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AN ADDITIVE DECOMPOSITION OF FUZZY NUMBERS

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Hong and Do [4] improved Mareš' [7] result about additive decomposition of fuzzy quantities concerning an equivalence relation. But there still exists an open question which is the limitation to fuzzy quantities on R (the set of real numbers) with bounded supports in the presented theory. In this paper we restrict ourselves to fuzzy numbers, which are fuzzy quantities of the real line R with convex, normalized and upper semicontinuous membership function and prove this open question.

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1. INTRODUCTION

Perspective applications of the algebra of fuzzy quantities lie in multicriterial decision-making theory [11], fuzzy analogies of the network analysis [9] and PERT method [10]. The result on group properties of the addition operation over fuzzy quantities presented in a previous paper by Mareš [8] imply some interesting consequences concerning the equivalence relation between fuzzy quantities. Mareš' detailed analysis of the equivalence properties implies some conclusions regarding the interpretation of different sources of fuzziness entering the fuzzy quantities model.

In [7], Mareš used the method demanding to limit the investigation to fuzzy quantities with finite support to prove the main theorem. Mareš mentioned at the end of [7] that one of the open problems is its limitation to fuzzy quantities with finite supports. Recently Hong and Do [4] defined a more refined equivalent relation than Mareš [7] and improved Mareš' result. In this paper we restrict ourselves to fuzzy numbers, which are fuzzy quantities of the real line R with convex, normalized and upper semicontinuous membership function and prove the open question.

2. DEFINITIONS

Let R denote the sets of all real numbers. A fuzzy quantity $\mathbf{a}$ with real values is described by its membership function $f_{\mathbf{a}} : R \to [0,1]$ with the usual interpretation. We suppose that the support set of $\mathbf{a}$, i.e. the set

$$\{x \in R : f_{\mathbf{a}}(x) > 0\}$$

is non–empty and limited. By $\mathcal{R}$ we denote the set of all fuzzy quantities with real values and limited non–empty supports.

If $\mathbf{a} \in \mathcal{R}$, $\mathbf{b} \in \mathcal{R}$ are fuzzy quantities with membership functions $f_{\mathbf{a}}$ and $f_{\mathbf{b}}$, respectively, then we write that $\mathbf{a} = \mathbf{b}$ iff $f_{\mathbf{a}}(x) = f_{\mathbf{b}}(x)$ for all $x \in R$.

The sum $\mathbf{a} + \mathbf{b}$ is also a fuzzy quantity from R with a membership function $f_{\mathbf{a}+\mathbf{b}} : R \to [0,1]$ such that

$$\begin{aligned} f_{\mathbf{a}+\mathbf{b}}(x) &= \sup_{y \in R}(\min(f_{\mathbf{a}}(y), f_{\mathbf{b}}(x-y))) \\ &= \sup_{z \in R}(\min(f_{\mathbf{a}}(x-z), f_{\mathbf{b}}(z))) \end{aligned}$$

(cf. [1, 2]). If $\mathbf{a} \in \mathcal{R}$ then the fuzzy quantity $-\mathbf{a} \in \mathcal{R}$ such that

$$f_{-\mathbf{a}}(x) = f_{\mathbf{a}}(-x), \qquad x \in R$$

is called the opposite element to $\mathbf{a}$. The fuzzy quantity $\mathbf{0} \in \mathcal{R}$ such that

$$f_{\mathbf{0}}(x) = \begin{cases} 1, & \text{for } x = 0, \\ 0, & \text{for } x \neq 0, \end{cases}$$

is called the zero element of $\mathcal{R}$.

We say that a fuzzy quantity $\mathbf{s} \in \mathcal{R}$ is symmetric iff

$$f_{\mathbf{s}}(x) = f_{\mathbf{s}}(-x)$$

for all $x \in R$, i.e. iff $\mathbf{s} = -\mathbf{s}$. The set of all symmetric fuzzy quantities will be denoted by $\mathcal{S} \subset \mathcal{R}$. Obviously $\mathbf{a} + (-\mathbf{a}) \in \mathcal{S}$ for any $\mathbf{a} \in \mathcal{R}$.

If $\mathbf{a} \in \mathcal{R}$ is a fuzzy quantity then we denote

$$\sigma_{\mathbf{a}} = \sup(f_{\mathbf{a}}(x) : x \in R)$$

and by $\mathbf{a}^*$ we denote the fuzzy quantity from R for which

$$f_{\mathbf{a}^*}(x) = f_{\mathbf{a}}(x) \cdot \sigma_{\mathbf{a}}^{-1}.$$

The fuzzy quantity $\mathbf{a}^*$ will be called the normalization of the fuzzy quantity $\mathbf{a}$. Fuzzy quantity $\mathbf{a}$ such that $\sigma_{\mathbf{a}} = 1$ is called normalized.

We denote by $\mathcal{R}^*$ the set of all normalized fuzzy quantities and generally if $\mathcal{T} \subset \mathcal{R}$ is a set of fuzzy quantities then

$$\mathcal{T}^* = \mathcal{T} \cap \mathcal{R}^*$$

is the subset of $\mathcal{T}$ containing exactly all normalized elements from $\mathcal{T}$.

Let us denote the set of fuzzy quantities with finite supports by $\mathcal{R}_0 \subset \mathcal{R}$, and generally if $\mathcal{T} \subset \mathcal{R}$ is a set of fuzzy quantities then $\mathcal{T}_0 = \mathcal{T} \cap \mathcal{R}_0$ is the set of exactly all elements from $\mathcal{T}$ with finite support.

In [7], Mareš used the following definitions and proved a theorem.

Definition 1. (Mareš [7]) If $\mathbf{a} \in \mathcal{R}$ and $\mathbf{b} \in \mathcal{R}$ are fuzzy quantities then we say that $\mathbf{a}$ is equivalent to $\mathbf{b}$ and write $\mathbf{a} \sim_M \mathbf{b}$ iff there exist symmetric fuzzy quantities $\mathrm{s} \in \mathcal{S}, \mathbf{t} \in \mathcal{S}$ such that

$$\mathbf{a}^* + \mathbf{s}^* = \mathbf{b}^* + \mathbf{t}^*.$$

Definition 2. (Mareš [7]) Let $\mathbf{a} \in \mathcal{R}_0$, let $\mathbf{a}^*$ be the normalization of $\mathbf{a}$, and let $[\mathbf{a}] \in \mathcal{R}_0^*$ be a fuzzy quantity such that $\mathbf{a}^* = [\mathbf{a}] + \mathbf{s}$ for some $\mathbf{s} \in \mathcal{S}^*$, if $[\mathbf{a}] = \mathbf{b} + \mathbf{s}'$ for some $\mathbf{b} \in \mathcal{R}^*$, $\mathbf{s}' \in \mathcal{S}^*$ then $\mathbf{s}' = \mathbf{0}$. Then the fuzzy quantity $[\mathbf{a}]$ will be called the core of the fuzzy quantity $\mathbf{a}$.

Theorem 1. (Mareš [7]) For every fuzzy quantity $\mathbf{a} \in \mathcal{R}_0$ its core $[\mathbf{a}]$ exists. If $[\mathbf{a}_1]$ and $[\mathbf{a}_2]$ are cores of the same fuzzy quantity $\mathbf{a} \in \mathcal{R}_0$ then they are equivalent, $[\mathbf{a}_1] \sim_M [\mathbf{a}_2]$.

Mareš mentioned at the end of [7] that one of the open problems is its limitation to fuzzy quantities with finite supports. In [4], Hong and Do used the following definitions and improved Mareš result.

Definition 3. (Hong and Do [4]) If $\mathbf{a} \in \mathcal{R}$ and $\mathbf{b} \in \mathcal{R}$ are fuzzy quantities then we say that $\mathbf{a}$ is equivalent to $\mathbf{b}$ and write $\mathbf{a} \sim_{DH} \mathbf{b}$ iff there exist symmetric fuzzy quantities $\mathbf{s} \in \mathcal{S}_c, \mathbf{t} \in \mathcal{S}_c$ such that

$$\mathbf{a}^* + \mathbf{s}^* = \mathbf{b}^* + \mathbf{t}^*,$$

where $S_c = \{s \colon s \text{ is symmetric and } f_s(0) = \sigma_s\}$.

Definition 4. (Hong and Do [4]) Let $\mathbf{a} \in \mathcal{R}$, let $\mathbf{a}^*$ be the normalization of $\mathbf{a}$, and let $[\mathbf{a}] \in \mathcal{R}^*$ be a fuzzy quantity such that $\mathbf{a}^* = [\mathbf{a}] + \mathbf{s}$ for some $\mathbf{s} \in \mathcal{S}_c^*$, if $[\mathbf{a}] = \mathbf{b} + \mathbf{s}'$ for some $\mathbf{b} \in \mathcal{R}^*$, $\mathbf{s} \in \mathcal{S}_c^*$ then $\mathbf{s}' = \mathbf{0}$. Then the fuzzy quantity $[\mathbf{a}]$ will be called the core of the fuzzy quantity $\mathbf{a}$.

We denote by $\mathcal{R}^\circ$ the set of fuzzy quantities with bounded supports.

Theorem 2. (Hong and Do [4]) For every fuzzy quantity $\mathbf{a} \in \mathcal{R}^\circ$ its core $[\mathbf{a}]$ exists. If $[\mathbf{a}_1]$ and $[\mathbf{a}_2]$ are cores of the same fuzzy quantity $\mathbf{a} \in \mathcal{R}^\circ$ then they are equivalent, $[\mathbf{a}_1] \sim_{DH} [\mathbf{a}_2]$.

3. ADDITIVE DECOMPOSITION OF FUZZY NUMBERS

In this section we restrict ourselves to fuzzy numbers. Actually fuzzy numbers cover most of theories and large scale of possible applications. Under this restriction, we consider the decomposition problem.

A fuzzy number $\mathbf{a}$ is a fuzzy quantity of the real line R with a convex, normalized and upper semicontinuous membership function [3]. The crisp set of reals that belong to the fuzzy number $\mathbf{a}$ at least to the degree α is called the α-level set :

$$\mathbf{a}_\alpha = \{x \in \mathcal{R}; f_{\mathbf{a}}(x) \geq \alpha\}, \quad 0 < \alpha \leq 1.$$

It is noted that $\mathbf{a}_\alpha$ is closed interval since $f_\mathbf{a}$ is upper semicontinuous and unimodal. It is well known that for any $x \in R$

$$f_\mathbf{a}(x) = \sup_{x \in (0,1]} \min(\alpha, \chi_{\mathbf{a}_\alpha}(x)) \tag{1}$$

where $\chi_{\mathbf{a}_\alpha}$ stands for the characteristic function of set $\mathbf{a}_\alpha$, $\alpha \in (0,1]$.

The following proposition is easy to check, noting that $\mathcal{S}_c = \mathcal{S}$ and $\mathcal{T}^* = \mathcal{T}$, $\mathbf{a}^* = \mathbf{a}$ for any $\mathcal{T} \subset \mathcal{R}$ and $\mathbf{a} \in \mathcal{R}$.

Proposition 1. $\sim_M$ is same as $\sim_{DH}$.

There two equivalence can be represented simply as follows:

Definition 5. If $\mathbf{a}$ and $\mathbf{b}$ are fuzzy numbers then we say that $\mathbf{a}$ is equivalent to $\mathbf{b}$ and write $\mathbf{a} \sim \mathbf{b}$ iff there exist symmetric fuzzy numbers $\mathbf{s}$ and $\mathbf{t}$ such that

$$\mathbf{a} + \mathbf{s} = \mathbf{b} + \mathbf{t}.$$

Similarly the core of fuzzy number $\mathbf{a}$ is defined as follows.

Definition 6. Let $\mathbf{a}$ be a fuzzy number, and let $[\mathbf{a}]$ be a fuzzy number such that $\mathbf{a} = [\mathbf{a}] + \mathbf{s}$ for some symmetric fuzzy number, if $[\mathbf{a}] = \mathbf{b} + \mathbf{s}'$ for some fuzzy number $\mathbf{b}$ and for some symmetric fuzzy number $\mathbf{s}'$ then $\mathbf{s}' = \mathbf{0}$. Then the fuzzy number $[\mathbf{a}]$ will be called the *core of the fuzzy number* $\mathbf{a}$.

We now prove the decomposition problem.

Theorem 3. For any fuzzy number $\mathbf{a}$ with bounded $\mathbf{a}_{\alpha_0}$ for some $\alpha_0 > 0$, its core $[\mathbf{a}]$ exists and any cores of the same fuzzy number $\mathbf{a}$ are equivalent.

Proof. Let $\mathbf{a}$ be a fuzzy number with bounded $\mathbf{a}_{\alpha_0}$ for some $\alpha_0 > 0$ and $\mathcal{A} = \{(\mathbf{b}, \mathbf{s}) : \mathbf{a} = \mathbf{b} + \mathbf{s}, \mathbf{b}$ is fuzzy number and $\mathbf{s}$ is symmetric fuzzy number$\}$. We write $(\mathbf{b}, \mathbf{s}) \le (\mathbf{b}', \mathbf{s}')$ iff $f_\mathbf{b}(x) \ge f_{\mathbf{b}'}(x)$ and $f_\mathbf{s}(x) \le f_{\mathbf{s}'}(x)$ for all $x \in R$. Then $(\mathcal{A}, \le)$ is a partially ordered set. Let $\mathcal{B}$ be a chain of $\mathcal{A}$ and let $\sup_{(\mathbf{b},\mathbf{s})\in\mathcal{B}} f_\mathbf{s}(x) = f^*(x)$ and $\inf_{(\mathbf{b},\mathbf{s})\in\mathcal{B}} f_\mathbf{b}(x) = g^*(x)$. Then g^* is a membership function of fuzzy number, since normality comes from $\{g^* = 1\} = \cap_{(\mathbf{b},\mathbf{s})\in\mathcal{B}}\{f_\mathbf{b} = 1\} \ne \emptyset$ by the finite intersection property [13], upper semicontinuity comes from the fact that $\{g^* \ge \alpha\} = \cap_{(\mathbf{b},\mathbf{s})\in\mathcal{B}}\{f_\mathbf{b} \ge \alpha\}$ is closed and unimodality is immediate. It is noted that for any $1 \ge \alpha \ge 0$, since $\{f_\mathbf{a} \ge \alpha\} = \{f_\mathbf{b} \ge \alpha\} + \{f_\mathbf{s} \ge \alpha\}$ (see [5,6]),

$$\begin{aligned} \{f_\mathbf{a} \ge \alpha\} &= \cap_{(\mathbf{b},\mathbf{s})\in\mathcal{B}}\{f_\mathbf{b} \ge \alpha\} + \overline{\cup_{(\mathbf{b},\mathbf{s})\in\mathcal{B}}\{f_\mathbf{s} \ge \alpha\}} \\ &= \{g^* \ge \alpha\} + \overline{\{f^* \ge \alpha\}}, \end{aligned} \tag{2}$$

where $\overline{A}$ is closure of set A and $A + B = \{x + y | x \in A, y \in B\}$. Now we define f^{**} as

$$f^{**}(x) = \sup_{\alpha \in (0,1]} \min\left(\alpha, \chi_{\overline{\{f^* \ge \alpha\}}}(x)\right).$$

Then clearly f^{**} is a membership function of a symmetric fuzzy number and $f^{**}(x) \geq f^{*}(x)$ for $x \in R$. From (1) and (2) we see that

$$f_{\mathbf{a}}(z) = \sup_{x+y=z} \min\{g^{*}(x), f^{**}(y)\}.$$

Let $\mathbf{b}^*$ and $\mathbf{s}^*$ be the fuzzy numbers with $f_{\mathbf{b}^*} = g^*$ and $f_{\mathbf{s}^*} = f^{**}$. Then clearly $(\mathbf{b}^*, \mathbf{s}^*) \in \mathcal{A}$ and $(\mathbf{b}^*, \mathbf{s}^*)$ is an upper bound of $\mathcal{B}$. Then by Zorn's Lemma ([12, Theorem 5,16]), there exists a maximal element, say $(\mathbf{b}', \mathbf{s}')$. Then $\mathbf{b}'$ is the core of the fuzzy number $\mathbf{a}$. For, let $\mathbf{b}' = \mathbf{b} + \mathbf{s}$, where $\mathbf{s}$ is a symmetric fuzzy number. Then $\mathbf{a} = \mathbf{b} + \mathbf{s} + \mathbf{s}'$ and $(\mathbf{b}, \mathbf{s} + \mathbf{s}') \geq (\mathbf{b}', \mathbf{s}')$ in $\mathcal{A}$. But $(\mathbf{b}', \mathbf{s}')$ is a maximal element, which imply $\mathbf{s} + \mathbf{s}' = \mathbf{s}'$, i. e., $\mathbf{s} = \mathbf{0}$. The proof is complete. □

The condition that $\mathbf{a}_{\alpha_0}$ is bounded for some $\alpha_0 > 0$ in Theorem 3 is essential. For this, see the following example.

Example 1. Let $\mathbf{a}$ be a fuzzy number with membership function

$$f_{\mathbf{a}}(x) = \chi_{[0,\infty)}(x) = \begin{cases} 1 & \text{if } x \in [0, \infty), \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbf{a} = [\mathbf{a}] + \mathbf{s}$ for some fuzzy number $[\mathbf{a}]$ and some symmetric fuzzy number $\mathbf{s}$. Then $[\mathbf{a}]$ and $\mathbf{s}$ should be of the forms that

$$f_{[\mathbf{a}]}(x) = \chi_{[t,\infty)}(x) \text{ for some } t \geq 0$$

and

$$f_{\mathbf{s}}(x) = \chi_{[-t,t]}(x).$$

Let $\mathbf{b}$ be the fuzzy number with $f_{\mathbf{b}}(x) = \chi_{[t+1,\infty)}(x)$ and $\mathbf{s}'$ be the fuzzy quantity with $f_{\mathbf{s}'}(x) = \chi_{[-1,1]}(x)$.

Then $[\mathbf{a}] = \mathbf{b} + \mathbf{s}'$. From this fact, we see that the core of $\mathbf{a}$ does not exist.

In some sense, the condition that $\mathbf{a}$ has a bounded $\mathbf{a}_{\alpha_0}$ for some $\alpha_0 \geq 0$ is necessary and sufficient condition for the existence of core of a fuzzy number.

Theorem 4. Let $\mathbf{a}$ be a fuzzy number such that $f_{\mathbf{a}} \neq \chi_R$. Then its core $[\mathbf{a}]$ exists if only if $\mathbf{a}_{\alpha_0}$ is bounded for some $0 \leq \alpha_0 \leq 1$.

Proof. Sufficiency is immediate from Theorem 3. Now suppose that $\mathbf{a}_\alpha$ is unbounded for any $0 \leq \alpha \leq 1$. Then $\{f_{\mathbf{a}} = 1\}$ is a set of the form either $[t, \infty)$ or $(-\infty, t]$ for some $t \in R$. We prove the case of $\{f_{\mathbf{a}} = 1\} = [t, \infty)$. The case of $\{f_{\mathbf{a}} = 1\} = (-\infty, t]$ is similar. Let $\{f_{\mathbf{a}} = 1\} = [t_0, \infty)$ and let $\mathbf{a} = \mathbf{a}' + \mathbf{s}$ for some fuzzy number $\mathbf{a}'$ and some symmetric fuzzy number $\mathbf{s}$. Then $f_{\mathbf{a}'}(x) = f_{\mathbf{a}}(x - \beta)$ and $f_{\mathbf{s}}(x) = \chi_{[-\beta,\beta]}(x)$ for some $\beta \geq 0$. But $\mathbf{a}' = \mathbf{b} + \mathbf{s}'$ where $f_{\mathbf{b}}(x) = f_{\mathbf{a}}(x - (\beta + 1))$, $f_{\mathbf{s}'}(x) = \chi_{[-1,1]}(x)$, which means that $\mathbf{a}'$ not a core of $\mathbf{a}$. This completes the proof. □

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