cap \{\mathtt{P} \odot^m p, \mathtt{P} \odot^m \overline{p} \mid m \in \mathbb{Z}\} \neq \varnothing$.

4. There exists $\ell_p \in \{p, \overline{p}\}$ such that

$$\Gamma \cap \{\mathtt{F} \odot^m \ell_p \mid m \in \mathbb{Z}\} \neq \varnothing \quad \text{and} \quad \Gamma \cap \{\mathtt{P} \odot^m \ell_p \mid m \in \mathbb{Z}\} \neq \varnothing$$

5. There exists $\ell_p \in \{p, \overline{p}\}$ such that

$$\Gamma \cap \{\mathtt{F} \odot^m \ell_p \mid m \in \mathbb{Z}\} \neq \varnothing \quad \text{and} \quad \Gamma \cap \{\mathtt{P} \odot^m \overline{\ell_p} \mid m \in \mathbb{Z}\} \neq \varnothing$$

For *case 1* we have that

$$\begin{aligned}\Gamma \subseteq \ & \{\bot, \mathtt{FG}p, \mathtt{GF}p, \mathtt{GH}p, \mathtt{PH}p, \mathtt{HP}p, \mathtt{FG}\overline{p}, \mathtt{GF}\overline{p}, \mathtt{GH}\overline{p}, \mathtt{PH}\overline{p}, \mathtt{HP}\overline{p}\} \cup \\ & \cup \{\odot^{r_1}p, \dots, \odot^{r_{n_1}}p\} \cup \{\mathtt{G} \odot^{s_1} p, \dots, \mathtt{G} \odot^{s_{n_2}} p\} \cup \\ & \cup \{\mathtt{H} \odot^{u_1} p, \dots, \mathtt{H} \odot^{u_{n_3}} p\} \cup \{\odot^{v_1}\overline{p}, \dots, \odot^{v_{n_4}}\overline{p}\} \cup \\ & \cup \{\mathtt{G} \odot^{w_1} \overline{p}, \dots, \mathtt{G} \odot^{w_{n_5}} \overline{p}\} \cup \{\mathtt{H} \odot^{z_1} \overline{p}, \dots, \mathtt{H} \odot^{z_{n_6}} \overline{p}\} \\ & r_1 < \dots < r_{n_1};\ s_1 < \dots < s_{n_2};\ u_1 < \dots < u_{n_3}; \\ & v_1 < \dots < v_{n_4};\ w_1 < \dots < w_{n_5};\ z_1 < \dots < z_{n_6}\end{aligned}$$

and as, by hypothesis, condition (i) is not satisfied, we can ensure that:

- $r_i \neq v_j$ for all $1 \leq i \leq n_1$ and for all $1 \leq j \leq n_4$.
- If $\mathtt{GF}\ell_p \in \Gamma$ with $\ell_p \in \{p, \overline{p}\}$ then $\mathtt{FG}\overline{\ell_p} \notin \Gamma$ and $\mathtt{GF}\overline{\ell_p} \notin \Gamma$.
- If $\mathtt{HP}\ell_p \in \Gamma$ with $\ell_p \in \{p, \overline{p}\}$ then $\mathtt{PH}\overline{\ell_p} \notin \Gamma$ and $\mathtt{HP}\overline{\ell_p} \notin \Gamma$.

We denote by $k_1 = \min\{r_1, u_1, v_1, z_1\}$ and by $k_2 = \max\{r_{n_1}, s_{n_2}, v_{n_4}, w_{n_5}\}$. Let us consider any interpretation h such that

$$h(p) = A^- \cup \{v_1, \dots, v_{n_4}\} \cup A^+$$

where

$$A^- = \begin{cases} \{k_1 - 1\} & \text{if } \mathtt{HP}p \in \Gamma \\ \{m \mid m < k_1 - 1\} & \text{if } \mathtt{HP}\overline{p} \in \Gamma \\ \{m \mid m < k_1 - 1 \text{ and } m \text{ is even}\} & \text{in other case} \end{cases}$$

$$A^+ = \begin{cases} \{k_2 + 1\} & \text{if } \mathtt{GF}p \in \Gamma \\ \{m \mid m > k_2 + 1\} & \text{if } \mathtt{GF}\overline{p} \in \Gamma \\ \{m \mid m > k_2 + 1 \text{ and } m \text{ is even}\} & \text{in other case} \end{cases}$$

It is easy to prove that $0 \notin h(\bigvee_{\ell \in \Gamma} \ell)$ and so, $\bigvee_{\ell \in \Gamma} \ell \neq \top$, that is $(\Gamma]_{Lit^\pm} \neq Lit^\pm$.

For *case 2*, that is, if

$$\Gamma \cap \{\mathtt{F} \odot^m p, \mathtt{F} \odot^m \overline{p} \mid m \in \mathbb{Z}\} \neq \varnothing \quad \text{and} \quad \Gamma \cap \{\mathtt{P} \odot^m p, \mathtt{P} \odot^m \overline{p} \mid m \in \mathbb{Z}\} = \varnothing$$

we have that, as by hypothesis, condition (i) is not satisfied, if $\mathrm{F}\odot^k \ell_p \in \Gamma$, then $\mathrm{FG}\overline{\ell_p} \notin \Gamma$, $\mathrm{GF}\overline{\ell_p} \notin \Gamma$ and, for all $m \in \mathbb{Z}$, $\mathrm{F}\odot^m \overline{\ell_p} \notin \Gamma$, and so:

$$\begin{aligned}\Gamma \subseteq\ & \{\bot, \mathrm{FG}\ell_p, \mathrm{GF}\ell_p, \mathrm{GH}\ell_p, \mathrm{PH}\ell_p, \mathrm{HP}\ell_p, \mathrm{GH}\overline{\ell_p}, \mathrm{PH}\overline{\ell_p}, \mathrm{HP}\overline{\ell_p}\} \cup \\ & \cup \{\odot^{r_1}\ell_p, \dots, \odot^{r_{n_1}}\ell_p\} \cup \{\mathrm{F}\odot^{s_1}\ell_p, \dots, \mathrm{F}\odot^{s_{n_2}}\ell_p\} \cup \\ & \cup \{\mathrm{G}\odot^{t_1}\ell_p, \dots, \mathrm{G}\odot^{t_{n_3}}\ell_p\} \cup \{\mathrm{H}\odot^{u_1}\ell_p, \dots, \mathrm{H}\odot^{u_{n_4}}\ell_p\} \cup \\ & \cup \{\odot^{v_1}\overline{\ell_p}, \dots, \odot^{v_{n_5}}\overline{\ell_p}\} \cup \{\mathrm{G}\odot^{w_1}\overline{\ell_p}, \dots, \mathrm{G}\odot^{w_{n_6}}\overline{\ell_p}\} \cup \\ & \cup \{\mathrm{H}\odot^{z_1}\overline{\ell_p}, \dots, \mathrm{H}\odot^{z_{n_7}}\overline{\ell_p}\}\end{aligned}$$

$$r_1 < \dots < r_{n_1};\ s_1 < \dots < s_{n_2};\ t_1 < \dots < t_{n_3};\ u_1 < \dots < u_{n_4};$$

$$v_1 < \dots < v_{n_5};\ w_1 < \dots < w_{n_6};\ z_1 < \dots < z_{n_7}$$

where:

- $r_i \neq v_j$ for all $1 \leq i \leq n_1$ and all $1 \leq j \leq n_5$.
- As $\overline{\mathrm{F}\odot^{s_1}\ell_p} = \mathrm{G}\odot^{s_1}\overline{\ell_p}$ we have that $v_{n_5} \leq s_1$ and $w_{n_6} < s_1$.
- If $\mathrm{HP}\ell_p \in \Gamma$ then $\mathrm{PH}\overline{\ell_p} \notin \Gamma$ and $\mathrm{HP}\overline{\ell_p} \notin \Gamma$.
- If $\mathrm{HP}\overline{\ell_p} \in \Gamma$ then $\mathrm{PH}\ell_p \notin \Gamma$ and $\mathrm{HP}\ell_p \notin \Gamma$.

and because condition (ii) is not satisfied, there exists $k_1 \in \mathbb{Z}$ with $w_{n_6} < k_1 \leq s_1$ such that $\odot^{k_1}\ell_p \notin \Gamma$.[13]

Let us consider any interpretation h such that

$$h(\ell_p) = A^- \cup \{v_1, \dots, v_{n_5}\} \cup \{k_1\}$$

where, if $k_2 = \min\{r_1, s_1, u_1, v_1, z_1\}$, then

$$A^- = \begin{cases} \{k_2 - 1\} & \text{if } \mathrm{HP}\ell_p \in \Gamma \\ \{m \mid m < k_2 - 1\} & \text{if } \mathrm{HP}\overline{\ell_p} \in \Gamma \\ \{m \mid m < k_2 - 1 \text{ and } m \text{ is even}\} & \text{in other case} \end{cases}$$

We have that $0 \notin h(\bigvee_{\ell \in \Gamma} \ell)$ and so, $\bigvee_{\ell \in \Gamma} \ell \neq \top$, that is $(\Gamma]_{Lit^\pm} \neq Lit^\pm$.

Case 3 is obtained by duality from case 2.

For *case 4*, that is, if there exists $\ell_p \in \{p, \overline{p}\}$ such that

$$\Gamma \cap \{\mathrm{F}\odot^m \ell_p \mid m \in \mathbb{Z}\} \neq \varnothing \quad \text{and} \quad \Gamma \cap \{\mathrm{P}\odot^m \ell_p \mid m \in \mathbb{Z}\} \neq \varnothing$$

[13]If does not exist $\mathrm{G}\odot^m \overline{\ell_p} \in \Gamma$, then exists $k_1 \leq s_1$ such that $\odot^{k_1}\ell_p \notin \Gamma$ because Γ is finite.

then, as condition (i) is not satisfied, we have that

$$\begin{aligned}\Gamma \subseteq\ & \{\bot, \mathrm{FG}\ell_p, \mathrm{GF}\ell_p, \mathrm{GH}\ell_p, \mathrm{PH}\ell_p, \mathrm{HP}\ell_p, \mathrm{GH}\overline{\ell_p}\} \cup \\ & \{\odot^{r_1}\ell_p, \dots, \odot^{r_{n_1}}\ell_p\} \cup \{\mathrm{F}\odot^{s_1}\ell_p, \dots, \mathrm{F}\odot^{s_{n_2}}\ell_p\} \cup \\ & \{\mathrm{G}\odot^{t_1}\ell_p, \dots, \mathrm{G}\odot^{t_{n_3}}\ell_p\} \cup \{\mathrm{P}\odot^{u_1}\ell_p, \dots, \mathrm{P}\odot^{u_{n_4}}\ell_p\} \cup \\ & \{\mathrm{H}\odot^{v_1}\ell_p, \dots, \mathrm{H}\odot^{v_{n_5}}\ell_p\} \cup \{\odot^{w_1}\overline{\ell_p}, \dots, \odot^{w_{n_6}}\overline{\ell_p}\} \cup \\ & \{\mathrm{G}\odot^{y_1}\overline{\ell_p}, \dots, \mathrm{G}\odot^{y_{n_7}}\overline{\ell_p}\} \cup \{\mathrm{H}\odot^{z_1}\overline{\ell_p}, \dots, \mathrm{H}\odot^{z_{n_8}}\overline{\ell_p}\}\end{aligned}$$

$$r_1 < \dots < r_{n_1};\ s_1 < \dots < s_{n_2};\ t_1 < \dots < t_{n_3};\ u_1 < \dots < u_{n_4};$$

$$v_1 < \dots < v_{n_5};\ w_1 < \dots < w_{n_6};\ y_1 < \dots < y_{n_7};\ z_1 < \dots < z_{n_8}$$

where:

- $r_i \neq w_j$ for all $1 \leq i \leq n_1$ and for all $1 \leq j \leq n_6$.
- Because $\overline{\mathrm{F}\odot^{s_1}\ell_p} = \mathrm{G}\odot^{s_1}\overline{\ell_p}$ we have that $w_{n_6} \leq s_1$ and $y_{n_7} < s_1$.
- Because $\overline{\mathrm{P}\odot^{u_{n_4}}\ell_p} = \mathrm{H}\odot^{u_{n_4}}\overline{\ell_p}$ we have that $u_{n_4} \leq w_1$ and $u_{n_4} < z_1$.

In this case, we have two subcases:

a) $\Gamma \subseteq Lit^{\pm}(\ell_p)$ and so, neither (ii) neither (iii) nor (iv) are satisfied. For any interpretation h such that $h(\overline{\ell_p}) = \mathbb{Z}$ we have that $0 \notin h(\bigvee_{\ell\in\Gamma} \ell)$ and so, $\bigvee_{\ell\in\Gamma} \ell \neq \top$, that is $(\Gamma]_{Lit^{\pm}} \neq Lit^{\pm}$.

b) $\Gamma \cap Lit^{\pm}(\overline{\ell_p}) \not\subseteq \{\bot\}$ and so, as condition (ii) is not satisfied, there exists $m_1 \in \mathbb{Z}$ such that $y_{n_7} < m_1 \leq s_1$ and $\odot^{m_1}\ell_p \notin \Gamma$; as condition (iii) is not satisfied, it is necessary that there exists $m_2 \in \mathbb{Z}$ such that $u_{n_4} \leq m_2 < z_1$ and $\odot^{m_2}\ell_p \notin \Gamma$ and, as condition (iv) is not satisfied, it is necessary that it exists $m_3 \in \mathbb{Z}$ such that $u_{n_4} \leq m_3 \leq s_1$ and $\odot^{m_3}\ell_p \notin \Gamma$.

   If we consider $k_1 = \max\{m_1, m_3\}$[14], $k_2 = \min\{m_2, m_3\}$[15] and any interpretation h such that $h(\ell_p) = \{w_1, \dots, w_{n_6}\} \cup \{k_1, k_2\}$ we have that $0 \notin h(\bigvee_{\ell\in\Gamma} \ell)$ and so, $\bigvee_{\ell\in\Gamma} \ell \neq \top$, that is $(\Gamma]_{Lit^{\pm}} \neq Lit^{\pm}$.

[14] this ensures us that $\odot^{k_1}\ell_p \notin \Gamma$ and $y_{n_7} < k_1 \leq s_1$ and $u_{n_4} \leq k_1 \leq s_1$

[15] this ensures us that $\odot^{k_2}\ell_p \notin \Gamma$ and $u_{n_4} \leq k_2 < z_1$ and $u_{n_4} \leq k_2 \leq s_1$

For *case 5*, that is, if it exists $\ell_p \in \{p, \overline{p}\}$ such that

$$\Gamma \cap \{\mathtt{F} \odot^m \ell_p \mid m \in \mathbb{Z}\} \neq \varnothing \quad \text{and} \quad \Gamma \cap \{\mathtt{P} \odot^m \overline{\ell_p} \mid m \in \mathbb{Z}\} \neq \varnothing$$

as condition (i) is not satisfied, we have that:

$$\begin{aligned}\Gamma \subseteq\ & \{\bot, \mathtt{FG}\ell_p, \mathtt{GF}\ell_p, \mathtt{GH}\ell_p, \mathtt{PH}\overline{\ell_p}, \mathtt{HP}\overline{\ell_p}, \mathtt{GH}\overline{\ell_p}\} \cup \\ & \{\odot^{r_1}\ell_p, \ldots, \odot^{r_{n_1}}\ell_p\} \cup \{\mathtt{F} \odot^{s_1}\ ell_p, \ldots, \mathtt{F} \odot^{s_{n_2}} \ell_p\} \cup \\ & \{\mathtt{G} \odot^{t_1} \ell_p, \ldots, \mathtt{G} \odot^{t_{n_3}} \ell_p\} \cup \{\mathtt{H} \odot^{u_1} \ell_p, \ldots, \mathtt{H} \odot^{u_{n_4}} \ell_p\} \cup \\ & \{\mathtt{P} \odot^{v_1} \overline{\ell_p}, \ldots, \mathtt{P} \odot^{v_{n_5}} \overline{\ell_p}\} \cup \{\odot^{w_1}\overline{\ell_p}, \ldots, \odot^{w_{n_6}}\overline{\ell_p}\} \cup \\ & \{\mathtt{G} \odot^{y_1} \overline{\ell_p}, \ldots, \mathtt{G} \odot^{y_{n_7}} \overline{\ell_p}\} \cup \{\mathtt{H} \odot^{z_1} \overline{\ell_p}, \ldots, \mathtt{H} \odot^{z_{n_8}} \overline{\ell_p}\}\end{aligned}$$

$$r_1 < \cdots < r_{n_1};\ s_1 < \cdots < s_{n_2};\ t_1 < \cdots < t_{n_3};\ u_1 < \cdots < u_{n_4};$$
$$v_1 < \cdots < v_{n_5};\ w_1 < \cdots < w_{n_6};\ y_1 < \cdots < y_{n_7};\ z_1 < \cdots < z_{n_8}$$

where:

- $r_i \neq w_j$ for all $1 \leq i \leq n_1$ and for all $1 \leq j \leq n_6$.
- Because $\overline{\mathtt{F} \odot^{s_1} \overline{\ell_p}} = \mathtt{G} \odot^{s_1} \overline{\ell_p}$ we have that $w_{n_6} \leq s_1$ and $y_{n_7} < s_1$.
- Because $\overline{\mathtt{P} \odot^{v_{n_5}} \overline{\ell_p}} = \mathtt{H} \odot^{v_{n_5}} \ell_p$ we have that $v_{n_5} \leq r_1$ and $v_{n_5} < u_1$.

In this case, condition (iv) is not satisfied. Because condition (ii) is not satisfied, it is necessary that it exists $m_1 \in \mathbb{Z}$ such that $y_{n_7} < m_1 \leq s_1$, $\odot^{m_1}\ell_p \notin \Gamma$ and, because condition (iii), it is necessary that it exists $m_2 \in \mathbb{Z}$ such that $v_{n_5} \leq m_2 < u_1$ and $\odot^{m_2}\overline{\ell_p} \notin \Gamma$. If we consider any interpretation h such that

$$h(\ell_p) = (-\infty, v_{n_5}] \cup \{w_1, \ldots, w_{n_6}\} \cup \{m_1\}$$

we have that $0 \notin h(\bigvee_{\ell \in \Gamma} \ell)$ and so $\bigvee_{\ell \in \Gamma} \ell \neq \top$, that is $(\Gamma]_{Lit^{\pm}} \neq Lit^{\pm}$.

The following corollary of Theorem 7.2 allows us to describe the finite sets $\Gamma \subseteq Lit^+$ such that $(\Gamma]_{Lit^+} = Lit^+$.

Corollary 7.3. Let $\varnothing \neq \Gamma \subseteq Lit^+$ a non-unitary set. $(\Gamma]_{Lit^+} = Lit^+$ if and only if it is satisfied one of this two conditions:

(i). There exist $\ell_1, \ell_2 \in \Gamma$ such that $\overline{\ell_1} \trianglelefteq \ell_2$.

(ii). There exist $\ell_p \in \mathcal{V}^{\pm}$, and $m_1, m_2 \in \mathbb{N}$ with $m_1 < m_2$ such that:

$$\{\mathtt{G} \oplus^{m_1} \overline{\ell_p}, \mathtt{F} \oplus^{m_2} \ell_p\} \cup \{\oplus^m \ell_p \mid m_1 < m \leq m_2\} \subseteq \Gamma$$

As an immediate consequence of Theorem 7.2, we have that:

Corollary 7.4. Let $\varnothing \neq \Gamma \subseteq Lit^{\pm}$ and $\ell_0 \in Lit^{\pm}(\ell_p)$. Then $\ell_0 \in (\Gamma]_{Lit^{\pm}}$ if and only if it is satisfied one of the following conditions:

(i). $(\Gamma]_{Lit^{\pm}} = Lit^{\pm}$.

(ii). $\ell_0 \in \Gamma{\downarrow}_{Lit^{\pm}}$.

(iii). There exist $m_1, m_2 \in \mathbb{Z}$ such that $m_1 < m_2$, $\ell_0 = \mathtt{F} \odot^{m_1} \ell_p$ and $\{\mathtt{F} \odot^{m_2} \ell_p\} \cup \{\odot^m \ell_p \mid m_1 < m \leq m_2\} \subseteq \Gamma$.

(iv). There exist $m_1, m_2 \in \mathbb{Z}$ such that $m_1 < m_2$, $\ell_0 = \mathtt{G} \odot^{m_2} \ell_p$ and $\{\mathtt{G} \odot^{m_1} \ell_p\} \cup \{\odot^m \overline{\ell_p} \mid m_1 < m \leq m_2\} \subseteq \Gamma$.

(v). There exist $m_1, m_2, m_3 \in \mathbb{Z}$ such that $m_1 < m_3 \leq m_2$, $\ell_0 = \odot^{m_3} \ell_p$ and $\{\mathtt{G} \odot^{m_1} \ell_p, \mathtt{F} \odot^{m_2} \overline{\ell_p}\} \cup \{\odot^m \overline{\ell_p} \mid m_1 < m \leq m_2 \text{ y } m \neq m_3\} \subseteq \Gamma$.

(vi). There exist $m_1, m_2 \in \mathbb{Z}$ such that $m_1 > m_2$, $\ell_0 = \mathtt{P} \odot^{m_1} \ell_p$ and $\{\mathtt{P} \odot^{m_2} \ell_p\} \cup \{\odot^m \ell_p \mid m_1 > m \geq m_2\} \subseteq \Gamma$.

(vii). There exist $m_1, m_2 \in \mathbb{Z}$ such that $m_1 > m_2$, $\ell_0 = \mathtt{H} \odot^{m_2} \ell_p$ and $\{\mathtt{H} \odot^{m_1} \ell_p\} \cup \{\odot^m \overline{\ell_p} \mid m_1 > m \geq m_2\} \subseteq \Gamma$.

(viii). There exist $m_1, m_2, m_3 \in \mathbb{Z}$ such that $m_1 > m_3 \geq m_2$, $\ell_0 = \odot^{m_3} \ell_p$ and $\{\mathtt{H} \odot^{m_1} \ell_p, \mathtt{P} \odot^{m_2} \overline{\ell_p}\} \cup \{\odot^m \overline{\ell_p} \mid m_1 > m \geq m_2 \text{ y } m \neq m_3\} \subseteq \Gamma$.

(ix). There exist $m_1, m_2 \in \mathbb{Z}$ such that $m_1 \leq m_2$, $\ell_0 \trianglelefteq \mathtt{FP}\ell_p$, and $\{\mathtt{P} \odot^{m_1} \ell_p, \mathtt{F} \odot^{m_2} \ell_p\} \cup \{\odot^m \ell_p \mid m_1 \leq m \leq m_2\} \subseteq \Gamma$.

(x). There exist $m_1, m_2 \in \mathbb{Z}$ such that $m_1 \leq m_2$, $\ell_0 = \mathtt{H} \odot^{m_1} \ell_p$, $\Gamma \cap Lit^{\pm}(\ell_p) \not\subseteq \{\perp\}$ and $\{\mathtt{F} \odot^{m_2} \overline{\ell_p}\} \cup \{\odot^m \overline{\ell_p} \mid m_1 \leq m \leq m_2\} \subseteq \Gamma$.

(xi). There exist $m_1, m_2 \in \mathbb{Z}$ such that $m_1 \leq m_2$, $\ell_0 = \mathtt{G} \odot^{m_2} \ell_p$, $\Gamma \cap Lit^{\pm}(\ell_p) \not\subseteq \{\perp\}$ and $\{\mathtt{P} \odot^{m_1} \overline{\ell_p}\} \cup \{\odot^m \overline{\ell_p} \mid m_1 \leq m \leq m_2\} \subseteq \Gamma$.

(xii). There exist $m_1, m_2, m_3 \in \mathbb{Z}$ such that $m_1 \leq m_3 \leq m_2$, $\ell_0 = \odot^{m_3} \ell_p$, $\{\mathtt{P} \odot^{m_1} \overline{\ell_p}, \mathtt{F} \odot^{m_2} \overline{\ell_p}\} \cup \{\odot^m \overline{\ell_p} \mid m_1 \leq m \leq m_2 \text{ y } m \neq m_3\} \subseteq \Gamma$ and $\Gamma \cap Lit^{\pm}(\ell_p) \not\subseteq \{\perp\}$.

The following corollary particularizes for Lit^+ the results of Corollary 7.4.

Corollary 7.5. Let $\varnothing \neq \Gamma \subseteq Lit^{+}$ and $\ell_0 \in Lit^{+}(\ell_p)$. Then $\ell_0 \in (\Gamma]_{Lit^{+}}$ if and only if it is satisfied one of the following conditions:

(i). $(\Gamma]_{Lit^{+}} = Lit^{+}$.

(ii). $\ell_0 \in \Gamma{\downarrow}_{Lit^{+}}$.

(iii). There exist $m_1, m_2 \in \mathbb{N}$ such that $m_1 < m_2$, $\ell_0 = \mathtt{F} \oplus^{m_1} \ell_p$ and $\{\mathtt{F} \oplus^{m_2} \ell_p\} \cup \{\oplus^m \ell_p \mid m_1 < m \leq m_2\} \subseteq \Gamma$.

(iv). There exist $m_1, m_2 \in \mathbb{N}$ such that $m_1 < m_2$, $\ell_0 = \mathtt{G} \oplus^{m_2} \ell_p$ and $\{\mathtt{G} \oplus^{m_1} \ell_p\} \cup \{\oplus^m \overline{\ell_p} \mid m_1 < m \leq m_2\} \subseteq \Gamma$.

(v). There exist $m_1, m_2, m_3 \in \mathbb{N}$ such that $m_1 < m_3 \leq m_2$, $\ell_0 = \oplus^{m_3} \ell_p$ and $\{\mathtt{G} \oplus^{m_1} \ell_p, \mathtt{F} \oplus^{m_2} \overline{\ell_p}\} \cup \{\oplus^m \overline{\ell_p} \mid m_1 < m \leq m_2 \text{ y } m \neq m_3\} \subseteq \Gamma$.

Theorem 7.6. In FNext and FNext± we have that Lit^+ and $Lit^\pm$, respectively, are strong groupables.

Proof. We will only prove the second affirmation, because the first one is similar. Firstly, if $(\omega]_{Lit^\pm} = \{\omega\}\downarrow_{Lit^\pm}$ then $\Phi_*^{Lit^\pm}(\omega) = \Phi_*^{Lit^\pm}(\Phi_2^{Lit^\pm}(\ell_1, \ell_2)\omega_1)$ where $\ell_1, \ell_2 \in \omega$ y $\omega_1 = \omega \setminus \{\ell_1, \ell_2\}$.
If there exist $\ell_1, \ell_2 \in \omega$ such that $\overline{\ell_1} \trianglelefteq \ell_2$ then

$$\Phi_*^{Lit^\pm}(\omega) = \Phi_*^{Lit^\pm}(\top\omega_1) = \Phi_*^{Lit^\pm}(\Phi_2^{Lit^\pm}(\ell_1, \ell_2)\omega_1)$$

where $\omega_1 = \omega \setminus \{\ell_1, \ell_2\}$. In other cases, as a consequence of Corollary 7.4, we can ensure that there exists a permutation of ω such that we can apply one of the following possibilities:

1. Because $\odot^m \ell_p \vee \mathtt{F} \odot^m \ell_p = \mathtt{F} \odot^{m-1} \ell_p$ we have that

$$\Phi_*^{Lit^\pm}(\odot^m \ell_p \mathtt{F} \odot^m \ell_p \omega_1) = \Phi_*^{Lit^\pm}(\mathtt{F}\odot^{m-1} \ell_p \omega_1) = \Phi_*^{Lit^\pm}(\Phi_2^{Lit^\pm}(\odot^m \ell_p, \mathtt{F}\odot^m \ell_p)\omega_1)$$

2. Because $(\mathtt{G} \odot^m \ell_p, \odot^{m+1}\overline{\ell_p}]_{Lit^\pm} = \{\mathtt{G} \odot^{m+1} \ell_p, \odot^{m+1}\overline{\ell_p}\}\downarrow_{Lit^\pm}$ we have that

$$\Phi_*^{Lit^\pm}(\mathtt{G} \odot^m \ell_p \odot^{m+1} \overline{\ell_p}\omega_1) = \Phi_*^{Lit^\pm}(\mathtt{G} \odot^{m+1} \ell_p \odot^{m+1} \overline{\ell_p}\omega_1)$$

Corollary 7.4 ensures that

$$(\mathtt{G} \odot^{m+1} \ell_p \odot^{m+1} \overline{\ell_p}\omega_1]_{Lit^\pm} = (\mathtt{G} \odot^{m+1} \ell_p\omega_1]_{Lit^\pm} \cup (\odot^{m+1}\overline{\ell_p}\omega_1]_{Lit^\pm}$$

and so that

$$\Phi_*^{Lit^\pm}(\mathtt{G} \odot^m \ell_p \odot^{m+1} \overline{\ell_p}\omega_1) = \Phi_*^{Lit^\pm}(\Phi_2^{Lit^\pm}(\mathtt{G} \odot^{m+1} \ell_p, \odot^{m+1}\overline{\ell_p})\, \omega_1)$$

3. Because $(\mathtt{G} \odot^m \ell_p, \mathtt{F} \odot^{m+1} \overline{\ell_p}]_{Lit^\pm} = \{\odot^{m+1}\ell_p, \mathtt{F} \odot^{m+1} \overline{\ell_p}\}\downarrow_{Lit^\pm}$ we have that

$$\begin{aligned}\Phi_*^{Lit^\pm}(\mathtt{G} \odot^m \ell_p \mathtt{F} \odot^{m+1} \overline{\ell_p}\omega_1) &= \Phi_*^{Lit^\pm}(\odot^{m+1}\ell_p \mathtt{F} \odot^{m+1} \overline{\ell_p}\, \omega_1)\\ &= \Phi_*^{Lit^\pm}(\Phi_2^{Lit^\pm}(\odot^{m+1}\ell_p, \mathtt{F} \odot^{m+1} \overline{\ell_p})\, \omega_1)\end{aligned}$$

4. By duality from cases 1, 2 and 3, we have that

$$\Phi_*^{Lit^\pm}(\odot^m \ell_p \mathtt{P} \odot^m \ell_p\omega_1) = \Phi_*^{Lit^\pm}(\Phi_2^{Lit^\pm}(\odot^m \ell_p, \mathtt{P} \odot^m \ell_p)\omega_1)$$

$$\Phi_*^{Lit^\pm}(\mathtt{H} \odot^m \ell_p \odot^{m-1} \overline{\ell_p}\omega_1) = \Phi_*^{Lit^\pm}(\Phi_2^{Lit^\pm}(\mathtt{H} \odot^{m-1} \ell_p, \odot^{m-1}\overline{\ell_p})\omega_1)$$

$$\Phi_*^{Lit^\pm}(\mathtt{H} \odot^m \ell_p \mathtt{P} \odot^{m-1} \overline{\ell_p}\omega_1) = \Phi_*^{Lit^\pm}(\Phi_2^{Lit^\pm}(\odot^{m-1}\ell_p, \mathtt{P} \odot^{m-1} \overline{\ell_p})\omega_1)$$

5. If $m_1 < m_2$, because $\mathrm{F} \odot^{m_1} \ell_p \vee \mathrm{P} \odot^{m_2} \ell_p = \mathrm{FP}\ell_p$ we have that

$$\Phi_*^{Lit^\pm}(\mathrm{F}\odot^{m_1}\ell_p\mathrm{P}\odot^{m_2}\ell_p\omega_1) = \Phi_*^{Lit^\pm}(\mathrm{FP}\ell_p\omega_1) = \Phi_*^{Lit^\pm}(\Phi_2^{Lit^\pm}(\mathrm{F}\odot^{m_1}\ell_p, \mathrm{P}\odot^{m_2}\ell_p)\ \omega_1)$$

We finish this work hinting at the importance of the groupability property in the calculation of the unitary implicants and implicates of these and other temporal logics. The set of unitary implicants of a formula (ideal restricted to the set of literals) can be infinite, and therefore, difficult to manage. For example,

$$\begin{aligned}(\mathrm{F} \oplus p \vee \mathrm{P} \ominus q]_{Lit\pm} = \ & \{\perp, \mathrm{GH}p, \mathrm{GF}p, \mathrm{FG}p, \mathrm{GH}q, \mathrm{PH}q, \mathrm{HP}q\} \cup \{\mathrm{G} \odot^n p, \mathrm{H} \odot^n q \mid n \in \mathbb{Z}\} \\ & \cup\{\odot^n p \mid n \geq 2\} \cup \{\mathrm{F} \odot^n p \mid n \geq 1\} \cup \{\mathrm{H} \odot^n p \mid n \geq 3\} \\ & \cup\{\odot^n q \mid n \leq -2\} \cup \{\mathrm{P} \odot^n q \mid n \leq -1\} \cup \{\mathrm{G} \odot^n q \mid n \leq -3\}\end{aligned}$$

The tools we have developed [7] to solve this problem are based on their substitution by the set of maximal elements (which we called base) This set is always finite in the logics we have worked with. The main problem is how to determine it, and in that regard, the algebraic multisemilattice structure in [23] and the groupability property play an important role. For instance: How do we know if a set Γ of unitary formulae is a base? Is it sufficient that Γ is an antichain, i.e. that its elements are not comparable two by two? Obviously it is not, as it is also necessary that $(\Gamma]_{Lit} = \Gamma{\downarrow}_{Lit}$. But how can we determine effectively if this condition is met without having to calculate $(\Gamma]_{Lit}$? The following proposition gives us the answer, that we back with an example.

Proposition 7.7. Let $(A, \leq)$ be a lattice, $B \subseteq A$ and $\Gamma \subseteq B$. If B is *groupable* then the following two conditions are equivalent:

(i). $(\Gamma]_{Lit} = \Gamma{\downarrow}_{Lit}$

(ii). for all $x, y \in \Gamma$ if $a \leq x \vee y$ and $a \in B$ then $a \leq x$ or $a \leq y$

For example, the following set is a base in FNext±.

$$\{\mathrm{P} \ominus p, \oplus p, \mathrm{F} \oplus p, \mathrm{HP}q, \ominus q, q, \oplus q, \oplus^3 q, \mathrm{G} \oplus^5 q\}$$

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