APPROXIMATIONS FOR THE MAXIMUM OF STOCHASTIC PROCESSES WITH DRIFT[1]

ISTVÁN BERKES[2] AND LAJOS HORVÁTH[3]

If a stochastic process can be approximated with a Wiener process with positive drift, then its maximum also can be approximated with a Wiener process with positive drift.

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1. INTRODUCTION AND RESULTS

Let $X_1, X_2, \ldots$ be a sequence of independent, identically distributed random variables with

$$EX_1 = \mu > 0 \text{ and } 0 < \text{var} X_1 = \sigma^2 < \infty. \tag{1.1}$$

The motivation of our note is the following central limit theorem due to Teicher [6]. Let

$$S(j) = \sum_{1 \le i \le j} X_i$$

and

$$0 \le \alpha < 1. \tag{1.2}$$

Theorem 1.1. If (1.1) and (1.2) hold, then

$$\frac{1}{\sigma n^{1/2-\alpha}} \left\{ \max_{1 \le j \le n} \frac{S(j)}{j^\alpha} - \mu n^{1-\alpha} \right\} \xrightarrow{\mathcal{D}} N(0,1),$$

where $N(0,1)$ denotes a standard normal random variable.

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Since

$$\frac{1}{\sigma n^{1/2-\alpha}}\left\{\frac{S(n)}{n^\alpha} - \mu n^{1-\alpha}\right\} \xrightarrow{\mathcal{D}} N(0,1),$$

Theorem 1.1 strongly suggests that

$$\max_{1\le j\le n} \frac{S(j)}{j^\alpha} - \frac{S(n)}{n^\alpha} = o_P(n^{1/2-\alpha}),$$

i. e. $S(j)/j^\alpha$ reaches its largest value on $[1,n]$ nearly at $j=n$. Indeed, Chow and Hsiung [1] proved the following result:

Theorem 1.2. If (1.1) and (1.2) hold, then

$$\max_{1\le j\le n} \frac{S(j)}{j^\alpha} - \frac{S(n)}{n^\alpha} = o(n^{1/2-\alpha}) \quad \text{a.s.} \tag{1.3}$$

For generalizations of (1.3) we refer to Chow, Hsiung and Yu [2].

We show that (1.3) holds not only for partial sums of independent identically distributed random variables, but for any process if they can be approximated with a Wiener process with drift. Let $\Gamma(t)$ be a stochastic process on $\mathcal{D}[1,\infty)$.

Theorem 1.3. We assume that there exist a Wiener process $\{W(t), 1\le t<\infty\}$ and constants $\tau>0$, $\gamma>0$ such that

$$\Gamma(t) - (\tau W(t) + \gamma t) = o(t^{1/\nu}) \quad \text{a.s. } (t\to\infty) \tag{1.4}$$

with some $\nu>2$. If (1.2) holds, then

$$\sup_{1\le t\le T} \frac{\Gamma(t)}{t^\alpha} - \frac{\Gamma(T)}{T^\alpha} = o(T^{1/\nu-\alpha}) \quad \text{a.s. } (T\to\infty) \tag{1.5}$$

and

$$\sup_{1\le t\le T} \frac{\Gamma(t)}{t^\alpha} - \frac{\tau W(T)+\gamma T}{T^\alpha} = o(T^{1/\nu-\alpha}) \quad \text{a.s. } (T\to\infty). \tag{1.6}$$

Theorem 1.3 implies immediately an improvement of the rate in (1.3) under stronger moment conditions on X_1.

Theorem 1.4. If (1.1), (1.2) hold and

$$E|X_1|^\nu < \infty \text{ with some } \nu>2, \tag{1.7}$$

then

$$\max_{1\le j\le n} \frac{S(j)}{j^\alpha} - \frac{S(n)}{n^\alpha} = o(n^{1/\nu-\alpha}) \quad \text{a.s. } (n\to\infty). \tag{1.8}$$

Theorems 1.3 and 1.4 will be proven in the next section. The following two corollaries are immediate consequences of (1.6) and the properties of the Wiener process. Let $[\cdot]$ denote the integer part function.

Corollary 1.1. We assume that the conditions of Theorem 1.3 are satisfied.

(i) If $0 \le \alpha < 1/2$, then

$$n^{\alpha-1/2}\left\{\sup_{1\le t\le [nu]+1}\frac{\Gamma(t)}{t^\alpha}-\gamma([nu]+1)^{1-\alpha}\right\}\overset{\mathcal{D}[0,1]}{\longrightarrow}\frac{\tau W(u)}{u^\alpha}. \tag{1.9}$$

(ii) If $1/2 < \alpha < 1$, then

$$n^{\alpha-1/2}\left\{\sup_{1\le t\le [nu]+1}\frac{\Gamma(t)}{t^\alpha}-\gamma([nu]+1)^{1-\alpha}\right\}\overset{\mathcal{D}[1,\infty]}{\longrightarrow}\frac{\tau W(u)}{u^\alpha}. \tag{1.10}$$

(iii) For any $0 < c_1 < c_2 < \infty$

$$n^{\alpha-1/2}\left\{\sup_{1\le t\le [nu]+1}\frac{\Gamma(t)}{t^\alpha}-\gamma([nu]+1)^{1-\alpha}\right\}\overset{\mathcal{D}[c_1,c_2]}{\longrightarrow}\frac{\tau W(u)}{u^\alpha}. \tag{1.11}$$

Corollary 1.2. If the conditions of Theorem 1.3 are satisfied, then

$$\limsup_{T\to\infty}\frac{T^\alpha}{(2T\log\log T)^{1/2}}\left|\sup_{1\le t\le T}\frac{\Gamma(t)}{t^\alpha}-\gamma T^{1-\alpha}\right|=\tau \quad \text{a.s.}$$

2. PROOFS

The first two lemmas show that $\Gamma(t)/t^\alpha$ and $(\tau W(t)+\gamma t)/t^\alpha$ will reach their largest value on $[1,T]$ on the second half of this interval.

Lemma 2.1. If (1.2) holds and $\gamma > 0$, then there is a random variable T_1 such that

$$\sup_{1\le t\le T}\frac{\tau W(t)+\gamma t}{t^\alpha}=\sup_{T/2\le t\le T}\frac{\tau W(t)+\gamma t}{t^\alpha}, \quad \text{if } T\ge T_1. \tag{2.1}$$

Proof. By the law of iterated logarithm for W we have

$$\frac{1}{T^{1-\alpha}}\sup_{1\le t\le T}\frac{\tau W(t)+\gamma t}{t^\alpha}\longrightarrow\gamma \quad \text{a.s. } (T\to\infty) \tag{2.2}$$

and

$$\frac{1}{T^{1-\alpha}}\sup_{1\le t\le T/2}\frac{\tau W(t)+\gamma t}{t^\alpha}\longrightarrow\left(\frac{1}{2}\right)^{1-\alpha}\gamma \quad \text{a.s. } (T\to\infty), \tag{2.3}$$

implying the statement of Lemma 2.1. □

Lemma 2.2. If the conditions of Theorem 1.3 are satisfied, then there is a random variable T_2 such that

$$\sup_{1\le t\le T}\frac{\Gamma(t)}{t^\alpha}=\sup_{T/2\le t\le T}\frac{\Gamma(t)}{t^\alpha},\quad\text{if } t\ge T_2.$$

Proof. The approximation in (1.4) implies that

$$\sup_{1\le t\le T}\frac{|\Gamma(t)-(\tau W(t)+\gamma t)|}{t^\alpha}=O(\max(1,T^{1/\nu-\alpha}))\quad\text{a.s.}$$

and therefore (2.2) and (2.3) yield

$$\frac{1}{T^{1-\alpha}}\sup_{1\le t\le T}\frac{\Gamma(t)}{t^\alpha}\longrightarrow\gamma\quad\text{a.s. } (t\to\infty)\tag{2.4}$$

and

$$\frac{1}{T^{1-\alpha}}\sup_{1\le t\le T/2}\frac{\Gamma(t)}{t^\alpha}\longrightarrow\left(\frac{1}{2}\right)^{1-\alpha}\gamma\quad\text{a.s. } (T\to\infty).\tag{2.5}$$

Lemma 2.2 follows from (2.4) and (2.5). □

Let $F_0(t)$ be the uniform distribution function on $[0,1]$. For any $0<\alpha<1$, $F_\alpha(t)$ denotes the uniform distribution function on $[1,1/\alpha]$.

Lemma 2.3. Let $0\le\alpha<1$ and $Y_1,Y_2,\ldots$ be independent, identically distributed random variables with distribution function $F_\alpha(t)$. Then

$$\max_{1\le j\le n}\frac{1}{j^\alpha}\sum_{1\le i\le j}Y_i=\frac{1}{n^\alpha}\sum_{1\le i\le n}Y_i.$$

Proof. It is enough to show that

$$\left(1+\frac{1}{j}\right)^\alpha\sum_{1\le i\le j}Y_i\le\sum_{1\le i\le j+1}Y_i\quad\text{for all } 1\le j<\infty.\tag{2.6}$$

Since $Y_i\ge 0$, (2.6) holds if $\alpha=0$. If $0<\alpha<1$, we observe that $1\le Y_i\le 1/\alpha$ and

$$\left(1+\frac{1}{j}\right)^\alpha-1\le\frac{\alpha}{j}.$$

Hence

$$\left\{\left(1+\frac{1}{j}\right)^\alpha-1\right\}\sum_{1\le i\le j}Y_i\le\frac{\alpha}{j}\sum_{1\le i\le j}Y_i\le 1\le Y_{j+1},$$

completing the proof of (2.6). □

Lemma 2.4. If (1.2) holds and $\tau > 0$, $\gamma > 0$, then

$$\sup_{1\le t\le T} \frac{\tau W(t)+\gamma t}{t^\alpha} - \frac{\tau W(T)+\gamma T}{T^\alpha} = O\left(\frac{\log T}{T^\alpha}\right) \quad \text{a.s.}$$

Proof. Let $\mu_* = \mu_*(\alpha)$ and $\sigma_* = \sigma_*(\alpha)$ be the mean and standard deviation of a random variable with distribution function $F_\alpha(t)$. Next we define

$$c = \left(\frac{\mu_*}{\gamma}\frac{\tau}{\sigma_*}\right)^2. \tag{2.7}$$

Obviously,

$$\begin{aligned}\sup_{1\le t\le T} \frac{\tau W(t)+\gamma t}{t^\alpha} &= \tau \sup_{1/c\le s\le T/c} \frac{W(cs)+\frac{\gamma}{\tau}cs}{(cs)^\alpha} \\ &= \tau c^{1/2-\alpha} \sup_{1/c\le s\le T/c} \frac{W_1(s)+\frac{\gamma}{\tau}c^{1/2}s}{s^\alpha},\end{aligned} \tag{2.8}$$

where

$$W_1(s) = c^{-1/2}W(cs),\ \ 0\le s<\infty \tag{2.9}$$

is a Wiener process. By (2.7) and (2.8) we have

$$\sup_{1\le t\le T} \frac{\tau W(t)+\gamma t}{t^\alpha} = \frac{\tau}{\sigma_*}c^{1/2-\alpha} \sup_{1/c\le t\le T/c} \frac{\sigma_* W_1(t)+\mu_* t}{t^\alpha}. \tag{2.10}$$

Using the K–M–T approximation (cf. Komlós, Major and Tusnády [3, 4] and Major [5]) we can define $Y_1^*, Y_1^*, \dots$, a sequence of independent, identically distributed random variables with distribution function $F_\alpha(t)$ such that

$$\sum_{1\le i\le t} Y_i^* - (\sigma_* W_1(t)+\mu_* t) = O(\log t) \quad \text{a.s. } (t\to\infty). \tag{2.11}$$

By Lemmas 2.1, 2.2 and (2.10) there is random variable T_0 such that

$$\sup_{1/c\le t\le T/c} \frac{\sigma_* W_1(t)+\mu_* t}{t^\alpha} = \sup_{T/(2c)\le t\le T/c} \frac{\sigma_* W_1(t)+\mu_* t}{t^\alpha}$$

and

$$\sup_{1/c\le t\le T/c} \frac{1}{t^\alpha}\sum_{1\le i\le t} Y_i^* = \sup_{T/(2c)\le t\le T/c} \frac{1}{t^\alpha}\sum_{1\le i\le t} Y_i^*,$$

if $T\ge T_0$. Hence (2.11) yields, as $T\to\infty$,

$$\sup_{1/c\le t\le T/c} \frac{\sigma_* W_1(t)+\mu_* t}{t^\alpha} - \sup_{1/c\le t\le T/c} \frac{1}{t^\alpha}\sum_{1\le i\le t} Y_i^* = O(T^{-\alpha}\log T) \quad \text{a.s.} \tag{2.12}$$

Putting together Lemma 2.3 and (2.11) we conclude

$$\begin{aligned}\sup_{1/c\le t\le T/c}\frac{1}{t^\alpha}\sum_{1\le i\le t}Y_i^* &= \left(\frac{T}{c}\right)^{-\alpha}\sum_{1\le i\le T/c}Y_i^*\\ &= \left(\frac{T}{c}\right)^{-\alpha}\left\{\sigma_* W_1\left(\frac{T}{c}\right)+\mu_*\frac{T}{c}\right\}+O(T^{-\alpha}\log T)\quad\text{a.s.}\end{aligned}\tag{2.13}$$

$(T\to\infty)$. Next we use (2.7), (2.9) and (2.10) to obtain

$$\begin{aligned}&\left(\frac{T}{c}\right)^{-\alpha}\left\{\sigma_* W_1\left(\frac{T}{c}\right)+\mu_*\frac{T}{c}\right\}\\ &\qquad=\left(\frac{T}{c}\right)^{-\alpha}\left\{\sigma_* c^{-1/2}W(T)+\mu_*\frac{T}{c}\right\}\\ &\qquad=\frac{1}{T^\alpha}c^{\alpha-1/2}\sigma_*\left\{W(T)+\frac{\mu_*}{\sigma_*}c^{-1/2}T\right\}\\ &\qquad=\frac{1}{T^\alpha}c^{\alpha-1/2}\frac{\sigma_*}{\tau}\left\{\tau W(T)+\gamma T\right\}.\end{aligned}\tag{2.14}$$

Lemma 2.4 now follows from (2.8) and (2.12) – (2.14). □

P r o o f o f T h e o r e m 1.3. Using (1.4) and Lemmas 2.1 and 2.2 we get that

$$\sup_{1\le t\le T}\frac{\Gamma(t)}{t^\alpha}-\sup_{1\le t\le T}\frac{\tau W(t)+\gamma t}{t^\alpha}=o(T^{1/\nu-\alpha})\quad\text{a.s.}$$

Hence Theorem 1.3 follows from Lemma 2.4. □

P r o o f o f T h e o r e m 1.4. By the K–M–T approximation there is a Wiener process $\{W(t), 0\le t<\infty\}$ such that

$$S(t)-(\sigma W(t)+\mu t)=o(t^{1/\nu})\quad\text{a.s.}\ (t\to\infty).$$

Hence (1.4) holds and the result follows from Theorem 1.3. □

P r o o f o f C o r o l l a r y 1.1. Assume that $0\le\alpha<1/2$. By Theorem 1.3 there is a Wiener process $\{W(t), 0\le t<\infty\}$ such that

$$n^{\alpha-1/2}\sup_{0\le u\le 1}\left|\sup_{1\le t\le[nu]+1}\frac{\Gamma(t)}{t^\alpha}-\frac{\tau W([nu]+1)+\gamma([nu]+1)}{([nu]+1)^\alpha}\right|=o(n^{1/\nu-1/2})\quad\text{a.s.}$$

Hence (1.9) is proven if

$$n^{\alpha-1/2}\frac{W([nu]+1)}{([nu]+1)^\alpha}\ \xrightarrow{\mathcal{D}[0,1]}\ \frac{W(u)}{u^\alpha}.\tag{2.15}$$

Obviously,

$$\sup_{0\le u\le\epsilon}\frac{|W([nu]+1)|}{([nu]+1)^\alpha}\le\sup_{0\le u\le[n\epsilon]+1}\frac{|W(u)|}{u^\alpha}$$

and by the scale transformation of W we have

$$n^{\alpha-1/2}\sup_{0\le u\le[n\epsilon]+1}\frac{|W(u)|}{u^\alpha}\stackrel{\mathcal{D}}{=}\sup_{0\le u\le[n\epsilon]+1}\frac{|W(u/n)|}{(u/n)^\alpha}=\sup_{0\le u\le([n\epsilon]+1)/n}\frac{|W(u)|}{u^\alpha}.$$

By the law of the iterated logarithm for W at 0 we have

$$\lim_{\epsilon\to0}\limsup_{n\to\infty}P\left\{\sup_{0\le u\le([n\epsilon]+1)/n}\frac{|W(u)|}{u^\alpha}>\delta\right\}=0\ \text{ for all }\ \delta>0. \tag{2.16}$$

The scale transformation of W and the almost sure continuity of $W(u)/u^\alpha$ on $[c_1,c_2]$, $0<c_1\le c_2$ yield

$$n^{\alpha-1/2}\frac{W([nu]+1)}{([nu]+1)^\alpha}\stackrel{\mathcal{D}[c_1,c_2]}{\longrightarrow}\frac{W(u)}{u^\alpha}. \tag{2.17}$$

Clearly, (2.15) follows from (2.16) and (2.17).

Assume that $1/2<\alpha<1$. Using again Theorem 1.3 there is a Wiener process $\{W(t),0\le t<\infty\}$ such that

$$n^{\alpha-1/2}\sup_{1\le u<\infty}\left|\sup_{1\le t\le[nu]+1}\frac{\Gamma(t)}{t^\alpha}-\frac{\tau W([nu]+1)+\gamma([nu]+1)}{([nu]+1)^\alpha}\right|=o(1)\ \text{ a.s.}$$

Hence (1.10) is proven if we show that

$$n^{\alpha-1/2}\frac{W([nu]+1)}{([nu]+1)^\alpha}\stackrel{\mathcal{D}[1,\infty]}{\longrightarrow}\frac{W(u)}{u^\alpha}. \tag{2.18}$$

For any $T>0$ we have that

$$\sup_{T\le u<\infty}\frac{|W([nu]+1)|}{([nu]+1)^\alpha}\le\sup_{[nT]\le u<\infty}\frac{|W(u)|}{u^\alpha}$$

and by the scale transformation of W we have

$$n^{\alpha-1/2}\sup_{[nT]\le u<\infty}\frac{|W(u)|}{u^\alpha}\stackrel{\mathcal{D}}{=}\sup_{[nT]/n\le u<\infty}\frac{|W(u)|}{u^\alpha}.$$

The law of the iterated logarithm for W at ∞ yields that

$$\sup_{T\le u\le\infty}\frac{|W(u)|}{u^\alpha}\ \to\ 0\ \text{a.s.}\ (T\to\infty). \tag{2.19}$$

Now (2.18) follows from (2.17) and (2.19).

Theorem 1.3 and (2.17) imply immediately (1.11). □

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Dr. István Berkes, A. Rényi Institute of Mathematics, Hungarian Academy of Sciences, P. O. Box 127, H-1364 Budapest. Hungary.
e-mail: berkes@renyi.hu

Dr. Lajos Horváth, Department of Mathematics, University of Utah, 155 South 1440 East, Salt Lake City, UT 84112-0090. U. S. A.
e-mail: horvath@math.utah.edu