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OUTPUT FEEDBACK PROBLEMS FOR A CLASS OF NONLINEAR SYSTEMS

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The paper deals with the construction of the output feedback controllers for the systems that are transformable into a simpler form via coordinate change and static state feedback and, at the same time, via (possibly different) coordinate change and output injection. Illustrative examples are provided to stress the major obstacles in applying the above scheme, especially as far as its global aspects are concerned. The corresponding results are then applied to the problem of the real-time control of the water-storing plant. Using the methods developed in the theoretical part of the paper, the control of the water levels is designed to handle the unknown influx of the water into the first tank using measurements of water levels only. Simulations results are presented showing good performance of the designed controller. Some preliminary laboratory experiments have shown promising results of the real time implementation as well.

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1. INTRODUCTION

The nonlinear techniques for the control systems design has been developed intensively during the last three decades, [8, 13, 15, 17, 18, 21]. As a matter of fact, one of the basic corner stones of the so-called geometrical approach is to study the possibility of transforming the nonlinear system into a simpler form. The transformations used are typically the nonlinear change of state coordinates, the nonlinear static state feedback (i. e., the change of coordinates in the input space depending also on the state variable) and/or the output injection. The ideal situation is when the system in question may be transformed into the linear one, in this case it is said to be exactly linearizable. For majority of nonlinear classes of systems that is too much to expect, but various types of partially exactly linearized systems, available under less restrictive conditions, might be considered as a suitable option.

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The present paper applies these control techniques to output feedback problems being the prominent topic in both nonlinear control theory and their applications, [5, 8, 10, 14, 16, 18, 19, 20, 22, 23]. While the static state feedback linearization facilitates the design of any kind of static state feedback controller, it is not convenient for output feedback design since the full knowledge of the state is required to compute the corresponding transformations. On the other hand, the so-called output injection, [12], combined with change of coordinates might be used for the observer design but not for the controller design. To put both approaches together and obtain dynamic output feedback controllers, a nonlinear version of the well-known separation principle has to be engaged, [1, 2, 9, 20]. Contrary to the majority of known publications, rather than involving numerous *ad hoc* tailored incongruous observability definitions, this paper understands the term *separation principle* in a clear and simple way: how to combine asymptotic observer and static state feedback controller to obtain dynamical output feedback doing the same job.

Some of these results are repeated and extended here. As a consequence, the solution for the output feedback stabilization problem is suggested for a class of nonlinear systems and the so-called output regulation problem is addressed as well.

To illustrate these results, a laboratory model of the water storing plant has been studied. It consists of two cascade connected water tanks having unknown water influx to the first of them, while outflux of both tanks is controlled via electromechanical valves. The system has two inputs being the valve voltages and two outputs being the water levels. First, it is shown that the corresponding mathematical model of that laboratory plant is exact linearizable and decouplable via nonlinear change of coordinates and feedback. Then, such an improved structure is used to control level of the water in tanks.

More exactly, it is assumed that only the outputs are available for measurements and the water influx is unknown. That represents an important situation for more realistic applications, like control of water level in dam cascades, etc.

Using the theoretical result presented in this paper, the algorithm to control water levels without measurements of the influx and the full system state is provided. Its efficiency and good performance is demonstrated via computer simulations.

The paper is organized as follows. The next section develops stabilizability and detectability concepts and suggests their design based on exact system transformations. Section 3 is devoted to the nonlinear separation principle and provides some results on dynamic output feedback stabilization and output regulation. Section 4 briefly discusses the problem of output regulation. Section 5 introduces the water storing plant and its model while Section 6 develops linearizing transformations and demonstrates ability to control water levels independently. Section 7 deals with the case of an unknown water influx and presents numerical simulations demonstrating that, indeed, the water levels in tanks might be kept at any prescribed level without any knowledge of water influx and valves states. Final section draws the conclusions and gives some outlooks for further research.

2. NONLINEAR CONTROLLERS AND OBSERVERS

In this section we repeat some known results on nonlinear stabilization and detection and give a new result for a certain class of systems generalizing the well-known result on systems with linearizable error dynamics. To motivate it and keep the paper self-contained some basics facts are given in detail. This is the result on stabilizer construction via exact state linearization and observer construction for systems linearizable via output injection (sometimes called as the systems with linearizable error dynamics). Finally, we mention an elegant result of [6], showing the simple way to construct global or semi-global observer for a certain class of nonlinear systems with inputs and provide its alternative simpler proof. All these results rely, nevertheless, on observability of approximate linearization and full observer scheme. Our contribution then is the reduced observer for a more general class of systems that need not have observable approximate linearization.

2.1. Stabilizers and observers via exact linearization

First, let us repeat some straightforward definitions and results on nonlinear stabilizability and detectability via exact feedback and output injection linearization. We assume that the reader is familiar with the basics of the theory of stability and stabilization of nonlinear systems, the corresponding terminology and main results. The interested reader may consult the monographs [3, 4, 13, 17, 21] for more details. We aim to concentrate here on the approach based on using the exact transformation of nonlinear system into a simpler form, [8, 11, 17].

Consider the following controlled nonlinear system

$$\dot{x} = f(x) + G(x)u \tag{1}$$

$$y = H(x) = \begin{bmatrix} h_1(x) \\ h_2(x) \\ \vdots \\ h_p(x) \end{bmatrix}, \tag{2}$$

where the state $x \in \mathbb{R}^n$, the output $y \in \mathbb{R}^p$ and the input $u \in \mathbb{R}^m$. Further, $f(x) = (f_1(x), \dots, f_n(x))^\top$ and the columns of the matrix $G(x)$ are the vector fields, while rows of $H(x)$ are smooth functions. The system is usually considered globally on $\mathbb{R}^n$ or locally around an equilibrium working point x_0 which is supposed to satisfy standard assumption that

$$f(x_0) = 0, \quad H(x_0) = 0$$

$$\operatorname{rank}[G(x_0)] = m, \ \operatorname{rank}[\mathrm{d}H(x_0)] := \operatorname{rank} \begin{bmatrix} \mathrm{d}h_1(x_0) \\ \mathrm{d}h_2(x_0) \\ \vdots \\ \mathrm{d}h_p(x_0) \end{bmatrix} = \operatorname{rank} \left[\frac{\partial H}{\partial x}(x_0)\right] = p. \tag{3}$$

Such a point is often referred to as the *regular equilibrium point* of the system (1) – (2).

Suppose that (1) – (2) has the locally smoothly state and static state feedback linearizable dynamical part (1), see e. g. [8] for the necessary and sufficient conditions and further bibliography. This means that there exists a smooth change of coordinates

$$z = \Phi(x), \quad \Phi(x_0) = 0, \quad \Phi : I\!R^n \mapsto I\!R^n, \tag{4}$$

and a smooth static state feedback (i. e. the change of coordinates of input space depending on the state)

$$v = \alpha(x) + \beta(x)u, \; v \in I\!R^m, \tag{5}$$

both defined on a neighbourhood of the equilibrium $\mathcal{N}_{x_0}$, such that in a new coordinates (z, v) system takes the form

$$\dot{z} = Fz + Gv, \quad y = \tilde{H}(z). \tag{6}$$

In particular, $\Phi(x)$, $\Phi(x_0) = 0$, is required to be smoothly invertible map and the matrix $\beta(x)$ is required to be nonsingular on the above neighbourhood of the equilibrium working point $\mathcal{N}_{x_0}$. If $\mathcal{N}_{x_0} = I\!R^n$, the linearization will be referred to as the *global* one.

To proceed with the approach based on using the exact transformations, as indicated above, suppose in the sequel that (F, G) is the stabilizable pair. The natural idea is to use that property together with the above known linearizing transformations to construct a static state feedback stabilizer for the original system. Let us put that as the following simple

Proposition 1. Consider nonlinear system (1) which is locally (globally) exact feedback linearizable via smooth nonlinear change of coordinates and static state feedback (4,5). Suppose that the resulting linear system (6) is stabilizable, i. e. there exists $(n \times m)$ matrix K, such that the matrix $F + GK$ is the Hurwitz one. Then, the smooth nonlinear static state feedback

$$u_{\text{stab}}(x) = [\beta(x)]^{-1} [K\Phi(x) - \alpha(x)] \tag{7}$$

locally (globally) asymptotically stabilizes the original nonlinear system (1).

Proof. By the definition of the exact linearizing transformations (4,5) one can easily see that the nonlinear closed loop system

$$\dot{x} = f(x) + G(x) [\beta(x)]^{-1} [K\Phi(x) - \alpha(x)] \tag{8}$$

is transformed by the smooth nonlinear change of coordinates (4) into the asymptotically stable linear system

$$\dot{z} = (F + GK)z, \quad y = Hz. \tag{9}$$

Since the smooth change of coordinates takes every trajectory of (8) into a trajectory of (9) and vice versa, the system (8) is locally asymptotically stable around the equilibrium working point x_0. The basin of attraction of x_0 obviously contains a maximal invariant subset of its neighbourhood where $\Phi(x)$ is the smoothly invertible mapping, so that the global aspects of the proposition follow as well. □

For the observers construction, the dual role is played by the smooth change of coordinates and *output injection.* The smooth output injection consists in adding an arbitrary vector field $\gamma(y,u) = \gamma(h(x),u)$ to the right hand side of (1). In other words, the system (1) is said to be exact linearizable via output injection if there exists smooth mapping $\gamma(y,u)$ (output injection) and the smooth coordinate change $z = \Phi(x)$ such that the system

$$\dot{x} = f(x) + G(x)u, \quad y = h(x), \tag{10}$$

takes in new coordinates the form

$$\dot{z} = Fz + \gamma(y,u), \quad y = Hz. \tag{11}$$

Necessary and sufficient conditions for the state equivalence of a given nonlinear system to the form (11) were obtained for the first time in [11]. Other more general forms related to observability and detectability problems are studied in [12, 14].

Definition 2. Consider the nonlinear dynamical system (1). Its asymptotic (exponential) observer is another nonlinear dynamical system of the form

$$\dot{\widehat{x}} = \widehat{f}(\widehat{x}) + G(\widehat{x})\widehat{u}, \quad \widehat{y} = \widehat{H}(\widehat{x},\widehat{u}), \tag{12}$$

with its input $\widehat{u}$ being the variables y, u of the observed system (1) and its output $\widehat{y}$ estimating asymptotically (exponentially) the state of the system to-be-observed (1), [10]. We call the observer as the *full* one if

$$\dim[x] = \dim[\widehat{x}], \quad \widehat{H}(\widehat{x},\widehat{u}) \equiv \widetilde{H}(\widehat{x}) \to x \text{ as } t \to \infty,$$

and the *reduced* one if

$$\dim[x] = \dim[\widehat{x}] + \dim[y], \quad \widehat{H}(\widehat{x},\widehat{u}) \equiv \begin{bmatrix} \widetilde{H}(\widehat{x}) \\ \overline{H}(y,u) \end{bmatrix} \to x \text{ as } t \to \infty.$$

Global on $\mathcal{B} \subset I\!R^n$ observer is the one ensuring convergence for any initial observation error and any observed trajectory from $\mathcal{B} \subset I\!R^n$. If $\mathcal{B} = I\!R^n$ we call it simply global, if $\mathcal{B}$ exists but is unknown we call the observer as the local one. Finally, we say that system has the *semiglobal observer* if it has a family of observers, each of them is global on a different set and union of all these sets equals to $I\!R^n$.

For the systems linearizable via output injection the observers may be constructed in a straightforward way as follows, [17]. Here and in the sequel, $||\cdot||$ stands for the usual Euclidean norm of vector in an appropriate vector space.

Proposition 3. Suppose (1) is locally state equivalent on a neighbourhood $\mathcal{N}$ of the equilibrium x_0 via smooth coordinate change $\Phi(x)$ to the system (11) with (F, H) being detectable. Let the matrix L be such that the matrix $F + LH$ is the Hurwitz one. Then the following system having the state $\hat{z} \in I\!R^n$ and the output $\hat{x} \in I\!R^n$

$$\dot{\hat{z}} = (F + LH)\hat{z} - Ly + \gamma(y, u), \quad \hat{x} = \Phi^{-1}(\hat{z}), \tag{13}$$

is the local exponential observer for the system (1). More precisely, there exist constants K_1, K_2 such that for any bounded solution $x(t) \in \mathcal{N}$ of (1) and $\hat{x}(t)$ of (13) with $||\hat{x}(0) - x(0)||$ being sufficiently small it holds

$$||\hat{x} - x|| < K_1 \exp(-K_2 t)||\hat{x}(0) - x(0)||, \quad K_1 > 0, \; K_2 > 0.$$

In particular, for $\mathcal{N} \in I\!R^n$, (13) is the global exponential observer for (1).

Proof. Consider any solution $x(t)$ of (1) such that $x(t) \in \mathcal{N}$ $\forall t \geq 0$. Comparing (13) and (11) one has

$$\dot{\hat{z}} - \dot{z} = (F + LH)(\hat{z} - z). \tag{14}$$

Therefore, $\forall t \in I\!R$, $\forall z(0) \in \Phi(\mathcal{N})$ and $\hat{z}(0)$ sufficiently close to $z(0)$ it holds $\hat{z} \in \Phi(\mathcal{N})$ $\forall t \geq 0$ and therefore $\forall t \geq 0$

$$||\hat{z}(t) - z(t)|| = ||\exp((F + LH)t)(\hat{z}(0) - z(0))|| < \tilde{K}_1 \exp(-\tilde{K}_2 t)||\hat{z}(0) - z(0)||,$$

where $\tilde{K}_{1,2} > 0$ are suitable constants existing since $F + LH$ is Hurwitz. Moreover, since $x(t) = \Phi^{-1}(z(t))$ and $\hat{x}(t) = \Phi^{-1}(\hat{z}(t))$, it holds

$$||\hat{x}(t) - x(t)|| = ||\Phi^{-1}(z(t)) - \Phi^{-1}(\hat{z}(t))|| \leq \max_{\hat{z} \in \overline{\Phi}(\mathcal{N})} ||\frac{\partial \Phi^{-1}}{\partial z}(\hat{z})|| ||\hat{z}(t) - z(t)||$$

$$< K_1 \exp(K_2 t)||\hat{x}(0) - x(0)||,$$

where $K_{1,2} > 0 = \tilde{K}_{1,2} \max_{\hat{z} \in \overline{\Phi}(\mathcal{N})} ||\frac{\partial \Phi^{-1}}{\partial z}(\hat{z})||$ due to the straightforward observation that both $\hat{z}(t)$ and $z(t)$ belong to $\Phi(\mathcal{N})$. The global statement of the proposition is straightforward. □

The necessary and sufficient conditions both for the exact feedback linearization and linearization using output injection were discussed thoroughly in the literature, see [8, 11, 12, 14, 17] and references within there.

2.2. Observers for nonlinearizable systems

Here we aim to give a generalization of the previous results to some classes of systems that can not be linearized using exact transformations.

First, let us give our modification of the result of [6].

Proposition 4. Consider the following nonlinear system

$$\dot{x} = \begin{bmatrix} x_2 \\ \vdots \\ x_n \\ \phi(x) \end{bmatrix} + u \begin{bmatrix} g_1(x_1) \\ g_2(x_1, x_2) \\ \vdots \\ g_n(x_1, x_2, \dots, x_n) \end{bmatrix}, \quad y = x_1 \tag{15}$$

where $x \in I\!R^n$, $u \in I\!R$. Let all mappings g_i, $i = 1, \dots n$ and ϕ be Lipschitzean on the ball of radius $R > 0$ centred at the origin $\mathcal{B}_R$ with common Lipschitz constant $L > 0$ and let the input u be uniformly bounded by some constant u_0. Then there exists sufficiently large real number $\theta > 0$ such that the following system

$$\dot{\hat{x}} = \begin{bmatrix} \hat{x}_2 \\ \vdots \\ \hat{x}_n \\ \phi(\hat{x}) \end{bmatrix} + u \begin{bmatrix} g_1(\hat{x}_1) \\ g_2(\hat{x}_1, \hat{x}_2) \\ \vdots \\ g_n(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n) \end{bmatrix} - \begin{bmatrix} \theta \\ \theta^2 \\ \vdots \\ \theta^n \end{bmatrix} (y - \hat{y}) \tag{16}$$

is the global on $\mathcal{B}_R$ exponential observer.

Remark 5. Notice that the above proposition can be used for both global and semiglobal observer construction. Namely, if all the mappings are globally Lipschitzean on $I\!R^n$ with common Lipschitz constant $L > 0$, the above constructed observer is the global one. Otherwise, due to the smoothness of all mappings defining systems right hand side, one can apply the above proposition on any ball and obtain observer depending on its radius, so that semiglobal observer is obtained.

Remark 6. Necessary and sufficient conditions for a given single input single output nonlinear system (1) to be transformable via smooth change of coordinates into the form (15) are discussed e.g. in [6, 17]. Briefly, these are

— System (1) is observable for any uniformly bounded input.

— Functions $h, L_h, \dots, L_h^{n-1}$ form a change of coordinates on the corresponding subset of the state space.

We provide here alternative proof of Proposition 4 that is more simple than the the one of [6].

P r o o f. Substracting the equation (16) from (15) and using Lipschitzean property one has for all observed trajectories and all initial observation errors

$$\frac{\mathrm{d}(x - \hat{x})}{\mathrm{d}t} = [H(\theta) + N(t)]\,(x - \hat{x})$$

where

$$H(\theta) = \begin{bmatrix} l_1\theta & 1 & 0 & \cdots & 0 & 0 \\ l_2\theta^2 & 0 & 1 & 0 & \cdots & 0 \\ l_3\theta^3 & 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & 0 \\ l_{n-1}\theta^{n-1} & 0 & 0 & \cdots & 0 & 1 \\ l_n\theta^n & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \qquad l_1, l_2, \dots, l_n \in I\!R$$

$$N(t) = [n_{ij}(t)]_{i,j=1,\dots,n}, \; n_{ij}(t) \equiv 0 \;\; \forall j > i, \quad |n_{ij}(t)| \leq Lu_0 \;\; \forall t \geq 0, \; \forall j \leq i.$$

Denote

$$e(t) = \begin{bmatrix} e_1(t) \\ \vdots \\ e_{n-1}(t) \\ e_n(t)) \end{bmatrix} = \begin{bmatrix} \theta^{n-1}(x_1(t) - \hat{x}_1(t)) \\ \vdots \\ \theta(x_{n-1}(t) - \hat{x}_{n-1}(t)) \\ (x_n(t) - \hat{x}_n(t)) \end{bmatrix}.$$

Obviously, it is sufficient to prove that for some $\theta > 0$ it holds $\lim_{t\to\infty} e(t) = 0$ for all initial observation errors and all observed trajectories. Indeed, we have via straightforward computations

$$\dot{e} = (\theta H + \tilde{N}(t))\, e, \quad ||\tilde{N}(t)|| \leq ||N(t)|| \leq Lu_0\sqrt{n(n+1)/2}, \quad \forall t \geq 0,$$

where $||\cdot||$ is the matrix norm compatible with the Euclidean vector norm and $l_1, \dots, l_n \in I\!R$ are such that

$$H := H(1) = \begin{bmatrix} l_1 & 1 & 0 & \cdots & 0 & 0 \\ l_2 & 0 & 1 & 0 & \cdots & 0 \\ l_3 & 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & 0 \\ l_{n-1} & 0 & 0 & \cdots & 0 & 1 \\ l_n & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

is the Hurwitz matrix. Therefore, there exists positive definite nonsingular symmetric matrix P such that

$$H'P + HP = -I_n, \quad I_n := \mathrm{diag}[1, 1, \dots, 1].$$

Therefore

$$\frac{\mathrm{d}[e'Pe]}{\mathrm{d}t} = -\theta||e||^2 + e'[\tilde{N}(t)'P + P\tilde{N}(t)]e \leq -||e||^2(\theta - Lu_0||P||)$$

so that choosing

$$\theta > Lu_0||P||$$

the proof follows by Lyapunov-like argument. □

Remark 7. Let us underline that the key point of the above proof is that indeed upper bound on matrix $\tilde{N}(t)$ can be taken the same as for $N(t)$, i. e. independent of θ. That is thanks to lower triangular structure of $N(t)$, which in turn is due to the form of nonlinearities on the right hand side of (15).

In many situations, the second condition mentioned in Remark 6 is not satisfied, since that requires the observability of the linear approximation of a studied nonlinear system, [17]. In particular, this is the case of the water storing plant model studied later on.

Therefore, we aim to give here a more general version of the above Proposition 3. For a certain class of systems, more general than the one of linearizable via output injection, the *reduced* observers may be constructed. Let us recall, cf. [10] and Definition 2 above, that the reduced observer gives the estimate having dimension of the state space minus dimension of the output and that the estimate combined with the direct measurement of the output of the original system provides the estimate of its full state. As a matter of fact, one can provide modification of Propositions 3, 4 in a very standard way to obtain reduced observers as well. The purpose of the next result is more general, namely, to treat the situation, when the studied system need not be observable.

More precisely, let us consider the class of nonlinear systems

$$\begin{array}{rcl} y & = & \eta_D \\ \dot{\eta}_D & = & f_D(\eta, u) \\ \dot{\eta}_R & = & F_R \eta_R + \gamma(y, u) \end{array} \quad , \quad \eta = \begin{bmatrix} \eta_D \\ \eta_R \end{bmatrix} \in I\!R^n, \;\; \eta_R \in I\!R^{n-p}, \;\; \eta_D \in I\!R^p \qquad (17)$$

where F_R is a Hurwitz matrix. Notice that the system (17) need not be observable, neither in its linear approximation, nor in any known nonlinear sense, cf. e. g. [6, 17].

First, let us show that the above class of systems is indeed a generalization of the systems linearizable using the smooth coordinate change and output injection.

Proposition 8. Suppose that the system (1) – (2) is locally exact linearizable via smooth coordinate change and output injection and the resulting system (11) is detectable. Then the system (1) – (2) is state equivalent to the form (17) with

$$f_D = \overline{F}_D \eta_D + \overline{F}_{DR} \eta_R + \overline{\gamma}(y, u),$$

$\overline{F}_D$ being a Hurwitz matrix.

Proof. First, let us prove the assertion for the case (F, H) being an observable pair. This means that there exists $(n \times p)$ matrix L such that $\tilde{F} = F + LH$ has n mutually distinct real negative eigenvalues and $\tilde{F}$ has n linearly independent eigenvectors. The system (11) can be easily rewritten as

$$\dot{z} = \tilde{F} z + \tilde{\gamma}(y, u), \quad y = Hz$$

where

$$\tilde{\gamma}(y, u) := \gamma(y, u) - Ly.$$

Since $\operatorname{rank} H = p$ and eigenvectors of $\tilde{F}$ span the $I\!R^n$, there exist p eigenvectors of $\tilde{F}$, denoted as $e_1, \dots, e_p$ such that $\operatorname{span}\{He_1, \dots, He_p\} = I\!R^p$. Take a basis of $I\!R^n$ composed of $e_1, \dots, e_p$ complemented to the dimension n. In this basis we obviously have (for the brevity, let us keep the previous notation):

$$\tilde{F} \to \begin{bmatrix} \tilde{F}_D & \tilde{F}_{DR} \\ 0 & \tilde{F}_R \end{bmatrix}, \quad H \to \begin{bmatrix} H_D & H_R \end{bmatrix}, \ \operatorname{rank} H_D = p, \ z \to \begin{bmatrix} \tilde{z}_D \\ \tilde{z}_R \end{bmatrix}. \tag{18}$$

Finally, put $\eta_D = H_D \tilde{z}_D + H_R \tilde{z}_R$, $\eta_R = \tilde{z}_R$ to obtain the form (17).

Summarizing, the proposition has been proved for the case of observable (F, H). To complete the proof, let us indicate how to treat unobservable, but detectable pair (F, H). Since the pair (F, H) in (11) is detectable, there exists a linear change of coordinates $\overline{z} = Tz$ such that

$$TFT^{-1} = \begin{bmatrix} F_{\text{obs}} & 0 \\ F_{21} & F_{\text{unobs}} \end{bmatrix}, \quad HT^{-1} = \begin{bmatrix} H_{\text{obs}} & 0 \end{bmatrix}$$

where $(F_{\text{obs}}, H_{\text{obs}})$ is the observable pair and F_{unobs} is Hurwitz. Now, the observable subsystem may be treated as above resulting in the suitable coordinates into the following pair

$$\begin{bmatrix} \hat{F}_D & \hat{F}_{DR} & 0 \\ 0 & \hat{F}_R & 0 \\ F_{21\text{obs}} & F_{21\text{unobs}} & F_{\text{unobs}} \end{bmatrix}, \quad [H_D \ \ H_R \ \ 0], \ \ \operatorname{rank} H_D = p,$$

giving the required form. □

Now, let us give the asymptotic observer for the class of systems being smoothly state equivalent to (17).

Proposition 9. Suppose (1) is locally state equivalent on $\mathcal{N}$ via smooth coordinate change $\Phi(x)$ to the system (17) with F_R being Hurwitz. Then the following system having the state $\hat{\eta} = (\hat{\eta}_D, \hat{\eta}_R)^\top \in I\!R^n$ and the output $\hat{x} \in I\!R^n$

$$\hat{\eta}_D = y, \quad \dot{\hat{\eta}}_R = F_R \hat{\eta}_R + \gamma(y, u), \quad \hat{x} = \Phi^{-1}(\hat{\eta})$$

is the local asymptotic observer for the system (1). More precisely, there exist constants K_1, K_2 and a neighbourhood $\mathcal{N}$ of the equilibrium working point x_0 such that for any bounded solutions $x(t)$ of (1) and $\hat{x}(t)$ of (13) belonging $\forall t \geq 0$ to $\mathcal{N}$ it holds

$$||\hat{x} - x|| < K_1 \exp(-K_2 t) ||\hat{x}(0) - x(0)||, \quad K_1 > 0, \ K_2 > 0.$$

Proof. Analogous to that of Proposition 3. □

3. NONLINEAR SEPARATION PRINCIPLE AND DYNAMIC OUTPUT FEEDBACK STABILIZATION

Previous section suggests for a suitable class of systems both the construction of the static state asymptotically stabilizing feedback and asymptotic observers. The natural idea, traditionally referred as the *separation principle* for the linear systems, is to combine these two constructions to obtain asymptotically stabilizing dynamical output feedback. Nevertheless, the situation is no more simple in the case of nonlinear systems, especially when the global aspects are concerned, [16, 1, 2, 10, 5, 9, 20, 22].

For the reader convenience, we present here some of those known facts and ideas on the nonlinear separation principle in a self-contained manner.

Proposition 10. Consider on $\mathbb{R}^n$ the smooth mappings $\phi(x,\varepsilon), \psi(x,\varepsilon)$, $x \in \mathbb{R}^n, \varepsilon \in \mathbb{R}^n$, and let the system

$$\dot{x} = \phi(x,0), \quad x \in \mathbb{R}^n$$

be locally asymptotically stable around the origin. Then the system

$$\dot{x} = \phi(x,\varepsilon), \quad \dot{\varepsilon} = \psi(x,\varepsilon)$$

is locally asymptotically stable around the origin if the system

$$\dot{\varepsilon} = \psi(x,\varepsilon)$$

is locally asymptotically stable around the origin uniformly with respect to any smooth $x(t)$.

Proof. By inverse Lyapunov's function theorem there exists Lyapunov function $V(x)$ and the neighbourhood of the origin $\mathcal{N}_0$ such that for any $\rho > 0$

$$V_x\phi(x,0) < 0$$

for all $x \in \overline{\mathcal{N}_0} \setminus \{x \in \mathbb{R}^n |\ V(x) < \rho\}$. Clearly, as the previous set is compact, there exists $\delta = \delta(\rho) > 0$ such that $\forall \varepsilon$, $\|\varepsilon\| < \delta$ and for all $x \in \overline{\mathcal{N}_0} \setminus \{x \in \mathbb{R}^n |\ V(x) < \rho\ \}$ it holds

$$V_x\phi(x,\varepsilon) < 0.$$

As the dynamics of the $\varepsilon(t)$ is locally asymptotically stable uniformly for every smooth $x(t)$, there exists a neighbourhood of the origin $\mathcal{N}_1$, independent of the particular trajectory $x(t)$, such that for all $\varepsilon(0) \in \mathcal{N}_1$ the trajectory $\|\varepsilon(t)\| < \delta$ for all $t > 0$. Therefore, the trajectory $x(t)$ enters after some time the set $\{x \in \mathbb{R}^n |\ V(x) < \rho\}$ and remains within there.

As $\rho > 0$ can be taken arbitrarily, we have that the total system is Lyapunov stable. Now, having the local Lyapunov stability of the total system proved, the local asymptotical stability is easily concluded, since $\varepsilon \to 0$ as $t \to \infty$. □

Remark 11. Actually, the only nontrivial point of the above proof is to ensure by choosing $\varepsilon(0)$ that $x(t)$ indeed stays for all $t \geq 0$ in the region of asymptotical stability of the x-system to use the obvious argument $\phi(x(t), \varepsilon(t)) \to \phi(x(t), 0)$ as $t \to \infty$.

The key role during the above proof is played by the fact that $\varepsilon(t)$ has the locally asymptotically stable dynamics, in particular, it is Lyapunov stable. Clearly, ε may be used as the error of observation in the previously mentioned separation principle scheme, i. e. $\varepsilon = x - \hat{x}$. Nevertheless, it is not sufficient to have that $x(t) \to \hat{x}(t) \to 0$, as $t \to \infty$, one has to be sure about possibility to make ε small enough by choosing its initial condition for *all* $t \geq 0$, not only converging to 0.

In the global case, the situation is not easier despite the fact that we need not care about region of asymptotical stability of the system $\dot{x} = \phi(x, 0)$. There are two reasons. First, as the whole $\mathbb{R}^n$ is unbounded, it is not assured that $\phi(x, \varepsilon)$ converge *uniformly* to $\phi(x, 0)$ as $\varepsilon \to 0$. For this reason, $V_x\phi(x, 0) < 0, \forall x \in \mathbb{R}^n, \|x\| > \rho$ does not guarantee that for ε small enough also $V_x\phi(x, \varepsilon) < 0, \forall x \in \mathbb{R}^n, \|x\| > \rho$. The second reason is the well-known finite-time escape phenomenon, when a solution $x(t)$ may escape to infinity in a finite time, thereby not giving opportunity for convergence of $\phi(x(t), \varepsilon(t))$ to $\phi(x(t), 0)$.

Example 12. As an illustrative example, consider the system

$$\dot{x} = -x + \varepsilon x^2, \quad \dot{\varepsilon} = -\varepsilon, \quad x, \varepsilon \in \mathbb{R}.$$

Locally, due to Proposition 10, the total (ε, x)-system is asymptotically stable. Nevertheless, for the $\varepsilon(0), x(0) > 0$ large enough it is easy to see that $x(t) \to \infty$ as $t \to t_{esc}, t_{esc} > 0$, despite $\varepsilon \to 0$ as $t \to \infty$ and $\lim_{\varepsilon \to 0}(-x + \varepsilon x^2) = -x$. Notice also that the last limit is not the uniform one. Moreover, one can actually see that no matter how small $\varepsilon(0)$ is, there always exists $x(0)$ large enough to produce trajectory escaping to infinity in a finite time.

Example 13. The finite time escape phenomenon may occur even if $\phi(x, \varepsilon) \to \phi(x, 0)$ as $\varepsilon \to 0$ uniformly for $x \in \mathbb{R}$. Actually, consider the system

$$\dot{x} = -x + \xi(\varepsilon)(x + x^2), \quad \dot{\varepsilon} = -\varepsilon, \quad x, \varepsilon \in \mathbb{R}$$

where $\xi(\varepsilon)$ is a smooth function such that

$$\xi(\varepsilon) = \begin{cases} 1 \text{ for } |\varepsilon| \geq a > 0 \\ 0 \text{ for } |\varepsilon| \leq b < a. \end{cases}$$

Obviously, $-x + \xi(\varepsilon)x^2 \to -x$ as $\varepsilon \to 0$ uniformly for $x \in \mathbb{R}$. At the same time, for $\varepsilon(t) \geq a$ the solution $x(t) = (1/x(0) - t)^{-1}$, so that for $\varepsilon(0) > a\exp(1/x(0))$ and $x(0) > 0$ the solution $x(t)$ escapes to infinity in the finite time $1/x(0)$. Nevertheless, if the initial condition $\varepsilon(0) < b$, the solution obviously $x(t)$ converge to 0 for any initial condition $x(0)$.

In view of the above remark and examples, the following proposition can be obtained in the analogous way as Proposition 10.

Proposition 14. Consider on $I\!R^n$ the mappings $\phi(x,\varepsilon), \psi(x,\varepsilon),\ \varepsilon \in I\!R^n, x \in I\!R^n$ and let the system

$$\dot{x} = \phi(x,0), \quad x \in I\!R^n$$

be globally asymptotically stable around the origin. Suppose that

$$\dot{\varepsilon} = \psi(x,\varepsilon)$$

is locally asymptotically stable around the origin uniformly with respect to any smooth $x(t)$ and let

$$\phi(x,\varepsilon) \to \phi(x,0)$$

uniformly for $x \in I\!R^n$ as $\varepsilon \to 0$. Then the system

$$\dot{x} = \phi(x,\varepsilon), \quad \dot{\varepsilon} = \psi(x,\varepsilon)$$

is locally asymptotically stable around the origin and there exists a neighbourhood of the origin in $I\!R^n$, say $\mathcal{N}^\varepsilon$, such that $I\!R^n \times \mathcal{N}^\varepsilon$ is its basin of attraction. In other words, the above asymptotic stability is global with respect to $x \in I\!R^n$.

Proof. It is indeed the very same as in the case of Proposition 10, since the initial condition for ε might be used to keep $\varepsilon(t)$ for all time moments inside any prescribed neighbourhood of the origin and converging to it. Moreover, assumption on uniform convergence of $\phi(x,\varepsilon) \to \phi(x,0)$ makes it possible to repeat the corresponding Lyapunov function arguments. □

The following proposition is a standard global result using the so-called growth condition.

Proposition 15. Consider on $I\!R^n$ the mappings $\phi(x,\varepsilon), \psi(x,\varepsilon),\ \varepsilon \in I\!R^n, x \in I\!R^n$ and let the system

$$\dot{x} = \phi(x,0), \quad x \in I\!R^n,$$

be globally asymptotically stable around the origin. Suppose that

$$\dot{\varepsilon} = \psi(x,\varepsilon), \quad \varepsilon \in I\!R^n,$$

is globally asymptotically stable around the origin uniformly with respect to any smooth $x(t)$ and let the so-called growth condition

$$||\phi(x,\varepsilon)|| \leq K_1(\varepsilon) + K_2(\varepsilon)||x||$$

holds, where the functions $K_1(\varepsilon), K_2(\varepsilon)$ are bounded on any compact set. Then the system

$$\dot{x} = \phi(x,\varepsilon), \quad \dot{\varepsilon} = \psi(x,\varepsilon)$$

is globally asymptotically stable around the origin.

Proof. Due to the well known Winter Theorem from the theory of ordinary differential equations, [7], the above growth condition guarantees the existence of the solution of the investigated equation on the time interval $[0, \infty)$ for every initial condition. The rest of the proof may therefore repeat the proof of Proposition 14 as any trajectory $\varepsilon(t)$ ultimately enters arbitrarily small neighbourhood of the origin uniformly with respect to $x(t)$. As the trajectory $x(t)$ is guaranteed to exist during any time interval, one can wait until ε is small enough that the corresponding Lyapunov function stays negative. □

Remark 16. Using the above propositions, one may combine the detectability and stabilizability results obtained earlier. Namely, one can obtain output feedback stabilization result for the system which is static state feedback and coordinate change equivalent to a linear controllable system and at the same time via possibly different coordinate change and output injection equivalent to an observable system. As a more general result, state equivalence to the form (17) may be used as well. The detailed formulations of these results are left to the reader. Later on, we apply the above propositions directly when combining particular static state feedback controllers and state observers to obtain dynamical output feedback controllers for the case study of the water storing plant.

4. OUTPUT REGULATION PROBLEM

We recall here the definition of the output regulation problem via dynamical output feedback. We use here slightly different and more general version than in [8].

Definition 17. The local output regulation problem is said to be solvable via dynamic output feedback for the nonlinear system (1) if there is a dynamic feedback such that for the resulting closed loop extended system it holds

1. It is Lyapunov stable around the working equilibrium point and all its solutions are bounded on its neighbourhood,
2. $y(t) \to 0$ as $t \to \infty$ for every initial condition from a neighbourhood of the equilibrium working point.

The semiglobal output regulation problem is said to be solvable via dynamic output feedback for the nonlinear system (1) if for every $R > 0$ there is a dynamic feedback such that for the resulting closed loop extended system it holds

1. It is Lyapunov stable around the working equilibrium point and all its solutions are bounded on the ball of radius R centred at the equilibrium working point,
2. $y(t) \to 0$ as $t \to \infty$ for every initial condition from the ball of radius R centred at the equilibrium working point.

The methods developed in the previous sections for the stabilization can be extended in a straightforward way to the case of output regulation. Actually, the standard approach is to find and stabilize output zeroing dynamics to which the previously discussed output feedback stabilization methods could be applied.

5. WATER STORING SYSTEM

We aim here to demonstrate the design methods introduced earlier on the case study of the laboratory model of the water storing plant. More precisely, we consider two cascade connected water-storing tanks system depicted in Figure 1.

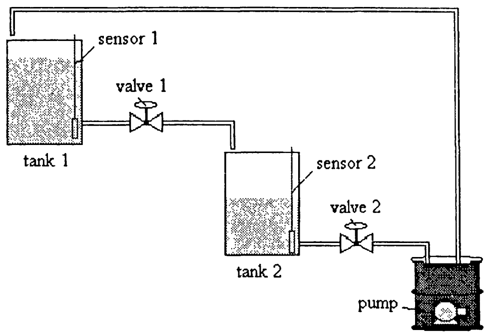

Fig. 1. Scheme of the system.

This prototype is composed by:

— A PC computer with A/D and D/A cards.

— Two prismatic acrylic tanks of dimensions $L = 40\,\text{cm}$, $W = 40\,\text{cm}$, and $H = 60\,\text{cm}$.

— Two electromechanical valves

— Two differential pressure sensors

— One pump which provides a constant input flow of $0.10395 \times 10^{-3}\ \text{m}^3/\text{s}$.

The nonlinear model for this plant is given by

$$
\begin{aligned}
\dot{h}_1(t) &= \tfrac{f_e}{A_t} - \tfrac{w_1(t)\sqrt{h_1(t)}}{A_t} \\
\dot{w}_1(t) &= \tfrac{K_{e1}}{T} v_1(t) - \tfrac{1}{T} w_1(t) \\
\dot{h}_2(t) &= \tfrac{w_1(t)\sqrt{h_1(t)}}{A_t} - \tfrac{w_2(t)\sqrt{h_2(t)}}{A_t} \\
\dot{w}_2(t) &= \tfrac{K_{e2}}{T} v_2(t) - \tfrac{1}{T} w_2(t)
\end{aligned}
\tag{19}
$$

where h_1 and h_2 are the levels in tank 1 and tank 2 respectively; w_1 and w_2 are valve 1 and valve 2 positions; v_1 and v_2 are the voltage control signals applied to valve 1 and valve 2; A_t is each tank area; T is the valve time constant; K_{e1} and K_{e2} are the valve gains and f_e is a constant input flow to tank 1.

6. EXACT LINEARIZATION AND THE CONTROLLERS FOR THE WATER STORING PLANT: THE CASE OF KNOWN WATER INFLUX

Let us consider the system (19) as the nonlinear controlled system having the state $(h_1, w_1, h_2, w_2)^\top$, the input $v = (v_1, v_2)^\top$ and the output $y = (h_1, h_2)^\top$.

We illustrate here some concepts introduced in Sections 2 and 3. First, we will show in this section that the above nonlinear model is exact feedback linearizable using smooth change of coordinates and static state feedback. That will enable us later on to show how the static state feedback controllers may be derived in case of exact knowledge of the water influx.

To proceed with, let us define new coordinates as

$$\begin{aligned} x_1 &= h_1 - l_1, \qquad & x_2 &= h_2 - l_2, \\ x_3 &= \tfrac{f_e}{A_t} - \tfrac{w_1\sqrt{h_1}}{A_t}, \qquad & x_4 &= \tfrac{w_1\sqrt{h_1}}{A_t} - \tfrac{w_2\sqrt{h_2}}{A_t} \end{aligned} \tag{20}$$

where l_1, l_2 are desired water levels. Therefore, in a new x-coordinates, outputs are x_1, x_2 and their desired values are equal to zero.

Inverse transformations to (20) have the following form

$$\begin{aligned} h_1 &= x_1 + l_1, \qquad & h_2 &= x_2 + l_2, \\ w_1 &= \left(\tfrac{f_e - A_t x_3}{\sqrt{x_1 + l_1}}\right), \qquad & w_2 &= \left(\tfrac{f_e - A_t(x_3 + x_4)}{\sqrt{x_2 + l_2}}\right). \end{aligned} \tag{21}$$

Straightforward computations give

$$\begin{aligned} \dot{x}_1(t) &= x_3(t) \\ \dot{x}_2(t) &= x_4(t) \\ \dot{x}_3(t) &= -\tfrac{1}{A_t}\left(w_1\tfrac{\dot{h}_1}{2\sqrt{h_1}} + \sqrt{h_1}\dot{w}_1\right) \\ &= -\tfrac{1}{A_t}\left(w_1\tfrac{f_e}{2A_t\sqrt{h_1}} - \tfrac{w_1}{2A_t} + \tfrac{\sqrt{h_1}(K_{e_1}v_1 - w_1)}{T}\right) \\ &= -\tfrac{K_{e_1}\sqrt{x_1(t)+l_1}}{A_t T}v_1(t) + \tfrac{A_t x_3(t) - f_e}{A_t}\left(\tfrac{x_3(t)}{2(x_1(t)+l_1} - \tfrac{1}{T}\right) \\ \dot{x}_4(t) &= \tfrac{1}{A_t}\left(\tfrac{w_1\dot{h}_1}{2\sqrt{h_1}} + \sqrt{h_1}\dot{w}_1 - \tfrac{w_2\dot{h}_2}{2\sqrt{h_2}} - \sqrt{h_2}\dot{w}_2\right) \\ &= \left(\tfrac{f_e}{A_t} - x_3(t)\right)\left(\tfrac{x_3(t)}{2(x_1(t)+l_1)} - \tfrac{1}{T}\right) + \tfrac{K_{e_1}\sqrt{x_1(t)+l_1}}{A_t T}v_1(t) \\ &\quad + \left(\tfrac{f_e}{A_t} - x_3(t) - x_4(t)\right)\left(\tfrac{1}{T} - \tfrac{x_4(t)}{2(x_2(t)+l_2)}\right) - \tfrac{K_{e2}\sqrt{x_2(t)}}{A_t T}v_2(t). \end{aligned} \tag{22}$$

Therefore, using the following static state feedback (i. e. introducing the new input variables u_1, u_2):

$$\begin{aligned} u_1(t) = & -\tfrac{K_{e_1}\sqrt{x_1(t)+l_1}}{A_t T} v_1(t) + \tfrac{A_t x_3(t) - f_e}{A_t}\left(\tfrac{x_3(t)}{2x_1(t)} - \tfrac{1}{T}\right) \\ & + (k_{11}x_1(t) + k_{13}x_3(t)) \\ u_2(t) = & \left(\tfrac{f_e}{A_t} - x_3(t)\right)\left(\tfrac{x_3(t)}{2x_1(t)} - \tfrac{1}{T}\right) + \tfrac{K_{e_1}\sqrt{x_1(t)}+l_1}{A_t T} v_1(t) \\ & + \left(\tfrac{f_e}{A_t} - x_3(t) - x_4(t)\right)\left(\tfrac{1}{T} - \tfrac{x_4(t)}{2x_2(t)}\right) - \tfrac{K_{e_2}\sqrt{x_2(t)}+l_2}{A_t T} v_2(t) \\ & + (k_{22}x_2(t) + k_{24}x_4(t)), \end{aligned} \tag{23}$$

where $k_{11}, k_{13}, k_{22}, k_{24} > 0$, we obtain the following linearized and asymptotically stable system

$$\begin{aligned} \dot{x}_1(t) &= x_3(t) \\ \dot{x}_2(t) &= x_4(t) \\ \dot{x}_3(t) &= -k_{11}x_1(t) - k_{13}x_3(t) + u_1 \\ \dot{x}_4(t) &= -k_{22}x_2(t) - k_{24}x_4(t) + u_2 \\ y_1 &= x_1 \\ y_2 &= x_2. \end{aligned}$$

Taking the new inputs u_1, u_2 to be identically zero gives that $x(t) \to 0$ exponentially as $t \to \infty$. As a consequence, putting in the equations (23) and solving them with respect to $v_1(t), v_2(t)$ gives $v_1^{st}(x(t)), v_2^{st}(x(t))$ that, being expressed by virtue of (21) in terms of h_1, h_2, w_1, w_2, guarantees, after substitution into (19), globally for all initial conditions $h_1(0), h_2(0), w_1(0), w_2(0) > 0$ that

$$h_1(t) \to l_1,\ h_2(t) \to l_2,\ w_1(t) \to \frac{f_e}{\sqrt{l_1}}, w_2(t) \to \frac{f_e}{\sqrt{l_2}},$$

as $t \to \infty$.

The designed control law was applied to the physical laboratory system using the computer and the A/D and D/A cards. Figure 2 shows the performance of the real-time implementation of the designed controller based on the above studied mathematical model of the water storing plant. First, starting at 25 cm for each output, a new input is applied to the linearized system at $t = 0$ and $t = 150$ sec., so that the level of tank 2 moves respectively to the values of 27 cm and 30 cm. Observe that the output 1, the level of tank 1, practically does not suffer any modification, staying at 25 cm. That demonstrates practically the above linearized and, in fact, even decoupled structure. Then, at $t = 360$ sec., a new input is applied such that the output 1 presents a sinusoidal behavior, with peak values approximately at 26 cm and 28 cm. Observe that the output 2 is practically not modified, again nicely complying with the above input-output decoupled structure of the linearized system.

At the same time, we have thereby demonstrated theoretically well-known property of steady output response of internally exponentially stable system feeded by input produced by an autonomous neutrally stable system (e. g. harmonic or constant input signals).

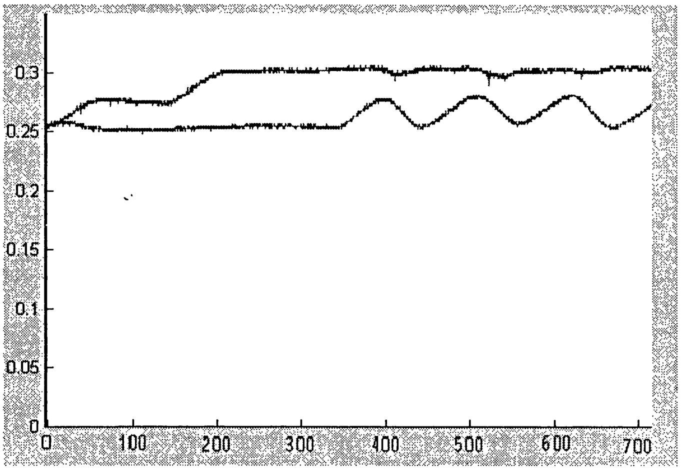

Fig. 2. Tank levels of the hydraulic system in real time.

Remark 18. The dynamic output feedback controller to preserve prescribed water levels can be obtained using the ideas of Sections 2 and 3. Actually, one can construct reduced asymptotic observer along Proposition 9, since the system (19) with the above static state feedback controller $v_{1,2}^{st}$ is exactly in the form required by the Proposition 9. Then, one can check that for all $h_{1,2}, w_{1,2} > 0$, the finite growth condition holds and apply nonlinear separation principle along Proposition 15. Summarizing, Proposition 1 is used first to compute asymptotically stabilizing static state feedback via change to new state and input coordinates, then Proposition 9 is applied in original coordinates to obtain asymptotic observer. Finally, Proposition 15 guarantees convergence of the dynamical output feedback controller obtained by replacing in a static state feedback controller states by their estimates.

Such a controller has been developed and successfully tested as well. For the sake of brevity, we skip the detailed computations and the corresponding simulations results. As a matter of fact, the more general version of that problem with unknown water influx will be analyzed in the following section, so that the interested reader may easily recover the corresponding algorithm for the case of known influx.

7. THE CASE OF UNCERTAIN INFLUX

Here we aim to consider the more involved case of the unknown influx f_e of water into the first tank. This obviously has a natural practical interpretation, when keeping the levels of water in the tanks constant despite unknown influx. Let us notice that from the practical point of view it is sufficient to have that levels of water in tanks approach asymptotically prescribed levels, while other state variables (valves openings) are kept as bounded, but possibly nonstationary functions of time. This corresponds to the well-known concept of output regulation, see Section 4 for the corresponding definition.

We aim to apply here the concepts of Sections 2 – 3. Rather than referring the appropriate propositions directly in their notation, we will make a self-contained exposition just for the water storing plant nonlinear model, thereby providing also illustration of the ideas discussed in Sections 2 – 3. These ideas provide tools for the dynamical output feedback stabilization. Therefore, to treat unknown influx f_e, a natural idea is to consider the system (19) as the system having 5 states $(h_1, w_1, h_2, w_2, f_e)^\top$ with dynamics

$$\begin{aligned}
\dot{h}_1(t) &= \tfrac{f_e}{A_t} - \tfrac{w_1(t)\sqrt{h_1(t)}}{A_t} \\
\dot{w}_1(t) &= \tfrac{K_{e1}}{T} v_1(t) - \tfrac{1}{T} w_1(t) \\
\dot{h}_2(t) &= \tfrac{w_1(t)\sqrt{h_1(t)}}{A_t} - \tfrac{w_2(t)\sqrt{h_2(t)}}{A_t} \\
\dot{w}_2(t) &= \tfrac{K_{e2}}{T} v_2(t) - \tfrac{1}{T} w_2(t) \\
\dot{f}_e &= 0.
\end{aligned} \tag{24}$$

Let l_1, l_2 be desired water levels. Obviously, due to the last row of (24), one can not stabilize such a system. Nevertheless, defining its outputs as

$$\begin{aligned} y_1 &= h_1 - l_1 \\ y_2 &= h_2 - l_2 \end{aligned} \quad . \tag{25}$$

one has exactly the output regulation concept, as introduced in Section 4.

To proceed with, apply to the above system the nonlinear coordinate change

$$\begin{aligned}
&x_1 = h_1 - l_1, \;\; x_2 = h_2 - l_2, \;\; x_3 = \tfrac{f_e}{A_t} - \tfrac{w_1\sqrt{h_1}}{A_t}, \\
&x_4 = \tfrac{w_1\sqrt{h_1}}{A_t} - \tfrac{w_2\sqrt{h_2}}{A_t}, \;\; x_5 = w_1,
\end{aligned} \tag{26}$$

having the inverse

$$\begin{aligned}
&h_1 = x_1 + l_1, \;\; h_2 = x_2 + l_2, \;\; w_1 = \left(\tfrac{f_e}{A_t} - x_3\right)\left(\tfrac{\sqrt{x_1 + l_1}}{A_t}\right)^{-1}, \\
&w_2 = \left[\left(\tfrac{f_e}{A_t} - x_3\right)\left(\tfrac{\sqrt{x_1 + l_1}}{A_t}\right)^{-1}\sqrt{x_1 + l_1} - A_t x_4\right](x_2 + l_2)^{-1/2}, \\
&f_e = A_t x_3 + x_5\sqrt{x_1 + l_1},
\end{aligned} \tag{27}$$

and static state feedback

$$\begin{aligned} u_1(t) = & -\frac{K_{e_1}\sqrt{x_1(t)+l_1}}{A_t T}v_1(t) + \frac{A_t x_3(t) - f_e}{A_t}\left(\frac{x_3(t)}{2(x_1(t)+l_1)} - \frac{1}{T}\right) \\ u_2(t) = & \left(\frac{f_e}{A_t} - x_3(t)\right)\left(\frac{x_3(t)}{2(x_1(t)+l_1)} - \frac{1}{T}\right) + \frac{K_{e_1}\sqrt{x_1(t)+l_1}}{A_t T}v_1(t) \\ & + \left(\frac{f_e}{A_t} - x_3(t) - x_4(t)\right)\left(\frac{1}{T} - \frac{x_4(t)}{2(x_2(t)+l_2)}\right) - \frac{K_{e_2}\sqrt{x_2(t)+l_2}}{A_t T}v_2(t) \end{aligned} \tag{28}$$

to obtain the system

$$\begin{aligned} & \dot{x}_1 = x_3, \quad \dot{x}_2 = x_4, \quad \dot{x}_3 = u_1, \quad \dot{x}_4 = u_2, \\ & \dot{x_5} = \frac{A_t}{\sqrt{x_1(t)+l_1}}u_1(t) + \frac{-x_5\sqrt{x_1}}{\sqrt{x_1(t)+l_1}}\left(\frac{x_3(t)}{2(x_1(t)+l_1)} - \frac{1}{T}\right) - \frac{1}{T}x_5 = \\ & \frac{K_{e_1}}{T}v_1(t) - \frac{1}{T}x_5, \\ & y_1 = x_1, \quad y_2 = x_2. \end{aligned} \tag{29}$$

Notice that we are exactly in the situation described by Proposition 9, so that all the states $x_{1,2,3,4,5}$ might be asymptotically reconstructed via reduced observer. Moreover, stabilizing the subsystem of (29) formed by its first 4 equations obviously solves the output regulation problem for (29). Summarizing, the following solution is proposed:

- Stabilize the $x_{1,2,3,4}$ subsystem of (29) via static state feedback.
- Construct the asymptotic observer for $x_{1,2,3,4,5}$.
- Using linear separation principle, stabilize the $x_{1,2,3,4}$ subsystem of (29) via dynamical output feedback.
- Recompute the obtained feedback into the original coordinates $h_{1,2}, w_{1,2}, f_e$, using during the feedback transformations the estimates of the states, obtaining in such a way the dynamical output feedback for the original system being the candidate to solve the output regulation problem.

Remark 19. The theoretical justification of the above algorithm is as follows. Re-computing of the dynamical feedback controller into the original coordinates corresponds to: 1) re-computing the static state feedback to obtain static state feedback controller in the original coordinates $h_{1,2}, w_{1,2}, f_e$, 2) re-computing the observer to obtain a certain observer in the original coordinates, 3) application of Proposition 10. Notice, that constructed observer may be only the semiglobal one, i. e. for any prescribed basin of attraction one can select sufficiently large observer gains to guarantee the estimates convergence. That is due to the fact that part of nonlinearities is not cancelled exactly when applying inverse static state feedback transformation with state replaced by the estimates. Nevertheless, these nonlinearities are of the higher-order what enables the mentioned semi-global observer design. As a consequence, the obtained dynamical output feedback controller is also a semiglobal one.

Let us derive that feedback explicitly. First, let us design the static state feedback

$$\begin{aligned} u_1 &= -c_1 x_1 - c_3 x_3, \\ u_2 &= -c_2 x_2 - c_4 x_4, c_{1,2,3,4} > 0, \end{aligned}$$

that guarantees that $x_{1,2,3,4} \to 0$ as $t \to \infty$.

The next step is to construct observer $\xi_{2,3}$ for the states $x_{3,4}$ dynamical feedback from output measurements. That is obviously possible using the simple asymptotic observer. Combining observer-controller in the usual way along separation principle we have the following closed loop system:

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= x_4 \\ \dot{x}_3 &= -c_1 x_1 - c_3 \xi_3 \\ \dot{x}_4 &= -c_2 x_2 - c_4 \xi_4 \\ \dot{\xi}_1 &= -l_1(\xi_1 - x_1) + \xi_3 \\ \dot{\xi}_2 &= -l_2(\xi_2 - x_2) + \xi_4 \\ \dot{\xi}_3 &= -l_3(\xi_1 - x_1) - c_1 x_1 - c_3 \xi_3 \\ \dot{\xi}_4 &= -l_4(\xi_2 - x_2) - c_2 x_2 - c_4 \xi_4 \end{aligned}$$

where $l_{1,2,3,4} > 0$. Actually, denoting by

$$(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)' := (x_1 - \xi_1, x_2 - \xi_2, x_3 - \varepsilon_3, x_4 - \xi_4)'$$

we have via straightforward computations:

$$\begin{aligned} \dot{\varepsilon}_1 &= -l_1 \varepsilon_1 + \varepsilon_3, & \dot{\varepsilon}_2 &= -l_2 \varepsilon_2 + \varepsilon_4, \\ \dot{\varepsilon}_3 &= -l_3 \varepsilon_1, & \dot{\varepsilon}_4 &= -l_4 \varepsilon_2. \end{aligned}$$

In other words,

$$\frac{\mathrm{d}}{\mathrm{d}t}(x_1, x_3, x_2, x_4, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)' = A(x_1, x_3, x_2, x_4, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)'$$

where the matrix A has the form

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -c_1 & -c_3 & 0 & 0 & 0 & c_3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -c_2 & -c_4 & 0 & 0 & 0 & c_4 \\ 0 & 0 & 0 & 0 & -l_1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -l_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -l_2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -l_4 & 0 \end{bmatrix}$$

and is Hurwitz by construction. Therefore we have that

$$x_1 \to 0, \quad x_2 \to 0, \quad x_3 \to 0, \quad x_4 \to 0, \text{ as } t \to \infty.$$

In other words, the dynamical feedback using only measurements of water levels of the form

$$u_1 = -c_1 x_1 - c_3 \xi_3,$$
$$u_2 = -c_2 x_2 - c_4 \xi_4,$$

with $\xi_{3,4}$ obtained from the dynamical part above, keeps asymptotically zero $x_{1,2}$ despite unknown constant water influx f_e. Notice that internally dynamics is not asymptotically stable, but it is bounded and has the favorable property that it is small for small f_e.

Now, inverting the coordinate transformations and feedback, one may compute dynamic feedback compensator for the original water storing system. Skipping all the tedious computations, the resulting closed loop system has the following form (to be self-contained, we repeat the water storing plant equations):

$$\dot{h}_1(t) = \frac{f_e}{A_t} - \frac{w_1(t)\sqrt{h_1(t)}}{A_t}$$

$$\dot{w}_1(t) = \frac{K_{e1}}{T} v_1(t) - \frac{1}{T} w_1(t)$$

$$\dot{h}_2(t) = \frac{w_1(t)\sqrt{h_1(t)}}{A_t} - \frac{w_2(t)\sqrt{h_2(t)}}{A_t}$$

$$\dot{w}_2(t) = \frac{K_{e2}}{T} v_2(t) - \frac{1}{T} w_2(t)$$

$$v_1 = \frac{A_t T}{K_{e1}\sqrt{h_1}} \left(-u_1 - \frac{w_1 \xi_3}{2A_t\sqrt{h_1}} + \frac{\sqrt{h_1}}{A_t T} w_1 \right),$$

$$v_2 = \frac{A_t T}{K_{e2}\sqrt{h_2}} \left[-u_2 - \frac{w_1 \xi_3}{2A_t\sqrt{h_1}} + \frac{\sqrt{h_1} K_{e1}}{A_t T} v_1 - \frac{\sqrt{h_1}}{A_t T} w_1 - \left(-\xi_4 + \frac{w_1\sqrt{h_1}}{A_t} \right) \frac{\xi_4}{2h_2} + \frac{1}{T} \left(-\xi_4 + \frac{w_1\sqrt{h_1}}{A_t} \right) \right]$$

$$\begin{aligned}
u_1 &= -c_1 x_1 - c_3 \xi_3, \\
u_2 &= -c_2 x_2 - c_4 \xi_4, \\
\dot{\xi}_1 &= -l_1(\xi_1 - x_1) + \xi_3, \\
\dot{\xi}_2 &= -l_2(\xi_2 - x_2) + \xi_4, \\
\dot{\xi}_3 &= -l_3(\xi_1 - x_1) - c_1 x_1 - c_3 \xi_3, \\
\dot{\xi}_4 &= -l_4(\xi_2 - x_2) - c_2 x_2 - c_4 \xi_4.
\end{aligned}$$

Finally, the unmeasurable variable w_1 can be replaced by the estimate using a copy of the fifth equation in (29), along the idea of Proposition 9. That results in the final form of the sought dynamical output feedback controller

$$\begin{aligned} v_1 &= \frac{A_t T}{K_{e1}\sqrt{h_1}}\left(-u_1 - \frac{\hat{w}_1\xi_3}{2A_t\sqrt{h_1}} + \frac{\sqrt{h_1}}{A_t T}\hat{w}_1\right), \\ v_2 &= \frac{A_t T}{K_{e2}\sqrt{h_2}}\left[-u_2 - \frac{\hat{w}_1\xi_3}{2A_t\sqrt{h_1}} + \frac{\sqrt{h_1}K_{e1}}{A_t T}v_1 - \frac{\sqrt{h_1}}{A_t T}\hat{w}_1 \right. \\ &\quad \left. - \left(-\xi_4 + \frac{\hat{w}_1\sqrt{h_1}}{A_t}\right)\frac{\xi_4}{2h_2} + \frac{1}{T}\left(-\xi_4 + \frac{\hat{w}_1\sqrt{h_1}}{A_t}\right)\right] \end{aligned} \tag{30}$$

where

$$\begin{aligned} u_1 &= -c_1 x_1 - c_3\xi_3 + a_1, \\ u_2 &= -c_2 x_2 - c_4\xi_4 + a_2, \\ \dot{\xi}_1 &= -l_1(\xi_1 - x_1) + \xi_3, \\ \dot{\xi}_2 &= -l_2(\xi_2 - x_2) + \xi_4, \\ \dot{\xi}_3 &= -l_3(\xi_1 - x_1) - c_1 x_1 - c_3\xi_3 + a_1, \\ \dot{\xi}_4 &= -l_4(\xi_2 - x_2) - c_2 x_2 - c_4\xi_4 + a_2, \\ \dot{\hat{w}}_1 &= \frac{K_{e1}}{T}v_1 - \frac{1}{T}\hat{w}_1, \end{aligned}$$

This controller has been successfully tested via computer simulations. Figure 3 shows the simulation of the system using parameter values

$$\begin{aligned} c_{1,2} &= 0.0025 \\ c_{3,4} &= 0.1 \\ l_{1,2,3,4} &= 1 \\ a_1 &= 0.00065 \\ a_2 &= 0.000675 \\ w &= 6.496875 \times 10^{-4} \end{aligned}$$

and initial conditions for the observer equal to zero. The values of a_1 and a_2 have been calculated so that the tank levels reach steady state values of $h_1 = 26\,\text{cm}$ and $h_2 = 27\,\text{cm}$. As we can see, the tank levels tend to the desired values, despite that we have no knowledge of the influx, and system states, showing that the designed control scheme performs very well.

Figure 4 shows the simulation results for the initial conditions $\xi_{1,2} = 0.25$, $\xi_{3,4} = 0$, which can actually be used since we have knowledge of this information. As expected, the tank levels reach faster the desired values and the performance of the system is better.

The actual influx of the real system is constant and unknown. For simulation purposes and in order to test the performance of the system for a different unknown influx, this quantity has been varied. Figure 5 shows the simulation results for the

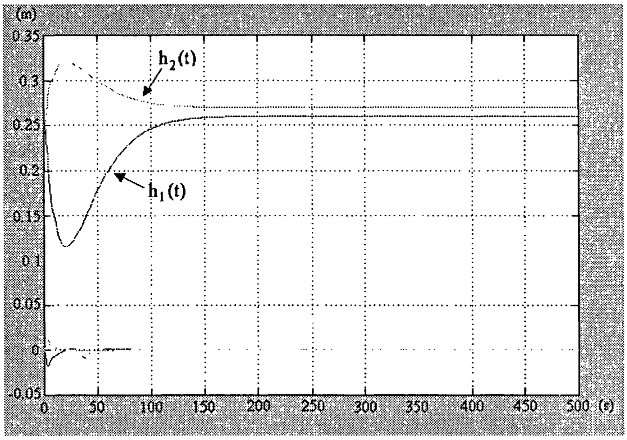

Fig. 3. Simulation with the nominal influx and zero initial conditions.

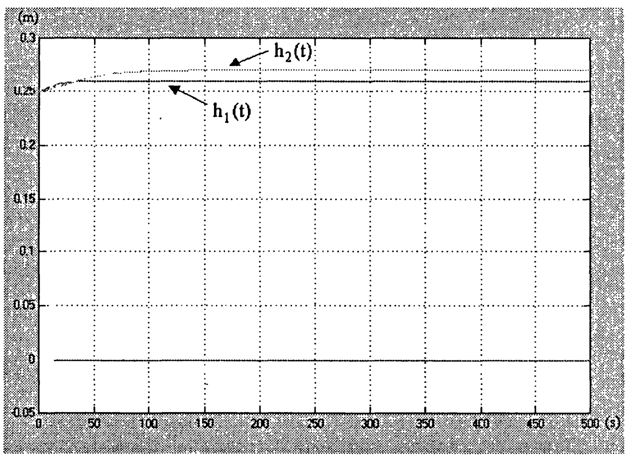

Fig. 4. Simulation with the nominal influx and initial conditions different from zero.

"real" influx of value 0.012993, i. e. 20 times bigger than the nominal influx. The values of $c_{1,2,3,4}$ and $l_{1,2,3,4}$ are the same as in the previous simulations, and the initial conditions of the observer are equal to zero. Again, the results confirm the previously theoretically justified convergence.

The values of a_1 and a_2 were calculated so that the tank levels reach the steady state values of $h_1 = 25$ cm, and $h_2 = 27$ cm. Then, at time $t = 230$ sec., a_2 is changed so that h_2 tends to 30 cm. Finally, at time $t = 440$ sec., a_1 is changed so that h_1 reaches a steady state value of 28 cm.

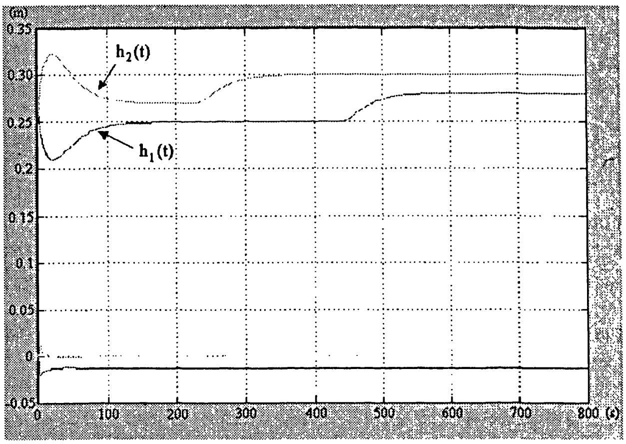

Fig. 5. Simulation with influx 20 times bigger than the nominal influx.

8. CONCLUSIONS AND OUTLOOKS

Some results on dynamic feedback stabilization have been generalized here combining the state feedback and coordinate change transformation with the output injection and, possibly different, state coordinates transformation.

As an application, the water storing system is studied and output feedback controller designed based on these nonlinear techniques. Various schemes were applied, up to the one that requires measurement of the water levels only. In other words, it performs well despite unknown constant water influx and without access to the full system state. The performance has been tested via computer simulations as well as in real time laboratory experiment. The latter one, though basically successful, requires further refinement and will be presented in detail in subsequent publications.

The interesting topic of the possible future research is the case of unknown and time-varying water influx. Three basic simplifications might be treated using the well-known design concepts here: 1) slowly varying influx, 2) bounded variation influx and 3) influx generated by known exogenous dynamical systems.

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