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A CONVERGENCE OF FUZZY RANDOM VARIABLES

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In this paper, a general convergence theorem of fuzzy random variables is considered. Using this result, we can easily prove the recent result of Joo et al, which gives generalization of a strong law of large numbers for sums of stationary and ergodic processes to the case of fuzzy random variables. We also generalize the recent result of Kim, which is a strong law of large numbers for sums of levelwise independent and levelwise identically distributed fuzzy random variables.

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1. INTRODUCTION

In recent years, strong laws of large numbers for sums of fuzzy random variables have received much attention by several people. A SLLN for sums of independent and identically distributed fuzzy random variables was obtained by Kruse [10], and a SLLN for sums of independent fuzzy random variables was obtained by Miyakoshi and Shimbo [11], Klement, Puri and Ralescu [15]. Also, Inoue [5] obtained a SLLN for sums of independent tight fuzzy random sets, and Hong and Kim [4] proved Marcinkiewicz-type law of large numbers. Many other papers [1, 3, 7, 12, 13, 14, 15, 16, 17, 18] are related to this topic. Recently, Joo, Lee and Yoo [6] generalized a strong law of large numbers for sums of stationary and ergodic processes to the case of fuzzy random variables and Kim [8] obtained a strong law of large numbers for sums of levelwise independent and levelwise identically distributed fuzzy random variables.

In this paper, we consider a general convergence theorem of fuzzy random variables, Using this result, we can easily prove the result of Joo et al [6] and generalize the result of Kim[8]. Section 2 is devoted to describe some basic concepts of fuzzy random variables. Main results are given in Section 3.

2. PRELIMINARIES

Let R denote the real line. A fuzzy number is a fuzzy set $\tilde{u} : R \longrightarrow [0, 1]$ with the following properties;

(1) $\tilde{u}$ is normal, i. e., there exists $x \in R$ such that $\tilde{u}(x) = 1$.

(2) $\tilde{u}$ is upper semicontinuous.

(3) $\operatorname{supp} \tilde{u} = cl\{x \in R | \tilde{u}(x) > 0\}$ is compact.

(4) $\tilde{u}$ is a convex fuzzy set, i. e., $\tilde{u}(\lambda x + (1-\lambda)y) \geq \min(\tilde{u}(x), \tilde{u}(y))$ for $x, y \in R$ and $\lambda \in [0, 1]$.

Let $F(R)$ be the family of all fuzzy numbers. For a fuzzy set $\tilde{u}$, if we define

$$L_\alpha \tilde{u} = \begin{cases} \{x | \tilde{u}(x) \geq \alpha\}, & 0 < \alpha \leq 1, \\ \operatorname{supp} \tilde{u}, & \alpha = 0, \end{cases}$$

then, $\tilde{u}$ is a fuzzy number if and only if $L_1 \tilde{u} \neq \phi$ and $L_\alpha \tilde{u}$ is a closed bounded interval for each $\alpha \in [0, 1]$. If we use this characteristic of fuzzy number, a fuzzy number $\tilde{u}$ is completely determined by the endpoints of the intervals $L_\alpha \tilde{u} = [u^1_\alpha, u^2_\alpha]$.

The following theorem(see Goetschel and Voxman [2]) implies that we can identify a fuzzy number $\tilde{u}$ with the parameterized representation

$$\{(u^1_\alpha, u^2_\alpha) \,|\, 0 \leq \alpha \leq 1\}.$$

Theorem 2.1. For $\tilde{u} \in F(R)$, denote $u^1(\alpha) = u^1_\alpha$ and $u^2(\alpha) = u^2_\alpha$ as functions of $\alpha \in [0, 1]$. Then

(1) u^1 is a bounded increasing function on [0,1].

(2) u^2 is a bounded decreasing function on [0,1].

(3) $u^1(1) \leq u^2(1)$.

(4) u^1 and u^2 are left continuous on [0,1] and right continuous at 0.

(5) If $v^1$and v^2 satisfy above (1) – (4), then there exists a unique $\tilde{v} \in F(R)$ such that $v^1_\alpha = v^1(\alpha), v^2_\alpha = v^2(\alpha)$.

The addition and scalar multiplication on $F(R)$ are defined as usual;

$$\begin{aligned} (\tilde{u} + \tilde{v})(z) &= \sup_{x+y=z} \min(\tilde{u}(x), \tilde{v}(y)), \\ (\lambda \tilde{u})(z) &= \begin{cases} \tilde{u}(z/\lambda), & \lambda \neq 0, \\ \tilde{0}, & \lambda = 0, \end{cases} \end{aligned}$$

for $\tilde{u}, \tilde{v} \in F(R)$ and $\lambda \in R$, where $\tilde{0} = I_{\{0\}}$ is the characteristic function of $\{0\}$. It follows that if $\tilde{u} = \{(u^1_\alpha, u^2_\alpha) \,|\, 0 \leq \alpha \leq 1\}$ and $\tilde{v} = \{(v^1_\alpha, v^2_\alpha) \,|\, \leq \alpha \leq 1\}$, then

$$\begin{aligned} \tilde{u} + \tilde{v} &= \{(u^1_\alpha + v^1_\alpha,\ u^2_\alpha + v^2_\alpha) \,|\, 0 \leq \alpha \leq 1\} \\ \lambda \tilde{u} &= \{(\lambda u^1_\alpha, \lambda u^2_\alpha) \,|\, 0 \leq \alpha \leq 1\} \text{ for } \lambda \geq 0. \end{aligned}$$

Now, we define the metric d_∞ on $F(R)$ by

$$d_\infty(\tilde{u},\tilde{v}) = \sup_{0\le\alpha\le1} h(L_\alpha\tilde{u}, L_\alpha\tilde{v}),$$

where h is Hausdorff metric defined as

$$h(L_\alpha\tilde{u}, L_\alpha\tilde{v}) = \max(|u_\alpha^1 - v_\alpha^1|, |u_\alpha^2 - v_\alpha^2|).$$

The norm of $\tilde{u} \in F(R)$ is defined by

$$||\tilde{u}|| = d_\infty(\tilde{u},\tilde{0}) = \max(|u_0^1|, |u_0^2|).$$

Then it is well-known that $F(R)$ is complete but nonseparable with respect to the metric d_∞. Joo and Kim [7] introduced a metric d_s in $F(R)$ which makes it a separable metric space as follows.

Definition 2.1. Let T denote the class of strictly increasing, continuous mappings of $[0,1]$ onto itself. For $\tilde{u},\tilde{v} \in F(R)$, we define

$$d_s(\tilde{u},\tilde{v}) = \inf\Big\{\varepsilon : \text{there exists a } t \text{ in } T \text{ such that} \sup_{0\le\alpha\le1}|t(\alpha)-\alpha| \le \varepsilon \text{ and } d_\infty(\tilde{u}, t\circ\tilde{v}) \le \varepsilon\Big\},$$

where $t \circ \tilde{v}$ denotes the composition of $\tilde{v}$ and t.

3. MAIN RESULTS

Throughout this section, we assume that the space $F(R)$ is considered as the metric space endowed with the metric d_s, unless otherwise stated. Also, we denote by $\mathcal{B}_s$ the Borel σ-field of $F(R)$ generated by the metric d_s.

Let $(\Omega, \mathcal{A}, P)$ be a probability space. A fuzzy number valued function $\tilde{X} : \Omega \to F(R)$ is called a fuzzy random variable if it is measurable, i. e.,

$$\tilde{X}^{-1}(B) = \{\omega : \tilde{X}(\omega) \in B\} \in \mathcal{A} \text{ for every } B \in \mathcal{B}_s.$$

If we denote $\tilde{X}(\omega) = \{(X_\alpha^1(\omega), X_\alpha^2(\omega))|0 \le \alpha \le 1\}$, then it is known that $\tilde{X}$ is a fuzzy random variable if and only if for each $\alpha \in [0,1]$, X_α^1 and X_α^2 are random variables in the usual sense. A fuzzy random variable $\tilde{X} = \{(X_\alpha^1, X_\alpha^2)|0 \le \alpha \le 1\}$ is called integrable if for each $\alpha \in [0,1]$, X_α^1 and X_α^2 are integrable, equivalently, $\int ||\tilde{X}||\,\mathrm{d}P < \infty$. In this case, the expectation of $\tilde{X}$ is the fuzzy number $E\tilde{X}$ defined by

$$E\tilde{X} = \{(EX_\alpha^1, EX_\alpha^2)\,|\,0 \le \alpha \le 1\}$$

Theorem 3.1. Let $\{\tilde{X}_n\} = \{(X^1_{n\alpha}, X^2_{n\alpha}) \,|\, 0 \le \alpha \le 1\}$ be a sequence of fuzzy random variables and $\tilde{u} = \{(u^1_\alpha, u^2_\alpha) \,|\, 0 \le \alpha \le 1\}$ be a fuzzy number with $\|\tilde{u}\| < \infty$. Suppose that

(1) $X^1_{n\alpha} \to u^1_\alpha$ a. s. and $X^2_{n\alpha} \to u^2_\alpha$ a. s. for any $\alpha \in [0,1]$

(2) $X^1_{n\alpha^+} \to u^1_{\alpha^+}$ a. s. and $X^2_{n\alpha^-} \to u^2_{\alpha^-}$ a. s. for every discontinuity point of u^α_1 and u^α_2, respectively.

Then we have

$$\lim_{n\to\infty} d_\infty(\tilde{X}_n, \tilde{u}) = 0 \ a.\,s.$$

We need the following lemma given in [6].

Lemma 3.1. Let $u = \{(u^1_\alpha, u^2_\alpha) \,|\, 0 \le \alpha \le 1\}$ with $\|u\| < \infty$ and $\varepsilon > 0$ be given.

(1) Then there exists a partition $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_r = 1$ of $[0,1]$ such that $u^1_{\alpha_i} - u^1_{\alpha^+_{i-1}} \le \varepsilon$ for all $i = 1, 2, \ldots, r$.

(2) Similar statements hold for u^2_α.

Proof of Theorem 3.1. Let $\varepsilon > 0$ be arbitrary fixed. By Lemma 3.1, there exists a partition $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_r = 1$ of $[0,1]$ such that $u^1_{\alpha_i} - u^1_{\alpha^+_{i-1}} \le \varepsilon$ for all $i = 1, 2, \ldots, r$. Let $A_k = \{X^1_{n\alpha_k} \longrightarrow u^1_{\alpha_k}$ and $X^1_{n\alpha^+} \longrightarrow u^1_{\alpha^+}$ for all discontinuity points of $u^1_\alpha\}$ and $A_\varepsilon = \cap^r_{k=1} A_k$, then by the assumption $P(A_k) = 1,\ k = 1, 2, \ldots, r$, and hence $P(A_\varepsilon) = 1$. Then for any given $w \in A_\varepsilon$, there exists $N(w)$ such that for $n \ge N(w)$

$$\sup_{k=1,2,\ldots,r} \{|X^1_{n\alpha_k}(w) - u^1_{\alpha_k}|, |X^1_{n\alpha_k^+}(w) - u^1_{\alpha_k^+}|\} \le \varepsilon.$$

Now, let $\alpha \in (\alpha_{k-1}, \alpha_k]$, then for $n \ge N(w)$,

$$X^1_{n\alpha}(w) - u^1_\alpha \le X^1_{n\alpha_k}(w) - u^1_{\alpha^+_{k-1}} \le u^1_{\alpha_k} + \varepsilon - u^1_{\alpha^+_{k-1}} \le 2\varepsilon$$

and

$$u^1_\alpha - X^1_{n\alpha}(w) \le u^1_{\alpha_k} - X^1_{n\alpha^+_{k-1}}(w) \le u^1_{\alpha_k} - (u^1_{\alpha^+_{k-1}} - \varepsilon) \le 2\varepsilon.$$

Hence

$$\sup_{\alpha\in(\alpha_{k-1},\alpha_k]} |X^1_{n\alpha}(w) - u^1_\alpha| \le 2\varepsilon.$$

Since k is arbitrary, we have

$$\sup_{\alpha\in[0,1]} |X^1_{n\alpha}(w) - u^1_\alpha| \le 2\varepsilon.$$

Let $A = \bigcap^\infty_{n=1} A_{\frac{1}{n}}$, then $P(A) = 1$ and for any $w \in A$

$$\lim_{n\to\infty} \sup_{0\le\alpha\le1} |X^1_{n\alpha}(w) - u^1_\alpha| = 0.$$

Similarly, it can be proved that

$$\lim_{n\to\infty} \sup_{0\le\alpha\le1} |X^2_{n\alpha} - u^2_\alpha| = 0, \quad \text{a.s.}$$

which completes the proof. □

Recently, Kim [8] proved a SLLN for sums of levelwise independent and identically distributed fuzzy random variables. But his result is a special case of Theorem 1. If $\tilde{X}_n$ is a sequence of levelwise independent and levelwise identically distributed random variables with $E\|\tilde{X}_1\| < \infty$, then, it is easy to check that both $\{X^1_{n\alpha+}\}$ and $\{X^2_{n\alpha-}\}$ for $\alpha \in [0,1]$ are independent and identically distributed random variables, respectively, with $E|\tilde{X}^1_{n\alpha+}| < \infty$ and $E|\tilde{X}^2_{n\alpha-}| < \infty$. And it is also easy to check that for any $\alpha \in [0,1]$

$$\frac{1}{n}\sum_{i=1}^{n} X^1_{i\alpha+} \longrightarrow EX^1_{\alpha+} \quad \text{a.s.}$$

and

$$\frac{1}{n}\sum_{i=1}^{n} X^2_{i\alpha-} \longrightarrow EX^2_{\alpha-} \quad \text{a.s.}$$

by Kolmogorov's strong law of large numbers and Monotone Convergence Theorem. It is also noted that the set of discontinuity point of EX^1_α and EX^2_α is at most countable. Now, using Theorem 1 we have the following generalized result of Kim [8] as a corollary.

Corollary 3.1. Let $\{\tilde{X}_n\}$ be a sequence of levelwise independent and levelwise identically distributed fuzzy random variables, with $E\|\tilde{X}_1\| < \infty$. Then we have

$$d_\infty\left(\frac{1}{n}\sum_{i=1}^{n} \tilde{X}_i, E\tilde{X}_1\right) \longrightarrow 0 \quad \text{a.s.}$$

Remark. The condition that $EX^1_{1\alpha}$ and $EX^2_{1\alpha}$ are continuous as functions of α in Kim's result is not needed.

Recently Joo et al [6] proved a SLLN for sums of stationary and ergodic fuzzy random variables. With similar arguments as above, noting that for each $\alpha \in [0,1]$, $\{X^1_{n\alpha}\}$, $\{X^1_{n\alpha+}\}$, $\{X^2_{n\alpha}\}$ and $\{X^2_{n\alpha-}\}$ are sequences of stationary and ergodic random variables under the assumption that $\{\tilde{X}_n\}$ is a sequence of stationary and ergodic fuzzy random variables, we also have Joo's result as a corollary by Theorem 1.

Corollary 3.2. Let X_n be a sequence of stationary fuzzy random variables. If $\{\tilde{X}_n\}$ is ergodic and $E\|\tilde{X}_1\| < \infty$, then

$$d_\infty\left(\frac{1}{n}\sum_{i=1}^{n} \tilde{X}_i, E\tilde{X}_1\right) \longrightarrow 0 \quad \text{a.s.}$$

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