KYBERNETIKA — VOLUME 39 (2003), NUMBER 3, PAGES 281–288

# $T$–EQUIVALENCES GENERATED BY SHAPE FUNCTION ON THE REAL LINE

DUG HUN HONG

This paper is devoted to give a new method of generating $T$-equivalence using shape function and finding the exact calculation formulas of $T$-equivalence induced by shape function on the real line. Some illustrative examples are given.

*Keywords:* fuzzy number, fuzzy relation, T-norm, T-equivalence, shape function
*AMS Subject Classification:* 26A21, 03E02

## 1. INTRODUCTION

For the fuzzy set-theoretical modelling of verbal quantities and computing with these quantities, it appears useful to part the class of real numbers into fuzzy equivalence classes. Jacas and Recasens [8] considered the idea of generating fuzzy numbers as equivalence classes of a $T$-indistinguishability operator based on a scale function. The theoretical approach suggested in [10] and further developed in [11] indicates that partitions based on the concept of a shape function can be especially significant. De Baets et al [2] and Marková [12] characterized that the shapes by means of which $T$-equivalences can be generated, are based on the knowledge of idempotents of the $T$-addition of fuzzy numbers.

In this paper, we give a new method of generating $T$-equivalence using shape function and finding the exact calculation formulas of $T$-equivalence induced by shape function on the real line. Some illustrative examples are given.

## 2. PRELIMINARIES

**Definition 1.** (Jacas and Recasens [8]) A fuzzy number is a mapping $A : R \to [0,1]$ such that there exists $a \in R$ with $A(a) = 1$ and $A$ is increasing on $(-\infty, a]$ and $A$ is decreasing on $[a, \infty)$.

**Definition 2.** (De Baets and Mesiar [3]) Consider a $t$-norm $T$. A binary fuzzy relation $E$ on an universe $X$ is called a $T$-equivalence on $X$ if and only if it is reflexive, symmetric and $T$-transitive, i.e. if and only if for any $(x, y, z)$ in $X^3$:

(i) $E(x, x) = 1$;

(ii) $E(x,y) = E(y,x)$;

(iii) $T(E(x,y),\ E(y,z)) \leq E(x,z)$.

**Definition 3.** (Jacas and Recasens [8]) A scale is a continuous non-decreasing surjective monotonic mapping $S : R \to R$.

**Definition 4.** A shape is a non-increasing mapping $\phi : R^+ \to [0,1]$ such that $\phi(0) = 1$.

**Definition 5.** A mapping $d : X^2 \to [0,\infty]$ is called a pseudo-metric on $X$ if and only if for any $(x,y,z)$ in $X^3$

(i) $d(x,x) = 0$;

(ii) $d(x,y) = d(y,x)$;

(iii) $d(x,z) \leq d(x,y) + d(y,z)$.

It is called a metric if it moreover satisfies, for any $(x,y) \in X^2$

(iv) $d(x,y) = 0 \Leftrightarrow x = y$.

Consider a scale $s$, then the mapping $d_s : R^2 \to R^+$ defined by

$$d_s(x,y) = |s(x) - s(y)|$$

is a pseudo-metric on $R$. Now consider a shape $\phi$, then we construct the binary fuzzy relation $E_{s,\phi}$ as follows:

$$E_{s,\phi}(x,y) = \phi(|s(x) - s(y)|).$$

**Definition 6.** A generator (or source of vagueness) $g$ is a scale such that $g(0) = 0$.

A function $T : [0,1] \times [0,1] \to [0,1]$ is said to be a triangular norm [9, 14] ($t$-norm for short) iff $T$ is symmetric, associative, non-decreasing in each argument, and $T(x,1) = x$ for all $x \in [0,1]$, and, in general, $T(x_1,\cdots,x_n) = T(T(\ldots T(T(x_1,x_2),x_3), \ldots,x_{n-1}),x_n)$. Some well-known continuous $t$-norms are the minimum operator $T_M$, the algebraic product $T_P$ and the Lukasiewicz $t$-norm $T_L$ defined by $T_L(x,y) = \max(x+y-1,0)$. The minimum operator $T_M$ is the strongest (greatest) $t$-norm. The weakest (smallest) $t$-norm $T_W$ is defined by

$$T_W(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1, \\ 0, & \text{elsewhere.} \end{cases}$$

We will call $t$-norm $T$ is Archimedean if and only if $T$ is continuous and $T(x,x) < x$ for all $x \in (0,1)$. Every Archimedean $t$-norm $T$ is representable by a continuous and decreasing function $f : [0,1] \to [0,\infty]$ with $f(1) = 0$ and

$$T(x_1,\cdots,x_n) = f^{[-1]}(f(x_1) + \cdots + f(x_n))$$

for all $x_i \in [0,1]$, $1 \le i \le n$, where $f^{[-1]}$ is the pseudo-inverse of $f$, defined by

$$f^{[-1]}(y) = \begin{cases} f^{-1}(y) & \text{if } y \in [0, f(0)], \\ 0 & \text{if } y \in [f(0), \infty]. \end{cases}$$

The function $f$ is the additive generator of $T$. If $T = T_P$, then $f(x) = \log x^{-1}$ and if $T = T_L$, then $f(x) = 1 - x$.

For arbitrary fuzzy numbers $A_i$, $i = 1, \cdots, n$, $n \in N$, on the real line, their $T$-sum is defined by means of the extension principle as follows:

$$A_1 \oplus_T \cdots \oplus_T A_n(z) = \sup_{x_1 + \cdots + x_n = z} T(A_1(x_1), \ldots, A_n(x_n)), \; z \in R.$$

**Definition 7.** Let $J$ be a finite or countable set. Let $\{T_i | i \in J\}$ be a collection of $t$-norms and $\{(a_i, b_i) | i \in J\}$ a collection of disjoint intervals in $[0,1]$. We call ordinal sum of $t$-norms $\{T_i | i \in J\}$ to the following $t$-norm :

$$T(x,y) = \begin{cases} a_i + (b_i - a_i) T_i \left( \dfrac{x - a_i}{b_i - a_i}, \dfrac{y - a_i}{b_i - a_i} \right) & \text{whenever } (x,y) \in (a_i, b_i)^2 \\ & \qquad = (a_i, b_i) \times (a_i, b_i), \\ \min(x,y) & \text{otherwise}, \end{cases}$$

which is denoted by $T = (\langle a_i, b_i, T_i \rangle | i \in J)$, and only if all $T_i$ are generated, then equivalently it can be used $T = (\langle a_i, b_i, f_i \rangle | i \in J)$ where $f_i$ is the additive generator of $T_i$.

The following theorem gives a general classification of continuous $t$-norms [9].

**Theorem 1.** (Ling [9]) Let $T$ be a continuous $t$-norm. Then $T$ is Archimedean or $T$-min or $T$ is an ordinal sum of Archimedean $t$-norms.

## 3. $T$-EQUIVALENCE GENERATED BY SHAPES

Consider a generator $g$ and a shape $\phi$, and the fuzzy relation $E_{g,\phi}$, which is always reflexive and symmetric. Let $T$ be a $t$-norm and $\phi_n = \phi \oplus_T \cdots \oplus_T \phi$ ($n$-fold $T$-sum of $\phi$). Then $\phi_n(x) \le \phi_{n+1}(x)$ for any $x \in R$ and for $n \in N$, the natural numbers. Hence the limit always exists. Let $\lim_{n\to\infty} \phi_n \equiv \phi^*$. We also note that if we define $|\phi| : R \to [0,1]$ such that $|\phi|(z) = \phi(|z|)$ and $|\phi|_n = |\phi| \oplus_T \cdots \oplus_T |\phi|$, then $\lim_{n\to\infty} |\phi|_n \equiv |\phi|^* = |\phi^*|$.

**Theorem 2.** For a continuous $t$-norm $T$, a generator $g$ and a shape $\phi$, the fuzzy relation $E_{f,\phi^*}$ is a $T$-equivalence on $R$.

Proof. We only need to show that for any $a, b, y \in R$

$$T(E_{g,\phi^*}(a,y), E_{g,\phi^*}(y,b)) \le E_{g,\phi^*}(a,b),$$

or equivalently

$$T(|\phi|^*(g(y)-g(a)), |\phi|^*(g(b)-g(y))) \leq |\phi|^*(g(b)-g(a)). \tag{1}$$

By the continuity of the $t$-norm $T$, we have

$$\begin{aligned} & T(|\phi|^*(g(y)-g(a)), |\phi|^*(g(b)-g(y))) \\ = & \lim_{n\to\infty} T(|\phi|_n(g(y)-g(a)), |\phi|_n(g(b)-g(y)) \end{aligned}$$

and

$$\begin{aligned} & T(|\phi|_n(g(y)-g(a)), |\phi|_n(g(b)-g(y)) \\ = \; & T\Bigg( \sup_{x_1+\cdots+x_n=g(y)-g(a)} T(|\phi|(x_1),\cdots,|\phi|(x_n)), \\ & \sup_{x_{n+1}+\cdots+x_{2n}=g(b)-g(y)} T(|\phi|(x_{n+1}),\cdots,|\phi|(x_{2n}))\Bigg) \\ = \; & \sup_{\substack{x_1+\cdots+x_n=g(y)-g(a) \\ x_{n+1}+\cdots+x_{2n}=g(b)-g(y)}} T(T(|\phi|(x_1),\cdots,|\phi|(x_n)), T(|\phi|(x_{n+1}),\cdots,|\phi|(x_{2n}))) \\ \leq \; & \sup_{x_1+\cdots+x_{2n}=g(b)-g(a)} T(|\phi|(x_1),\cdots,|\phi|(x_{2n})) \\ = \; & |\phi|_{2n}(g(b)-g(a)) \end{aligned}$$

where the second equality comes from the continuity of $T$ and the inequality comes from non-decreasing property of $T$, hence equation (1) is proved since $\lim_{n\to\infty}|\phi|_{2n}(g(b)-g(a)) = |\phi|^*(g(b)-g(a))$. □

The following theorem is due to B. De Baets et al [2]. Here, we give a new proof using the idea of Theorem 2.

**Theorem 3.** (De Baets et al [2]) Consider a $t$-norm $T$, a generator $g$ and a shape $\phi$. Let $H = \{|g(u)-g(v)| \,|\, (u,v) \in R^2\}$. If for any $x \in H$, $\phi \oplus_T \phi(x) = \phi(x)$, then the fuzzy relation $E_{g,\phi}$ is a $T$-equivalence on $R$.

Proof. Define $\phi_0$ as follows :

$$\phi_0(x) = \begin{cases} \phi(x) & \text{if } x \in H, \\ \inf\{\phi(w) | w < x,\, w \in H\} & \text{if } x \notin H. \end{cases}$$

Then $\phi_0$ is a shape with $E_{g,\phi}(x,y) = E_{g,\phi_0}(x,y)$ for $(x,y) \in R^2$. We can also show that for any $x \in R$, $\phi_0 \oplus_T \phi_0(x) = \phi_0(x)$. It is because $\phi_0 \oplus_T \phi_0(x) \geq \phi_0(x)$ is always true and for $x \notin H$, $w \in H$ and $w < x$,

$$\begin{aligned} \phi_0 \oplus_T \phi_0(x) & \leq \phi_0 \oplus \phi_0(w) \\ & = \phi(w) \end{aligned}$$

and hence

$$\begin{aligned} \phi_0 \oplus_T \phi_0(x) &\leq \inf\{\phi(w) | w < x,\, w \in H\} \\ &= \phi_0(x). \end{aligned}$$

We now note that $\phi_0 = \phi_0^*$ and can prove that $E_{g,\phi_0}$ is a $T$-equivalence on $R$ according to the exactly same method as Theorem 1 without the assumption of continuity of $T$ using $\phi_0 \oplus_T \phi_0 = \phi_0$. This completes the proof. □

Recently, many authors [5, 6, 7, 13] studied facts about $T$-sums of shape function and their limits.

**Theorem 4.** (Hong and Hwang [6], Hong and Ro [7], Mesiar [11]) Consider a continuous Archimedean $t$-norm $T$ with additive generator $f$ and a shape $\phi$. If $f \circ \phi$ is convex, then

$$\phi_n(x) = f^{[-1]}\left(nf \circ \phi\left(\frac{x}{n}\right)\right).$$

**Theorem 5.** (Hong and Hwang [5]) Consider a continuous Archimedean $t$-norm $T$ with additive generator $f$ and a shape $\phi$. If $f \circ \phi$ is convex, then $\phi^*(0) = 1$ and for $x > 0$,

$$\lim_{n\to\infty} \phi_n(x) = \phi^*(x) = f^{[-1]}(xf'_-(1)\phi'_+(0)).$$

**Definition 8.** Consider $(a, b) \in R$, $a \neq b$, then $\phi_{(a,b)}$ is the linear transformation defined by

$$\phi_{(a,b)}(x) = \frac{x-a}{b-a}$$

Note that the inverse mapping $\phi^{-1}_{(a,b)}$ of $\phi_{(a,b)}$ is given by $\phi^{-1}_{(a,b)}(x) = a + (b-a)x$.

**Definition 9.** Consider a fuzzy quantity $A$ and $(a, b) \in [0, 1]^2$, $a < b$.

(i) The fuzzy quantity $A^{[a,b]}$ is defined as $A^{[a,b]} = \mathrm{tr} \circ \phi_{(a,b)} \circ A$, i.e. $A^{[a,b]}(x) = \mathrm{tr}((A(x) - a)/(b - a))$, where $\mathrm{tr} : R \to [0, 1]$ is defined by

$$\mathrm{tr}(x) = \begin{cases} 0, & \text{if } x < 0, \\ x, & \text{if } 0 \leq x \leq 1, \\ 1, & \text{if } x > 1. \end{cases}$$

(ii) The fuzzy quantity $A_{[a,b]}$ is defined by

$$A_{[a,b]}(x) = \begin{cases} \phi^{-1}_{(a,b)}(A(x)), & \text{if } A(x) > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

We need the following result to generalize Theorem 5 to arbitrary continuous $t$-norm.

**Theorem 6.** (De Baets and Marková [1]) Consider an ordinal sum of continuous $t$-norm $T = (\langle a_i, b_i, f_i\rangle | i \in I)$ written in such a way that $\bigcup_{\lambda\in I}[a_i, b_i] = [0,1]$ and a shape $\phi$. If $f_i \circ \phi^{[a_i,b_i]}$ is convex for all $i \in I$, then

$$\phi_n(x) = \sup_{i\in I}\left\{(\phi_n^{T_i,[a_i,b_i]})_{[a_i,b_i]}(x)\right\}$$

where $\phi_n^{T_i,[a_i,b_i]}(x) = f_i^{[-1]}\left(nf_i \circ \phi^{[a_i,b_i]}\left(\frac{x}{n}\right)\right)$.

Theorem 5 can be easily generalized to arbitrary ordinal sums of continuous $t$-norm $T$.

**Theorem 7.** Consider an ordinal sums of continuous $t$-norm $T = (\langle a_i, b_i, f_i\rangle | i \in I)$ written in such a way that $\bigcup_{\lambda\in I}[a_i, b_i] = [0,1]$ and a shape $\phi$. If $f_i \circ \phi^{[a_i,b_i]}$ is convex for all $i \in I$, then

$$\begin{aligned}\phi^*(x) &= \lim_{n\to\infty}\phi_n(x)\\ &= \sup_{i\in I}\left\{(\phi^{T_i,[a_i,b_i]})_{[a_i,b_i]}(x)\right\},\end{aligned}$$

where $\phi^{T_i,[a_i,b_i]}(x) = \lim_{n\to\infty}\phi_n^{T_i,[a_i,b_i]}(x) = f_i^{[-1]}(x(f_i)'_-(1)\left(\phi^{[a_i,b_i]})'_+(0)\right)$.

## 4. EXAMPLES

**Example 1.** Consider the product $t$-norm $T_P$ with additive generator $f(x) = \log x^{-1}$, and a generator $g$ and a shape function $\phi$ defined by $\phi(x) = \max\{1-x, 0\}$. Then, by Theorem 5 (or see [5]), $\phi^*(x) = e^{-x}$, and hence $E_{g,\phi^*}(x,y) = e^{-|g(x)-g(y)|}$ is a $T$-equivalence on $R$.

**Example 2.** Consider the Lukasiewicz $t$-norm $T_L$ with additive generator $f(x) = 1-x$, and generator $g$ and a shape function $\phi$ defined by $\phi(x) = \max\{1-x, 0\}$. Then, by Theorem 5 (or see [5]), $\phi^*(x) = \phi(x)$, and hence $E_{g,\phi^*}(x,y) = \max\{1 - |g(x)-g(y)|, 0\}$ is a $T$-equivalence on $R$.

**Example 3.** Consider the ordinal sums $T = (\langle 0, \frac{1}{3}, \log x^{-1}\rangle, \langle\frac{1}{3}, 1, 1-x\rangle)$, a generator $g$ and a shape function $\phi$ defined by $\phi(x) = \max\{1-x, 0\}$. Then, by Theorem 7, $\phi^*(x) = \max\left\{1-x, \frac{1}{3}\right\}$, and hence $E_{g,\phi^*}(x,y) = \max\left\{1-|g(x)-g(y)|, \frac{1}{3}\right\}$ is a $T$-equivalence on $R$.

**Example 4.** Consider the ordinal sums $T = (\langle 0, \frac{1}{3}, 1-x\rangle, \langle\frac{1}{3}, 1, \log x^{-1}\rangle)$, a generator $g$ and a shape function $\phi$ defined by $\phi(x) = \max\{1-x, 0\}$. Then, by Theorem 7, $\phi^*(x) = \frac{1}{3} + \frac{2}{3}e^{-\frac{3}{2}x}$ since $f^{T_P,[\frac{1}{3},1]}(x) = e^{-\frac{3}{2}x}$ and $f^{T_L,[0,\frac{1}{3}]}(x) = 1$. Hence $E_{g,\phi^*}(x,y) = \frac{1}{3} + \frac{2}{3}e^{-\frac{3}{2}|g(x)-g(y)|}$ is a $T$-equivalence on $R$.

**Example 5.** Consider the ordinal sums $T = (\langle 0, \frac{1}{3}, \log x^{-1}\rangle, \langle \frac{1}{3}, 1, 1-x\rangle)$, a generator $g$ and a shape function $\phi$ defined by

$$\phi(x) = \begin{cases} 1 & \text{if } x = 0, \\ \frac{1}{3}(1-x) & \text{if } 0 < x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then, by Theorem 7,

$$\phi^*(x) = \begin{cases} 1 & \text{if } x = 0, \\ \frac{1}{3}e^{-x} & \text{otherwise,} \end{cases}$$

since

$$f^{T_L,[\frac{1}{3},1]}(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and $f^{T_P,[0,\frac{1}{3}]}(x) = e^{-x}$. Hence

$$E_{g,\phi^*}(x,y) = \begin{cases} 1 & \text{if } x = y, \\ \frac{1}{3}e^{-|g(x)-g(y)|} & \text{otherwise,} \end{cases}$$

is a $T$-equivalence on $R$.

**Example 6.** Consider the ordinal sums $T = (\langle 0, \frac{1}{3}, 1-x\rangle, \langle \frac{1}{3}, 1, \log x^{-1}\rangle)$, a generator $g$ and a shape function $\phi$ defined by

$$\phi(x) = \begin{cases} 1 & \text{if } x = 0, \\ \frac{1}{3}(1-x) & \text{if } 0 < x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then, by Theorem 7,

$$\phi^*(x) = \begin{cases} 1 & \text{if } x = 0, \\ \frac{1}{3}(1-x) & \text{if } |x| \leq 1, \\ 0 & \text{otherwise,} \end{cases}$$

since

$$f^{T_P,[\frac{1}{3},1]}(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and $f^{T_L,[0,\frac{1}{3}]}(x) = 1-x$. Hence

$$\phi_{g,\phi^*}(x,y) = \begin{cases} 1 & \text{if } g(x) = g(y), \\ \frac{1}{3}(1-|g(x)-g(y)|) & \text{if } |g(x)-g(y)| \leq 1, \\ 0 & \text{otherwise,} \end{cases}$$

is a $T$-equivalence on $R$.

## ACKNOWLEDGEMENTS

This work was supported by grant R01-2000-000-00011-0 from the Korea Science & Engineering Foundation.

(Received September 3, 2002.)

## REFERENCES

[1] B. De Baets and A. Marková: Analytical for the addition of fuzzy intervals. Fuzzy Sets and Systems *91* (1997), 203–213.

[2] B. De Baets, M. Mareš, and R. Mesiar: T-partition of the real line generated by impotent shapes. Fuzzy Sets and Systems *91* (1997), 177–184.

[3] B. De Baets and R. Mesiar: Pseudo-metrics and T-equivalence. J. Fuzzy Math. *5* (1997), 471–481.

[4] M. A. Harrison: Introduction to Switching and Automata Theory. McGraw–Hill, New York 1965.

[5] D. H. Hong and S. Y. Hwang: On the convergence of T-sum of L-R fuzzy numbers. Fuzzy Sets and Systems *63* (1994), 175–180.

[6] D. H. Hong and C. Hwang: A T-sum bound of L-R fuzzy numbers. Fuzzy Sets and Systems *91* (1997), 239–252.

[7] D. H. Hong and P. Ro: The law of large numbers for fuzzy numbers with bounded supports. Fuzzy Sets and Systems *116* (2000), 269–274.

[8] J. Jacas and J. Recasens: Fuzzy numbers and equality relations. In: Proc. 2nd IEEE International Conference on Fuzzy Systems, San Francisco 1993, pp. 1298–1301.

[9] C. Ling: Representation of associative function. Publ. Math. Debrecen *12* (1965), 189–212.

[10] M. Mareš and R. Mesiar: Composition of shape generators. Acta Mathematica et Informatica Universitatis Ostravienses *4* (1996), 37–46.

[11] M. Mareš and R. Mesiar: Fuzzy quantities and their aggregation, in aggregation operations. In: New Trends and Applications. Physica–Verlag, Heidelberg 2002, pp. 291–352.

[12] A. Marková–Stupňanová: Idempotents of T-addition of fuzzy numbers. Tatra Mt. Math. Publ. *12* (1997), 67–72.

[13] R. Mesiar: A note to T-sum of L-R fuzzy numbers. Fuzzy Sets and Systems *87* (1996), 259–261.

[14] B. Schweizer and A. Sklar: Probabilistic Metric Spaces. North–Holland, New York 1983.

*Dr. Dug Hun Hong, School of Mechanical and Automotive Engineering, Catholic University of Daegu, Kyungbuk 712–702. South Korea.*
*e-mail: dhhong@cuth.cataegu.ac.kr*