KYBERNETIKA — VOLUME 39 (2003), NUMBER 4, PAGES 461–481

NECESSARY AND SUFFICIENT CONDITIONS FOR STABILIZATION OF EXPANDING SYSTEMS SERVOMECHANISM PROBLEMS

DIBYENDU BAKSI, KANTI B. DATTA AND GOSHAIDAS RAY

The problem of designing realistic decentralized controller to solve a servomechanism problem in the framework of "large scale systems" is considered in this paper. As any large scale system is built by expanding construction of one subsystem being connected to the existing system. In particular, it is desired to find a local stabilizing controller in terms of a free parameter (belonging to the ring of proper stable transfer functions) so that desirable properties of the controlled system, such as tracking and/or disturbance rejection for any arbitrary deterministic signal along with stabilization of the expanded overall system occur. An algorithm for designing such a free controller parameter is presented. The necessary and sufficient conditions for the existence of solutions to the Expanding Systems with Tracking and/or Disturbance Rejection Problem are established here and characterized the corresponding full set of stabilizing controllers that solve the problem. A numerical example is presented to illustrate the design procedure of the proposed controller for the Expanded System.

Keywords: expanding system, stabilizing controller, disturbance rejection, tracking, stable rational functions

AMS Subject Classification: 93A15, 93D15, 93C95

NOTATION

- C_+ denotes the closed right half plane and C_{+e} denote the extended right half plane (i.e., $C_+ = C_+ U\{\infty\}$).
- R_p denotes the ring of proper rational functions.
- R_s denotes the ring of stable rational functions (i.e., those with no poles in C_+).
- R_{ps} denotes the ring of proper stable rational functions with real coefficients.
- $M(R_{ps}(s))$ denotes the set of matrices whose entries are in $R_{ps}(s)$
- A matrix $\mathcal{M} \in M(R_{ps}(s))$ is called $R_{ps}(s)$-unimodular iff $\mathcal{M}^{-1} \in M(R_{ps}(s))$.

1. INTRODUCTION

In spite of the active research carried out in decentralized control in the last three decades, there has been little attention paid to the controller design problem for decentralized systems to satisfy various constraints that are imposed on the performance of the closed-loop system. A systematic design method that solves the decentralized servomechanism problem using parametric optimization approach or a concurrent and/or sequentially stable synthesis approach is discussed in literature [2]. The methods of constructing connectively stable large scale systems, reported so far [5], are mostly those by state feedback or observer-based feedback. A new approach to the stabilization of connectively unstable expanding large scale systems using the proper stable factorization approach has drawn much attention in literature [13]. The most fundamental and significant result of the factorization approach is the parameterization of all centralized stabilizing controllers. This approach is adopted to each subsystem of an already operational interconnected system to define a local stabilizing controller with an unspecified parameter which can be tuned to make the composite system connectively stable [10].

In this paper, a decentralized servomechanism problem is considered, which is appropriate to the expanding construction of large scale systems. This problem is henceforth referred to as the Expanding Systems with Tracking and/or Disturbance Rejection Problem (*ESTR*) in general. As any large-scale system is built by expanding construction of one subsystem being connected to others, the local controller of the new subsystem has to be such that (i) it stabilizes the local subsystem, (ii) tracks and/or rejects arbitrary deterministic signals and (iii) stabilizes the overall system. The problem of finding conditions under which such local controllers exist, is called the existence of solution to *ESTR*. However, the existing literature

[11] - [12] provides only sufficient conditions for the local controllers that need to track and/or reject step signals and simultaneously stabilize the resultant expanding system. It is based on augmenting the local system with an integral compensator and therefore can only handle step functions as reference inputs or disturbances. Moreover, this augmented system is factorized to ascertain the existence of a local controller. Since the existing approach uses only sufficient conditions [10], it can only address a subset of the full class of compensators that solves *ESTR*.

The major contributions in this paper are, (a) establishment of necessary and sufficient conditions for the existence of solutions to *ESTR* and (b) parameterization of the set of local stabilizing controllers in terms of a free parameter belonging to the ring of proper and stable transfer matrices. The proposed method ensures the tracking and/or disturbance rejection properties for any arbitrary deterministic signals and also stabilizes of the expanded system. Naturally, because of generality on both counts, the full class of compensators solving *ESTR* is completely characterized.

The paper is organized in the following manner. In Section 2, the prerequisites of the expanding system problem are given. Section 3 contains the new results as it formulates to solve *ESTR* and establishes all the relevant theorems which solve this problem. In Section 4, a numerical example is considered to illustrate the effectiveness of the proposed controller for the *ESTR*. Finally, some concluding remarks are given in Section 5.

2. THE EXPANDING SYSTEM PROBLEM

Let us consider N number of subsystems

$\mathcal{S}_i$:

$$\begin{aligned} \dot{x}_i &= A_i x_i + B_i u_i + G_i v_i \\ y_i &= C_i x_i \\ w_i &= H_i x_i \qquad i = 1, 2, \ldots, N \end{aligned} \tag{1}$$

with local controllers described by

$\mathcal{LC}_i$:

$$\begin{aligned} \dot{z}_i &= M_i z_i + L_i y_i \\ u_i &= J_i z_i + K_i y_i \qquad i = 1, 2, \ldots, N \end{aligned} \tag{2}$$

Here, x_i is the state, u_i is the control input, y_i is the measured output, v_i is the interconnection input and w_i is the interconnection output of the i-th subsystem $\mathcal{S}_i$. Also, z_i is the state of the local controller $\mathcal{LC}_i$ and $A_i, B_i, C_i, H_i, G_i, F_{1i}, F_{2i}, F_{3i}, F_{4i}$, M_i, L_i, J_i and K_i are all constant matrices of appropriate dimensions. It is also assumed that (A_i, B_i) are stabilizable and (C_i, A_i) are detectable.

The closed loop subsystems are given by $\mathcal{S}_i^c$:

$$\begin{aligned} \begin{bmatrix} \dot{x}_i \\ \dot{z}_i \end{bmatrix} &= \begin{bmatrix} A_i + B_i K_i C_i & B_i J_i \\ L_i C_i & M_i \end{bmatrix} \begin{bmatrix} x_i \\ z_i \end{bmatrix} + \begin{bmatrix} G_i \\ 0 \end{bmatrix} v_i \\ w_i &= [H_i \quad 0] \begin{bmatrix} x_i \\ z_i \end{bmatrix} \qquad i = 1, 2, \ldots, N \end{aligned} \tag{3}$$

In the 'Expanding System Problem', there is already an interconnected system composed of $N-1$ subsystems, each of which is modelled as $\mathcal{S}_i^c$ above. The model of the interconnected system is given by

$\bar{\mathcal{S}}_{N-1}^c$:

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A + BKC + GEH & BJ \\ LC & M \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \tag{4}$$

where x, y, u, v, z are of the form $x = [x_1' \; x_2' \; \cdots x_{N-1}']'$ and A, B, C, G, H, J, K, L and M are all of the form $A = \text{block diag}\,[A_1, A_2, \ldots, A_{N-1}]$, $B = \text{block diag}\,[B_1, B_2, \ldots, B_{N-1}]$, $C = \text{block diag}\,[C_1, C_2, \ldots, C_{N-1}]$, $G = \text{block diag}\,[G_1, G_2, \ldots, G_{N-1}]$, $H = \text{block diag}\,[H_1, \; H_2, \; \ldots, H_{N-1}]$, $J = \text{block diag}\,[J_1, \; J_2, \; \ldots, J_{N-1}]$, $K = \text{block diag}\,[K_1, \; K_2, \; \ldots, K_{N-1}]$, $L = \text{block diag}\,[L_1, \; L_2, \; \ldots, L_{N-1}]$, and $M = \text{block diag}\,[M_1, \; M_2, \; \ldots, M_{N-1}]$. The matrix E represents the interconnection between v and w, i.e., $v = Ew$ where

$$E = \begin{bmatrix} 0 & E_{1,2} & E_{1,3} & \cdots & E_{1,N-1} \\ E_{2,1} & 0 & E_{2,3} & \cdots & E_{2,N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ E_{N-1,1} & E_{N-1,2} & E_{N-1,3} & \cdots & 0 \end{bmatrix} \tag{5}$$

with $v_i = \sum_{j=1,j\neq i}^{N-1} E_{i,j}w_j$.

Now a new subsystem $\mathcal{S}_N^c$ is connected to the already interconnected system $\bar{\mathcal{S}}_{N-1}^c$ and thereby we get the expanded system $\bar{\mathcal{S}}_N^c$.The problem is to design the controller for the new subsystem $\mathcal{S}_N$ so that both the closed loop new subsystem $\mathcal{S}_N^c$ as well as the expanded system $\bar{\mathcal{S}}_N^c$ become stable. Tan and Ikeda [10] found the most general result available to date regarding the condition for such decentralized stabilizability of expanding systems. Their condition is valid for any value of interconnection gains and does not depend on the pattern and/or strength of it. To prove their point, they used the factorization approach of controller design [3],[6]. Another remarkable feature is that all the conditions are in terms of the new subsystem's parameters only. The subsystem $\mathcal{S}_N$ can be viewed as a 2-input 2-output multi-variable system as follows.

$$\begin{bmatrix} w_N \\ y_N \end{bmatrix} = \begin{bmatrix} Z_{N,11} & Z_{N,12} \\ Z_{N,21} & Z_{N,22} \end{bmatrix} \begin{bmatrix} v_N \\ u_N \end{bmatrix} \tag{6}$$

where using the $[A, B, C, D]$ data structure notation of [4]

$$\begin{aligned} Z_{N,11} &= [A_N, G_N, H_N, 0] \\ Z_{N,12} &= [A_N, B_N, H_N, 0] \\ Z_{N,21} &= [A_N, G_N, C_N, 0] \\ Z_{N,22} &= [A_N, B_N, C_N, 0] \end{aligned} \tag{7}$$

To find the set of all stabilizing controllers, $Z_{N,22}$ can be factorized as

$$Z_{N,22} = N_N D_N^{-1} = \tilde{D}_N^{-1}\tilde{N}_N \tag{8}$$

where $N_N, D_N, \tilde{N}_N$ and $\tilde{D}_N \in M(\mathcal{R}_{ps}(s))$ so that they satisfy the *Bezout identity*

$$\begin{bmatrix} Q_N & P_N \\ -\tilde{N}_N & \tilde{D}_N \end{bmatrix} \begin{bmatrix} D_N & -\tilde{P}_N \\ N_N & \tilde{Q}_N \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{9}$$

for some P_N, Q_N, $\tilde{P}_N$ and $\tilde{Q}_N \in M(\mathcal{R}_{ps})$.

The set of all stabilizing controllers [13] for the subsystem $\mathcal{S}_N$ in (1) is given by

$$u_N = -(\tilde{P}_N + D_N R_N)(\tilde{Q}_N - N_N R_N)^{-1} y_N \tag{10}$$

where $R_N \in M(\mathcal{R}_{ps})$ is the free parameter. With a specified R_N, (2) becomes the time-domain realization of the controller. The closed loop N-th subsystem has the following transfer function matrix $T_N(R_N)$ from input v_N to output w_N, which can be seen to be affine in R_N.

$$T_N(R_N) = T_{N1} - T_{N2}R_N T_{N3} \tag{11}$$

where

$$\begin{aligned} T_{N1} &= Z_{N,11} - Z_{N,12}\tilde{P}_N\tilde{D}_N Z_{N,21} \\ T_{N2} &= Z_{N,12}D_N \\ T_{N3} &= \tilde{D}_N Z_{N,21} \end{aligned}$$

The feedback structure of an expanded system is shown in Figure 1 where $\bar{T}_{N-1}$ is the transfer matrix of $\bar{\mathcal{S}}^c_{N-1}$ from the interconnection input $\bar{v}_{N-1}$ to the interconnection output $\bar{w}_{N-1}$. The N-th subsystem $\mathcal{S}^c_N$ is connected to the existing system $\bar{\mathcal{S}}^c_{N-1}$ through

$$\begin{aligned} \bar{v}_{N-1} &= \bar{E}_{N-1,N} w_N \\ v_N &= \bar{E}_{N,N-1} \bar{w}_{N-1} \end{aligned} \tag{12}$$

where

$$\begin{aligned} \bar{E}_{N-1,N} &= [E'_{1,N}\ E'_{2,N}\ \cdots E'_{N-1,N}]' \\ \bar{E}_{N,N-1} &= [E_{N,1}\ E_{N,2}\ \cdots E_{N,N-1}] \end{aligned} \tag{13}$$

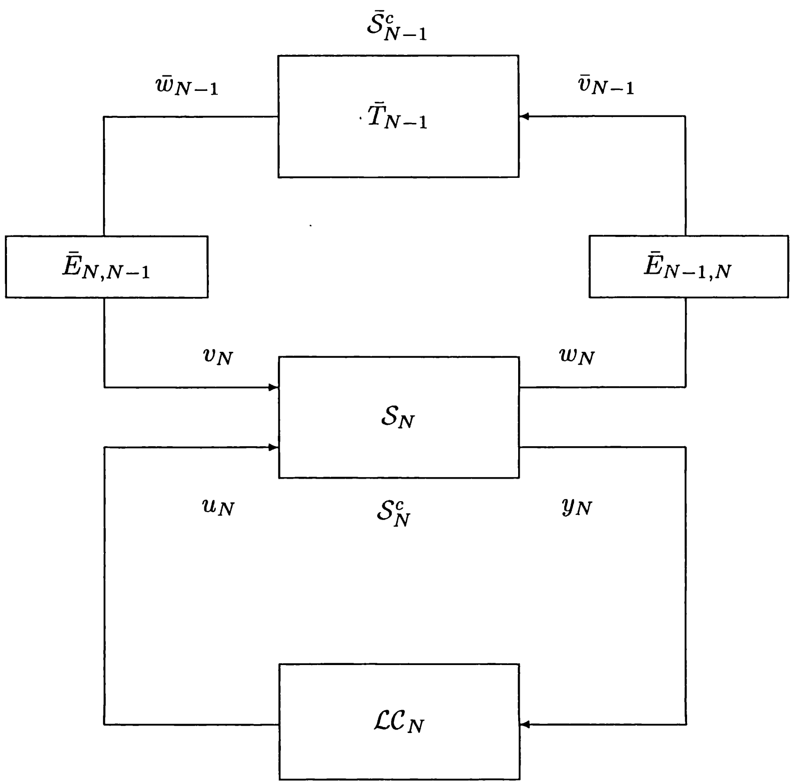

Fig. 1. Feedback structure of an expanded system.

The following theorem ensures the decentralized stabilizability of the expanding system problem which directly follows from the results given in [10].

Theorem 2.1. Let $\mathcal{LC}_N$ be the set of all stabilizing controllers for the new closed loop subsystem $\mathcal{S}_N^c$. A controller in $\mathcal{LC}_N$ will stabilize the expanded system $\bar{\mathcal{S}}_N^c$ iff the the following condition is valid.

$$\min_{R_N \in M(\mathcal{R}_{ps})} \delta(I - \bar{\psi}_{N-1}T_{N1} + \bar{\psi}_{N-1}T_{N2}R_N T_{N3}) = 0 \tag{14}$$

where $\bar{\psi}_{N-1} := \bar{E}_{N,N-1}\bar{T}_{N-1}\bar{E}_{N-1,N}$ and $\delta(t)$:= degree of $t \in M(\mathcal{R}_{ps}(s))$ = number of right half plane zeros of t including ∞. This degree function makes $M(\mathcal{R}_{ps}(s))$ a Proper Euclidean Domain.

Remark. For direct transmissions, the interconnection of state and input terms are included to the measured output y_i and the interconnection output w_i in equation (1) and the corresponding y_i, and w_i are then changed to $y_i = C_i x_i + F_{4i}u_i + F_{3i}v_i$ and in (3) $A_i + B_iK_iC_i, B_iJ_i, L_iC_i, M_i, G_i$ and 0 are replaced by $A_i + B_iK_i\Xi_iC_i$, $B_iJ_i + B_iK_i\Xi_iF_{4i}J_i$, $L_i\Xi_iC_i$, $M_i + L_i\Xi_iF_{4i}J_i$, $G_i + B_iK_i\Xi_iF_{3i}$, $L_i\Xi_iF_{3i}$ respectively, and in (7) $Z_{N,11}$, $Z_{N,12}$, $Z_{N,21}$ and $Z_{N,22}$ are modified to

$$\begin{aligned} Z_{N,11} &= [A_N, G_N, H_N, F_{1N}] \\ Z_{N,12} &= [A_N, B_N, H_N, F_{2N}] \\ Z_{N,21} &= [A_N, G_N, C_N, F_{3N}] \\ Z_{N,22} &= [A_N, B_N, C_N, F_{4N}] \end{aligned}$$

The expression for w_i in (3) is changed to $w_i = [H_i + F_{2i}K_i\Xi_iC_i]x_i + F_{2i}[J_i + K_i\Xi_iF_{4i}J_i]z_i + [F_{1i} + F_{2i}K_i\Xi_iF_{3i}]v_i$ where $\Xi_i = (I - F_{4i}K_i)^{-1}$ is assumed to exist, i.e., $(I - F_{4i}K_i)$ is nonsingular for well-posedness.The existence of a decentralized stabilizing controller for the expanding system problem in the multi-variable case can be checked using the theorem given below [1].

Theorem 2.2. Consider the fictitious plant $P = \bar{\psi}_{N-1}T_{N2}(I - \bar{\psi}_{N-1}T_{N1})^{-1}\ T_{N3}$ and define $\hat{B} := \bar{\psi}_{N-1}T_{N2}; \hat{A} := (I - \bar{\psi}_{N-1}T_{N1}); \hat{C} := T_{N3}$. Let F_1 be the greatest common right divisor of $(\hat{B}, \hat{A})$, i.e., $\hat{A} = \hat{A}_0F_1$ and $\hat{B} = BF_1$. Let F_2 be greatest common left divisor of $(\hat{A}_0, \hat{C})$, i.e., $\hat{A}_0 = F_2A$ and $\hat{C} = F_2C$. So, $(B, A, C, 0)$ is a bicoprime factorization [13]. Now consider a left and a right coprime factorization $(\tilde{A}_1, \tilde{B}C)$ and $(B\tilde{C}, \tilde{A}_2)$ respectively of $(B, A, C, 0)$. Let $\sigma_1, \sigma_2, \ldots, \sigma_l$ be the real non-negative distinct blocking zeros including ∞ of $B\tilde{C}$ (or, $\tilde{B}C$) arranged in the ascending order. Then the following statements are equivalent.

- $$\min_{R_N \in M(\mathcal{R}_{ps})} \delta(|I - \bar{\psi}_{N-1}T_{N1} + \bar{\psi}_{N-1}T_{N2}R_NT_{N3}|) = \delta(|F_1|) + \delta(|F_2|) \tag{15}$$

- The following real numbers have the same sign.

$$\{f(\sigma_1), f(\sigma_2), \ldots, f(\sigma_l)\}$$

where $f := |\tilde{A}_1|$ or $|\tilde{A}_2|$. Moreover, the stabilizing controller for N-th subsystem which also stabilizes the expanded system $\bar{\mathcal{S}}_N^c$ exists if and only if (i) $\delta(|F_1|) = \delta(|F_2|) = 0$ and (ii) the above sequence has the same sign.

3. THE PROBLEM (*ESTR*)

Suppose that there are N subsystems, each described by $\mathcal{S}_i$:

$$\begin{aligned} \dot{x}_i &= A_i x_i + B_i u_i + G_{1i} v_i + G_{2i} d_i + G_{3i} y_i^{ref} \\ y_i &= C_i x_i + F_{4i} u_i + F_{3i} v_i + G_{2i}^{y} d_i + G_{3i}^{y} y_i^{ref} \\ y_i^m &= C_i^m x_i + F_{4i}^m u_i + F_{3i}^m v_i + G_{2i}^m d_i + G_{3i}^m y_i^{ref} \\ w_i &= H_i x_i + F_{2i} u_i + F_{1i} v_i + G_{2i}^w d_i + G_{3i}^w y_i^{ref} \\ & \qquad i = 1, 2, \ldots, N \end{aligned} \tag{16}$$

It is to be noted that the above model is a very general one having direct feed forward terms whenever appropriate and therefore covers a broad class of systems.

Problem ESTR (Expanding Systems with Tracking and Disturbance Rejection):

Find the condition under which there exists a local stabilizing controller for the new subsystem which would locally track and reject any signal generated by the reference generators T and R, and at the same time stabilize the overall expanded system.

Our philosophy of attacking this problem is not of augmentation type but is based on the factorization approach. First, the local stability problems with tracking and/or disturbance rejection are solved using the theory advanced by Saeks and Murray [8]. Next, full use is made of the affine nature of the resulting closed loop feedback gain by utilizing the free parameter to achieve overall stability of the expanded system, if possible.

For the purpose of stabilization, d_N, y_N^{ref} and y_N are ignored for the time being. The transfer function matrix representation of $\mathcal{S}_N$ is as follows :

$$\begin{bmatrix} w_N \\ \hat{y}_N^m \end{bmatrix} = \begin{bmatrix} Z_{N,11} & Z_{N,12} \\ Z_{N,21} & Z_{N,22} \end{bmatrix} \begin{bmatrix} v_N \\ u_N \end{bmatrix} \tag{17}$$

where

$$\begin{aligned} Z_{N,11} &:= [\hat{A}_N,\ \hat{G}_N,\ \hat{H}_N,\ F_{1N}] \\ Z_{N,12} &:= [\hat{A}_N,\ \hat{B}_N,\ \hat{H}_N,\ F_{2N}] \\ Z_{N,21} &:= [\hat{A}_N,\ \hat{G}_N,\ \hat{C}_N^m,\ F_{3N}] \\ Z_{N,22} &:= [\hat{A}_N,\ \hat{B}_N,\ \hat{C}_N^m,\ F_{4N}] \end{aligned} \tag{18}$$

with

$$\begin{aligned} \hat{A}_N &:= \begin{bmatrix} A_N & 0 \\ C_N & 0 \end{bmatrix} \\ \hat{B}_N &:= \begin{bmatrix} B_N \\ 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \hat{C}_N^m &:= \begin{bmatrix} C_N^m & 0 \\ 0 & I \end{bmatrix} \\ \hat{G}_N &:= \begin{bmatrix} G_N \\ 0 \end{bmatrix} \\ \hat{H}_N &:= [H_N \quad 0] \end{aligned} \tag{19}$$

The same notations $w_N, \hat{y}_N^m, v_N$ and u_N are used in the s-domain as well as the time domain.The feedback structure of an expanded system is shown in Figure 1 where the N-th subsystem $\mathcal{S}_N$ is connected to the existing system $\bar{\mathcal{S}}_{N-1}^c$ through the equations (12) and (13) where $Z_{N,22}$ is factorized as

$$Z_{N,22} = p_r \tilde{p}_r^{-1} = \tilde{p}_l^{-1} p_l \tag{20}$$

such that

$$\begin{bmatrix} \tilde{q}_r & q_r \\ -p_l & \tilde{p}_l \end{bmatrix}^{-1} = \begin{bmatrix} \tilde{p}_r & -q_l \\ p_r & \tilde{q}_l \end{bmatrix} \tag{21}$$

The feedback structure of an expanded system with simultaneous local tracking and disturbance rejection is shown in Figure 2.

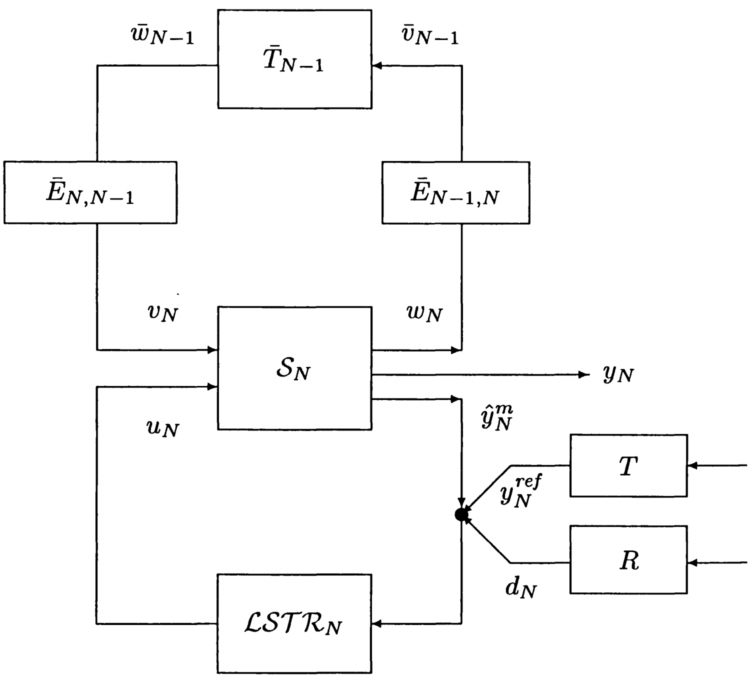

Fig. 2. *ESTR* Problem.

The set of all stabilizing controllers that solve the local tracking and disturbance rejection problem is

$$K^{tr} = \{[\tilde{t}\hat{t}(q_r m_r) - \tilde{r}\hat{r}(j_l \tilde{q}_l) + \tilde{a}_r d_N \tilde{e}_l]p_l + \tilde{q}_r\}^{-1}\{-[\tilde{t}\hat{t}(q_r m_r) - \tilde{r}\hat{r}(j_l \tilde{q}_l) + \tilde{a}_r d_N \tilde{e}_l]\tilde{p}_l + q_r\} \tag{22}$$

where $j_l, \tilde{e}_l, m_r, \tilde{a}_r, \hat{t}$ and $\hat{r}$ are defined in [9] and $d_N \in M(\mathcal{R}_{ps})$ is arbitrary. The transfer matrix from the interconnection input v_N to the interconnection output w_N is given by

$$T_N^{tr}(d_N) = T_{N1}^{tr} - T_{N2}^{tr} d_N T_{N3}^{tr} \tag{23}$$

where

$$\begin{aligned} T_{N1}^{tr} &= Z_{N,11} - Z_{N,12}(q_l - \tilde{p}_r \tilde{t}\hat{t} q_r m_r - \tilde{p}_r \tilde{r}\hat{r} j_l \tilde{q}_l)\tilde{p}_l Z_{N,21} \\ T_{N2}^{tr} &= Z_{N,12}\tilde{p}_r \tilde{a}_r \\ T_{N3}^{tr} &= \tilde{e}_l \tilde{p}_l Z_{N,21} \end{aligned}$$

Before presenting the main new results for the tracking and disturbance rejection problem *ESTR* established in this paper, the following proposition is stated. Intuitively, it shows the problem is equivalent to strong stabilization of a fictitious plant.

Proposition 3.1. Let $\mathcal{LSTR}_N$ be the set of all stabilizing controllers with guaranteed local tracking and disturbance rejection properties for the new closed loop subsystem $\mathcal{S}_N^c$. A controller in $\mathcal{LSTR}_N$ will stabilize the expanded system $\bar{\mathcal{S}}_N^c$ iff the the following condition is valid.

$$\min_{d_N \in M(\mathcal{R}_{ps})} \delta(I - \bar{\psi}_{N-1} T_{N1}^{tr} + \bar{\psi}_{N-1} T_{N2}^{tr} d_N T_{N3}^{tr}) = 0 \tag{24}$$

where

$$\bar{\psi}_{N-1} := \bar{E}_{N,N-1}\bar{T}_{N-1}\bar{E}_{N-1,N}.$$

and

$\delta(t)$:= degree of $t \in M(\mathcal{R}_{ps})$ = number of right half plane zeros of t including ∞. This degree function makes $M(\mathcal{R}_{ps})$ a Proper Euclidean Domain.

The main result is stated below:

Theorem 3.1. Consider the fictitious plant $P^{tr} = \bar{\psi}_{N-1} T_{N2}^{tr}(I - \bar{\psi}_{N-1} T_{N1}^{tr})^{-1}\, T_{N3}^{tr}$ and define $\hat{B} := \bar{\psi}_{N-1} T_{N2}^{tr}; \hat{A} := (I - \bar{\psi}_{N-1} T_{N1}^{tr}); \hat{C} := T_{N3}^{tr}$. Let F_1 be the greatest common right divisor of $(\hat{B}, \hat{A})$, i.e., $\hat{A} = \hat{A}_0 F_1$ and $\hat{B} = BF_1$. Let F_2 be greatest common left divisor of $(\hat{A}_0, \hat{C})$, i.e., $\hat{A}_0 = F_2 A$ and $\hat{C} = F_2 C$. So, $(B, A, C, 0)$ is a bicoprime factorization [13] of the plant P^{tr}. Now consider a left and a right coprime factorization $(\tilde{A}_1, \tilde{BC})$ and $(B\tilde{C}, \tilde{A}_2)$ respectively of $(B, A, C, 0)$. Let $\sigma_1, \sigma_2, \ldots, \sigma_l$ be the real non-negative distinct blocking zeros including ∞ of $B\tilde{C}$ (or, $\tilde{BC}$) arranged in the ascending order. Then the following statements are equivalent.

- $$\min_{d_N \in M(\mathcal{R}_{ps})} \delta(|I - \bar{\psi}_{N-1} T_{N1}^{tr} + \bar{\psi}_{N-1} T_{N2}^{tr} d_N T_{N3}^{tr}|) = \delta(|F_1|) + \delta(|F_2|) \tag{25}$$

- The following real numbers have the same sign.

$$\{f^{tr}(\sigma_1), f^{tr}(\sigma_2), \ldots, f^{tr}(\sigma_l)\}$$

where $f^{tr} := |\tilde{A}_1|$ or $|\tilde{A}_2|$. Moreover, the local stabilizing controller for the N-th subsystem with guaranteed tracking and disturbance rejection properties, which also stabilizes the expanded system $\bar{S}_N^c$, exists if and only if (i) $\delta(|F_1|) = \delta(|F_2|) = 0$ and (ii) the above sequence has the same sign.

The proof of the theorem follows directly from the following Theorem 3.2 proved below.

First, a few lemmas are first presented. The proof of the main Theorem utilizes all these results. These lemmas are believed to be of independent interest and can be used to solve other problems as well.

Lemma 3.1. Consider the following matrices $C^{n \times p}$, $R^{p \times m}$, $B^{m \times n}$, $A^{n \times n} \in M(R_{ps})$. The following identitiy holds.

$$a^p \; |aI_n - CRBA^{adj}| = a^n \; |aI_p - RBA^{adj}C|$$

where $a = |A|$.

Proof. Consider the matrix multiplication

$$\begin{bmatrix} I_n & -C \\ 0 & I_p \end{bmatrix} \begin{bmatrix} aI_n & C \\ RBA^{adj} & I_p \end{bmatrix} = \begin{bmatrix} aI_n - CRBA^{adj} & 0 \\ RBA^{adj} & I_p \end{bmatrix}$$

Taking determinants on both sides we have

$$\begin{vmatrix} aI_p & C \\ RBA^{adj} & I_p \end{vmatrix} = |aI_n - CRBA^{adj}|$$

Again consider,

$$\begin{bmatrix} I_n & 0 \\ -RBA^{adj} & aI_p \end{bmatrix} \begin{bmatrix} aI_n & C \\ RBA^{adj} & I_p \end{bmatrix} = \begin{bmatrix} aI_n & C \\ 0 & aI_p - RBA^{adj}C \end{bmatrix}$$

Taking determinants on both sides

$$a^p|aI_n - CRBA^{adj}| = a^n|aI_p - RBA^{adj}C|$$

□

Lemma 3.2. Consider the matrices $A^{n\times n}$, $C^{n\times p}$, $R^{p\times m}$ and $B^{m\times n} \in M(R_{ps})$. If (A, C) is left-coprime and (B, A) is right-coprime then there exist appropriate square matrices $A_1^{n\times n}, C_1^{n\times n}, R_1^{n\times n}$ and $B_1^{n\times n} \in M(R_{ps})$ such that $|A + CRB| = |A_1 + C_1R_1B_1|$, (A_1, C_1) is left-coprime and (B_1, A_1) is right-coprime.

Proof. The proof is divided into two parts. First, the cases of $p > n$ and $p < n$ are shown to be equivalent to $p = n$.

- Suppose $p > n$, then there exist unimodular matrices U and V in R_{ps} such that

$$UCV = [\overset{n}{C'} \quad \overset{p-n}{0}]\; n$$

As (A, C) is left-coprime

$$\begin{aligned} AX + CY &= I \\ UAX + UCVV^{-1}Y &= U \\ UAU^{-1}UX + [\overset{n}{C'} \quad \overset{p-n}{0}]\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} &= U \\ A'X_1 + C'Y_1 &= U \end{aligned}$$

where $A' := UAU^{-1}$, $X_1 := UX$ and $V^{-1}Y := \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$.

So, (A', C') is left-coprime. Hence,

$$\begin{aligned} |A + CRB| &= |UAU^{-1} + UCVV^{-1}RBU^{-1}| \\ &= |A' + [C' \quad 0]\begin{bmatrix} R' \\ R'' \end{bmatrix} B'| \\ &= |A' + C'R'B'| \end{aligned}$$

where $V^{-1}R$ is partitioned as $\begin{bmatrix} R' \\ R'' \end{bmatrix}$ and $B' := BU^{-1}$.

- Suppose $p < n$. As (A, C) is left coprime

$$\begin{aligned} AX + CY &= I \\ AX + [\overset{p}{C} \quad \overset{n-p}{0}]\begin{bmatrix} Y \\ 0 \end{bmatrix} &= I \\ A'X + C'Y' &= I \end{aligned}$$

where $C' := [\overset{p}{C} \quad \overset{n-p}{0}]$, $Y' := \begin{bmatrix} Y \\ 0 \end{bmatrix}$ and $A' := A$. So, (A', C') is left-coprime.

$$\begin{aligned} |A + CRB| &= |A' + [C \quad 0]\begin{bmatrix} R \\ 0 \end{bmatrix} B| \\ &= |A' + C'R'B'| \end{aligned}$$

where $C' := [C \quad 0]$, $R' := \begin{bmatrix} R \\ 0 \end{bmatrix}$ and $B' := B$.

Now, it is shown that the cases $m > n$ and $m < n$ are equivalent to $m = n$.

- Suppose $m > n$, then there exist unimodular matrices U and V in R_{ps} such that

$$UB'V = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \begin{array}{c} n \\ n-m \end{array}$$

As (B', A') is right-coprime

$$\begin{aligned} XA' + YB' &= I \\ XA'V + YB'V &= V \\ XA'V + YU^{-1}UB'V &= V \\ XA'V + YU^{-1}\begin{bmatrix} B_1 \\ 0 \end{bmatrix} &= V \\ XA'V + [Y_1 \quad Y_2]\begin{bmatrix} B_1 \\ 0 \end{bmatrix} &= V \\ XVV^{-1}A'V + Y_1B_1 &= V \\ X_1A_1 + Y_1B_1 &= V \end{aligned}$$

where $A_1 := V^{-1}A'V$, $X_1 := XV$ and YU^{-1} is partitioned as $[Y_1 \quad Y_2]$. Now,

$$\begin{aligned} |A + CRB| &= |A' + C'R'B'| \\ &= |V^{-1}A'V + V^{-1}C'R'B'V| \\ &= |A_1 + C_1R'U^{-1}UB'V| \\ &= |A_1 + C_1[R_1 \quad R_2]\begin{bmatrix} B_1 \\ 0 \end{bmatrix}| \\ &= |A_1 + C_1R_1B_1| \end{aligned}$$

where $C_1 := V^{-1}C'$ and $R'U^{-1}$ is partitioned as $[R_1 \quad R_2]$.

- Suppose $m < n$. As (B', A') is right-coprime,

$$\begin{aligned} XA' + YB' &= I \\ XA' + [Y \quad 0]\begin{bmatrix} B' \\ 0 \end{bmatrix} &= I \\ XA' + Y_1B_1 &= I \end{aligned}$$

where $Y_1 := [Y \quad 0]$ and $B_1 := \begin{bmatrix} B' \\ 0 \end{bmatrix}$. Now,

$$\begin{aligned} |A + CRB| &= |A' + C'R'B'| \\ &= |A' + C'[R' \quad 0]\begin{bmatrix} B' \\ 0 \end{bmatrix}| \\ &= |A' + C'R_1B_1| \\ &= |A_1 + C_1R_1B_1| \end{aligned}$$

where $A_1 := A'$, $C_1 := C'$, $R_1 := [R' \quad 0]$ and $B_1 := \begin{bmatrix} B' \\ 0 \end{bmatrix}$

□

Remark. Henceforth, only the case (C, A, B) all square is considered in all the results established below. This is due to the fact that the cases of $m \neq n$ and $p \neq n$ have been shown to be equivalent to $m = p = n$.

Lemma 3.3. Suppose $A^{n\times n}$, $C^{n\times n}$ and $B^{n\times n} \in M(R_{ps})$ are given such that (B, A) is right coprime and (A, C) is left coprime. In other words, (B, A, C) is a bicoprime factorization. Then there exists an $R \in R_{ps}$ such that $|A + CRB| \neq 0$.

Proof. If $|A| \neq 0$, let $R = 0$. Otherwise, proceed as follows. For any arbitrary unimodular matrices U and $V \in M(R_{ps})$ we have,

$$\begin{aligned} |A + CRB| &= |U||A + CRB||U^{-1}| \\ &= |UAU^{-1} + UCRBU^{-1}| \\ &= |UAU^{-1} + UCVV^{-1}RBU^{-1}| \\ &= |A_1 + C_1R_1B_1| \end{aligned}$$

where $A_1 := UAU^{-1}, C_1 := UCV, R_1 := V^{-1}R$ and $B_1 := BU^{-1}$. Since C and A are left-coprime, C_1 and A_1 are also left-coprime. In other words, for X and $Y \in M(R_{ps})$,

$$\begin{aligned} AX + CY &= I \\ &\Rightarrow UAU^{-1}UX + UCVV^{-1}Y = U \\ &\Rightarrow A_1X_1 + C_1Y_1 = U \end{aligned}$$

with $X_1 := UX, Y_1 := V^{-1}Y$ and U, a unimodular matrix in $M(R_{ps})$, i.e., $\exists X_2, Y_2 \in M(R_{ps})$ that satisfy the Bezout identity: $A_1X_2 + C_1Y_2 = I$ where $X_2 = X_1U^{-1}$ and $Y_2 = Y_1U^{-1}$. Using similar arguments, it can be shown that A_1 and B_1 are right-coprime if A and B are so. The unimodular matrices U and $V \in M(R_{ps})$ can now be chosen in such a way that $C \in R_{ps}^{n\times n}$ is converted to its Smith form C_1 :

$$C_1 := UCV$$

where $C_1 := \text{diag}(c_1, c_2, \ldots, c_k, c_{k+1}, \ldots, c_n)$ with elements c_{k+1} to c_n all zeros. Note that k is the rank of C alone. Since C_1 and A_1 are left-coprime, the matrix $[A_1 \quad C_1] \in R_{ps}^{n\times 2n}(s)$ has full row rank which implies that the $k+1$-st to n-th rows of A_1 are independent. Suppose that among the rows 1 to k of A_1, i_1 to i_r rows are dependent. Again, due to right-coprimeness of A_1 and B_1, the matrix $\begin{bmatrix} A_1 \\ B_1 \end{bmatrix}$ has full rank. As i_1 to i_r rows of A_1 are dependent, the minor of order n obtained by excluding i_1 to i_r rows of A_1 and including j_1 to j_r rows of B_1 is nonzero. Then, using Binet–Cauchy formula it can be shown that $|A_1 + C_1R_1B_1| \neq 0$ for R_1

with elements $r_{i_x,j_x} = 1$ where $x = 1$ to r and $r_{i,j} = 0$ for all other i's and j's. So, $|A_1 + C_1R_1B_1| = c_{i_1}, \ldots, c_{i_r}t$ where t is the determinant with all but $i_1, \ldots, i_r$ numbered rows of A and $j_1, \ldots, j_r$ numbered rows of B. So, $|A_1 + C_1R_1B_1| = |A + CRB|$, where $R = VR_1$. □

Lemma 3.4. Suppose $P \in M(R(s))$ and (B, A, C) is a bicoprime factorization, i.e., (i) $|A| \neq 0$ and $P = BA^{-1}C$, (ii) (B, A) is a right coprime factorization and (iii) (A, C) is left coprime factorization. Consider any left and right coprime factorizations $(\tilde{A}_1, \tilde{B})$ and $(\tilde{C}, \tilde{A}_2)$ of (B, A) and (A, C) respectively. Then $(B\tilde{C}, \tilde{A}_2)$ is right coprime and $(\tilde{A}_1, \tilde{B}C)$ is left coprime.

Proof.
$$P = BA^{-1}C = \tilde{A}_1^{-1}\tilde{B}C$$

Since (B, A) is right coprime and $(\tilde{A}_1, \tilde{B})$ is left coprime $\exists X, Y, \tilde{X}, \tilde{Y} \in M(R_{ps})$ such that

$$\begin{bmatrix} \tilde{A}_1 & \tilde{B} \\ \tilde{Y} & -\tilde{X} \end{bmatrix} \begin{bmatrix} X & -B \\ Y & A \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

or,

$$\begin{aligned} \tilde{A}_1X + \tilde{B}Y &= I \\ \tilde{A}_1B &= \tilde{B}A \\ \tilde{Y}B + \tilde{X}A &= -I \\ \tilde{Y}X &= \tilde{X}Y \end{aligned}$$

Since (A, C) is left coprime, $\exists X_1, Y_1 \in M(R_{ps})$

$$AX_1 + CY_1 = I$$

Now the problem is to find $X_2, Y_2 \in M(R_{ps})$ so that

$$\tilde{B}CX_2 + \tilde{A}_1Y_2 = I$$

i.e., the pair $(\tilde{A}_1, \tilde{B}C)$ is left coprime.

Define

$$\begin{aligned} X_2 &:= -Y_1Y \\ Y_2 &:= -X - BX_1Y \end{aligned}$$

So,

$$\begin{aligned} \tilde{B}CX_2 + \tilde{A}_1Y_2 & \\ &= -\tilde{B}CY_1Y - \tilde{A}_1X - \tilde{A}_1BX_1Y \\ &= -\tilde{B}Y + \tilde{B}AX_1Y - \tilde{A}_1X - \tilde{A}_1BX_1Y \\ &= -\tilde{B}Y - \tilde{A}_1X + (\tilde{B}A - \tilde{A}_1B)X_1Y \\ &= -I \end{aligned}$$

which implies that $(\tilde{A}_1, \tilde{B}C)$ is left coprime. □

Theorem 3.2. Suppose $(B, A, C) \in \mathcal{R}_{ps}^{n\times n}$ is a bicoprime factorization of $P = BA^{-1}C \in M(R(s))$. In other words, (i) $|A| \neq 0$ and $P = BA^{-1}C$, (ii) (B, A) is a right coprime factorization and (iii) (A, C) is a left coprime factorization. Consider any left and right coprime factorizations $(\tilde{A_1}, \tilde{B})$ and $(\tilde{C}, \tilde{A_2})$ of (B, A) and (A, C) respectively. Let 'a'$= |A|$; $\tilde{z_1}$ and z_1 be the smallest invariant factors of $\tilde{B}C$ and $B\tilde{C}$ respectively. Then, the sets

1. $$\min_{R\in M(\mathcal{R}_{ps})} \delta(|A + CRB|),$$

2. $$\min_{R\in M(\mathcal{R}_{ps})} \delta(|\tilde{A_2} + RB\tilde{C}|),$$

3. $$\min_{R\in M(\mathcal{R}_{ps})} \delta(|\tilde{A_1} + \tilde{B}CR|),$$

4. $$\min_{r\in\mathcal{R}_{ps}} \delta(a + rz_1) =: I(a, z_1),$$

5. $$\min_{r\in\mathcal{R}_{ps}} \delta(a + \tilde{z_1}r) =: I(a, \tilde{z_1})$$

are equal where $I(a, \tilde{z_1})$ denotes the number of sign changes in the sequence $\{a(\sigma_1),\ a(\sigma_2),\ \ldots,\ a(\sigma_n)\}$; $\{\sigma_1,\ \sigma_2, \ldots,\ \sigma_n\ \}$ being the distinct nonnegative real zeros of 'z_1' or '$\tilde{z_1}$' in ascending order including ∞.

Proof. From Theorem 4.4.1 in [13], statements 2 and 4 are known to be equivalent. Similarly, statements 3 and 5 are also equivalent. Only the equivalency of statements 1, 2 and 3 is left to be proved. The statements 1 and 3 are shown to be equivalent below.

$$\begin{aligned}
|A + CRB| &= |A||I + A^{-1}CRB| \\
&= |A||I + BA^{-1}CR| \\
&= |A||I + \tilde{A_1}^{-1}\tilde{B}CR| \\
&= |A||\tilde{A_1}^{-1}||\tilde{A_1} + \tilde{B}CR| \\
&= |\tilde{A_1} + \tilde{B}CR|
\end{aligned}$$

Since $|A| \sim |\tilde{A_1}|$ [13], without lack of generality, the equivalency is assumed to be an equality. Following the same procedure, statements 1 and 2 can also be shown to be equivalent. This in turn implies equivalency of statements 2 and 3. □

The constructive procedure for finding the free parameter R is given below:

Algorithm for designing the free parameter R in the multi-variable case

1. Find $z_1, \tilde{z_1}$ (smallest invariant factors) and a (characteristic determinant) from $B\tilde{C}$, $\tilde{B}C$ and A respectively, where $B, C, \tilde{B}, \tilde{C}$ and A are as defined in Theorem 4.1 of [1].

2. Find an r such that $a + rz_1$ or $a + \tilde{z_1}r$ is unimodular in R_{ps} using the interpolation algorithm given in [13, 14].

3. Convert $\tilde{A}_1{}^{adj}\tilde{B}C$ or $B\tilde{C}\tilde{A}_2{}^{adj}$ to its Smith form, i.e.,

$$X := U\tilde{A}_1{}^{adj}\tilde{B}CV = \operatorname{diag}(x_1, x_2, \ldots, x_k, 0, \ldots, 0)$$

4. Find $q_1, q_2, \ldots, q_k \in R_{ps}$ such that

$$q_1x_1a^{k-1} + q_2x_1x_2a^{k-2} + \cdots + q_kx_1\cdots x_k = a^{k-1}\tilde{z}_1r$$

5. Define the bordered companion matrix

$$R_0 := \begin{bmatrix} q_1 & q_2 & \cdots & q_k & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix} \in R_{ps}^{p\times m}(s)$$

This procedure is based on the proof of equivalency of statements 3 and 5 given in Section 4.4.4 of [13].

6. $R = V^{-1}R_0$.

The existence of a decentralized stabilizing controller as stated in Theorem 3.1 for the expanding system problem with simultaneous tracking and disturbance rejection *ESTR* in the multi-variable case can be easily checked using the theorem given above. Proof of the afore-mentioned main Theorem 3.1 follows directly from the one proved above.

4. EXAMPLE

Example: Simultaneous Tracking and Disturbance Rejection

An example is given below which illustrates the application of proposed Theorem as also the construction procedure to find the global stabilizing controller for the expanded system. The first open loop subsystem S_1 is described by S_1:

$$\begin{aligned} \dot{x}_1 &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_1 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v_1 \\ y_1 &= [1 \quad 2]\, x_1 \\ w_1 &= [1 \quad 1]\, x_1 \end{aligned} \tag{26}$$

As this is the only system to be stabilized, it is not required to satisfy any of the established conditions. Applying static output feedback $u_1 = -y_1$, we obtain the closed loop stable system $\bar{S}_1^c = S_1^c$:

$$\begin{aligned} \dot{x}_1 &= \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ w_1 &= [1 \quad 2]x_1 \end{aligned} \tag{27}$$

Now we connect the second subsystem given by
S_2 :

$$
\begin{aligned}
\dot{x}_2 &= \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_2 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v_2 \\
y_2 &= [-1 \quad 1] x_2 \\
w_2 &= [-1 \quad 1] x_2
\end{aligned} \tag{28}
$$

with interconnection

$$
\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{29}
$$

The transfer functions $Z_{2,11}, Z_{2,12}, Z_{2,21}$ and $Z_{2,22}$ are computed as

$$
Z_{2,11} = Z_{2,12} = Z_{2,21} = Z_{2,22} = \frac{s-1}{s^2+s+1}
$$

It is required to track e^{2t} and simultaneously reject a step disturbance. Results of factorization computations are given below :

$$
Z_{2,22} = \frac{(s-1)}{(s^2+s+1)} = p, \ \tilde{p} = 1, \ q = 0, \ \tilde{q} = 1
$$

$$
T = \frac{1}{(s-2)}, \ t = \frac{1}{(s+2)}, \ \tilde{t} = \frac{(s-2)}{(s+2)}, \ u = 4, \tilde{u} = 1
$$

$$
R = \frac{1}{s}, \ r = \frac{1}{(s+2)}, \ \tilde{r} = \frac{s}{(s+2)}
$$

$$
\varrho = \frac{4(s+1)}{(s+2)}, \ \tilde{\varrho} = \frac{s}{(s+2)}
$$

For expanding systems with tracking (EST), $\tilde{p}$ and $\tilde{t}$ must be coprime, i.e.,

$$
jp + \tilde{j}\tilde{p} = 1
$$

with

$$
j = \frac{(9s^2+9s+9)}{(s+1)^2}
$$

and

$$
\tilde{j} = \frac{(s^2-3s-10)}{(s+1)^2}
$$

Also,

$$
E := \tilde{p}\tilde{t}^{-1} = e\tilde{e}^{-1} = \frac{(s+2)}{(s-2)}
$$

i.e.,

$$
e = 1
$$

$$\tilde{e} = (\frac{s-2)}{(s+2)}$$

For expanding systems with disturbance rejection (*ESR*), $\tilde{p}$ and $\tilde{r}$ had to be coprime, i.e.,

$$m\tilde{p} + \tilde{m}\tilde{r} = 1$$

with

$$m = 1,\ \tilde{m} = 0$$

Also,

$$A = p\tilde{r}^{-1} = a\tilde{a}^{-1}$$

i.e.,

$$a = \frac{(s-1)}{(s^2+s+1)}$$

and

$$\tilde{a} = \frac{s}{(s+2)}$$

Now, for Expanding systems with tracking and disturbance rejection (*STR*), the additional solvability condition has to be checked, which is, $\tilde{r}$ and $\tilde{t}$ must be coprime, i.e.,

$$\hat{r}\tilde{r} + \hat{t}\tilde{t} = 1$$

is solvable in $\mathcal{R}_{ps}$. As,

$$\hat{r} = 2$$

and

$$\hat{t} = -1$$

is such a solution, *STR* problem can be solved. The expression for the free design parameter becomes,

$$w = \left[\frac{-2s}{(s+2)}\frac{(9s^2+9s+1)}{(s+1)^2}\right] + \left[\frac{s}{(s+2)}\frac{(s-2)}{(s+2)}d_2\right] \tag{30}$$

The gain from v_2 to w_2 is

$$T_2^{tr} = \left[\frac{(s-1)(s+2)(s+1)^2 + 18s(s-1)^2}{(s+1)^2(s+2)(s^2+s+2)}\right] - \left[\frac{s(s-2)(s-1)^2}{(s^2+s+1)(s+2)^2}d_2\right] \tag{31}$$

To check solvability of *ESTR*, define

$$f^{tr} = 1 - \bar{\psi}_1 T_{21}^{tr}$$

$$g^{tr} = \bar{\psi}_1 T_{22}^{tr} T_{23}^{tr}$$

$$\begin{aligned} f^{tr} &= 1 - \left[\frac{(s-1)(s+1)^2(s+2) + 18s(s-1)^2}{(s+1)^3(s+2)(s^2+s+1)}\right] \\ g^{tr} &= \frac{s(s-2)(s-1)^2}{(s+1)(s+2)^2(s^2+s+1)^2} \end{aligned} \tag{32}$$

It can be seen that g has zeros at $\{0, 1, 2, \infty\}$. The sequence $\{f^{tr}(0), f^{tr}(1)\}$, $\{f^{tr}(2), f^{tr}(\infty)\}$ is $\{2, 1, \frac{19}{21}, 1\}$, and has no sign change, confirming solvability of *ESTR* problem. To design the controller, the first step is that of finding a d_2 such that

$$\min_{d_N \in M(\mathcal{R}_{ps})} \delta(f^{tr} + g^{tr} d_2) = 0$$

Using interpolation algorithm, the interpolating unit

$$(f^{tr} + g^{tr} d_N) := \mathsf{Unit}$$

is computed and shown below.

Simultaneous tracking and disturbance rejection:

First term of the unit $(f^{tr} + g^{tr}\, d_N) :=$ Unit1

$$= \frac{(1.234568E-02)(9s^4 + 36s^3 + 61s^2 + 22s + 16)(3s^3 + 11s^2 + 5s + 5)\,(3s^2 + 4s + 5)(s^5 + 5s^4 + 10s^3 + 9.405s^2 + 6.19s + 0.405)}{(s+1)(s+1)(s+1)(s+1)(s+1)(s+1)(s+1)(s+1)(s+1)(s+1)\,(s+1)(s+1)(s+1)(s+1)}$$

$$= \frac{\begin{array}{l}(s + 7.289919E-02)[(s+0.1111594)^2 + 0.5512521^2] \\ [(s+0.1753327)^2 + 0.6869224^2][(s+0.4159896)^2 + 0.9509943^2] \\ [(s+0.6666667)^2 + 1.105542^2][(s+2.047561)^2 + 0.9817432^2] \\ [(s+1.88884)^2 + 1.433169^2](s+3.315601)\end{array}}{(s+1)^{14}}$$

Second term of the unit$(f^{tr} + g^{tr}\, d_N) :=$ Unit2

$$= \frac{(s^6 + 6s^5 + 15s^4 + 19.00593s^3 + 16.98814s^2 + 5.005932s + 1)}{(s+1)(s+1)(s+1)(s+1)(s+1)(s+1)}$$

$$= \frac{\begin{array}{l}[(s+0.167326)^2 + 0.2490306^2][(s+0.4736535)^2 + 1.170826^2] \\ [(s+2.355614)^2 + 1.170805^2]\end{array}}{(s+1)^6}$$

$$\text{The unit } (f^{tr} + g^{tr} d_N) := \text{ Unit} := \text{ Unit1 } * \text{ Unit2}$$

Due to this very high order of the unit, calculation of d_2 is difficult without Symbolic packages. A modified version of the interpolation algorithm developed in [7] however, can reduce the order of the interpolating unit to some extent. However, the exact lower bound of the order of the interpolating unit is not known. To make the exact contributions of this paper clear, Figure 3 is drawn.

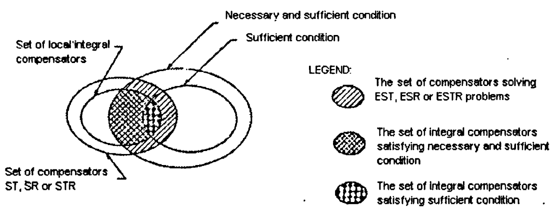

Fig. 3. Main contributions of the paper.

It shows that other than the intersection of the set of local integral compensators with that of sufficient condition, all other hatched regions were unexplored prior to the work in this paper. In other words, there was no way of asserting the existence of a decentralized stabilizing controller with guaranteed local tracking and/or disturbance rejection properties which might not satisfy the sufficient condition and could still tackle arbitrary deterministic signal.

5. CONCLUSIONS

In this paper, the necessary and sufficient conditions for the existence of solutions to *ESTR* are established for the first time. The local stabilizing controller for the N-th subsystem is parameterized, in terms of a free parameter belonging to the ring of proper stable transfer matrices, for guaranteed tracking and/or disturbance rejection properties for any arbitrary deterministic signal along with stabilization of the expanded system. This identifies the full class of compensators that achieves all the desired local and global properties. Moreover, the theory developed is applicable to a broader class of systems (i.e., systems having direct feed-through terms) than dealt with in existing literature. Interpolation technique and the results presented in this paper can be extended to find the lower bound of the controller order and this interesting topic is under investigation.

ACKNOWLEDGEMENT

The authors are grateful to the anonymous referees for their valuable comments and words of encouragement.

(Received June 6, 2002.)

REFERENCES

[1] D. Baksi, V. V. Patel, K. B. Datta, and G. D. Ray: Decentralized stabilization and strong stabilzation of a bicoprime factorized plant. Kybernetika *35* (1999), 235–253.
[2] E. J. Davision and W. Gesing: Sequential stability and optimization of large scale decentralized systems. Automatica *15* (1979), 307–324.
[3] E. J. Davison and U. Ozgüner: The expanding system problem. Systems and Control Letters *1* (1982), 255–260.
[4] B. A. Francis: A Course in H_∞ Control Theory. Springer Verlag, Berlin 1987.
[5] M. Ikeda: Decentralized control of large scale systems. In: Three Decades of Mathematical System Theory: A Collection of Surveys on the Occasion of the Fiftieth Birthday of Jan C. Willems (Lecture Notes in Control and Information Sciences 135, H. Nijmeijer and J. M. Schumacher, eds.). Springer Verlag, Berlin 1989, pp. 219–242.
[6] M. Ikeda and D. D. Siljak: On decentrally stabilizable large-scale systems. Automatica *16* (1980), 331–334.
[7] V. V. Patel and K. B. Datta: A dual look at unity interpolation in H_∞. Internat. J. Control *62* (1995), 813–829.
[8] R. Saeks and J. Murray: Feedback system design: The tracking and disturbance rejection problems. IEEE Trans. Automatic Control *26* (1981), 203–217.
[9] R. Saeks and J. J. Murray: Fractional representations, algebraic geometry and the simultaneous stabilization problem. IEEE Trans. Automatic Control *27* (1982), 895–903.
[10] X. L. Tan and M. Ikeda: Decentralized stabilization for expanding construction of large scale systems. IEEE Trans. Automatic Control *35* (1990), 664–651.
[11] X. L. Tan and M. Ikeda: Expanding construction of large scale servosystems. In: Proc. 2nd Japan–China Joint Symposium on System and Control Theory and Applications 1990.
[12] X. L. Tan and M. Ikeda: Expanding construction of large scale servosystems. In: IMACS World Congress, Dublin, Ireland 1991.
[13] M. Vidyasagar: Control System Synthesis: A Factorization Approach. MIT Press, Cambridge, MA 1985.
[14] D. C. Youla, J. J. Bongiorno, and C. N. Lu: Single loop feedback stabilization of linear multivariable dynamical plants. Automatica *10* (1974), 155–173.

Dr. Dibyendu Baksi, Scientific Technologies Corporation, 750 Commerce Drive, Suite 309, Decatur, GA 30030. U.S.A.
e-mail: Dibyendu_Baksi@stchome.com

Dr. Kanti Bhushan Datta, Dr. Goshaidas Ray, Electrical Engineering Department, Indian Institute of Technology, Kharagpur–721 302. India.
e-mails: kbd, gray@ee.iitkgp.ernet.in