ALTERNATIVE MODEL OF FUZZY NTU COALITIONAL GAME

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One of the possible models of fuzzification of non-transferable utility (NTU) coalitional games was extensively treated in [4]. In this paper, we suggest an alternative structure of fuzzification of the NTU games, where for every coalition a fuzzy class of (generally crisp) sets of its admissible pay-off vectors is considered. It is shown that this model of a fuzzy coalitional game can be represented by a fuzzy class of deterministic NTU games, and its basic concepts like the superadditivity or the core can be transparently introduced by means of that class of games.

Keywords: coalitional games, non-transferable utility, fuzzy quantity, fuzzy cooperative game, core of a game, superadditivity

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1. INTRODUCTION

The non-transferable utility (NTU) coalitional games, sometimes called also games without side-payments or general coalitional games, represent the model of cooperation in which the members of coalitions coordinate their strategical activities but each of them obtains his individual pay-off which either cannot be transferred to other members of the coalition or the transfer is subjected to some deformations of the utility (see, e. g., [2] and also [4, 5, 6]). In fact, the NTU coalitional model can be interpreted as a generalization of the classical transferable utility coalitional games as shown, e. g., in the publications referred above.

As the coalitional cooperation is negotiated before starting the practical realization of the game, the expected pay-offs can be only estimated. The hidden assumption of the classical deterministic game theory about the absolute exactness of this estimation appears to be unrealistic. In many practical cooperative situations the idea of coalitions about their expected outcome is more or less vague, which in the formal definition means fuzzy.

In the classical deterministic coalitional game theory, the coalitional pay-offs in an NTU game are represented by closed subsets of a multidimensional Euclidean space fulfilling certain quite natural conditions (briefly recalled in Section 2). Then the fuzzification of this concept can be done in two quite different ways. One of them is used in [3, 4, 5, 6] and it consists in the substitution of the deterministic pay-off sets by fuzzy subsets of the multidimensional space. The alternative approach, used below, is to consider fuzzy classes of the (crisp) pay-off subsets of the space. The interpretations of both approaches are evident. In the former case the expectation of particular pay-off vectors is considered as being vague, whereas in the latter case the complete expected sets of results are connected with uncertainty. The latter approach can be interpreted also in the way that several different coalitional games with different possibilities are potentially possible and the choice of the game which will be realized is connected with the possibilities (i. e. values of membership functions) of the corresponding sets of pay-offs. This interpretation also implies the method used for the processing of these fuzzy games in this paper.

In the following sections the original deterministic model is briefly recalled, and the principle of its fuzzification by means of the latter of the two approaches mentioned above is suggested. Some of its elementary properties are established and its relation to the fuzzification method described in [4] is briefly discussed.

2. DETERMINISTIC NTU COALITIONAL GAME

Let I be a nonempty finite set. Without loss of generality we assume that $I = \{1, 2, \ldots, n\}$ where n is a positive integer. The elements of I are called *players* and the subsets of I are called *coalitions*. The n-dimensional real vector space of all n-tuples $x = (x_1, x_2, \ldots, x_n)$ of real numbers is denoted by $\mathbf{R}^n$. We assume that $\mathbf{R}^n$ is equipped with a norm. If $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ from $\mathbf{R}^n$ are such that $y_i \le x_i$ for all $i \in I$, then we write $y \le x$ or $x \ge y$.

Definition 2.1. Let V be a mapping assigning to each coalition K a subset $V(K)$ of $\mathbf{R}^n$. The ordered pair (I, V) is said to be a deterministic NTU coalitional game (NTU game for short) if the mapping V has the following properties:

(2.1) Each $V(K)$ is a nonempty closed subset of $\mathbf{R}^n$.

(2.2) $V(K) = \mathbf{R}^n$ if and only if $K = \emptyset$.

(2.3) If $x \in V(K)$ and $y_i \le x_i$ for all $i \in K$, then $y \in V(K)$.

The elements of $V(K)$ are called $V(K)$-imputations.

Each set $V(K)$ can be described as the set of vectors in $\mathbf{R}^n$ for which coalition K is effective in the sense that if $x \in V(K)$ then coalition K can obtain pay-off x_i for each $i \in K$ no matter what the coalition $N \setminus K$ does.

An important class of NTU games is the class of superadditive games defined as follows.

Definition 2.2. A deterministic NTU coalitional game (I, V) is called superadditive if

$$V(K \cup L) \supset V(K) \cap V(L)$$

for each pair of disjoint coalition K and L.

We are also interested in one of the basic solution concepts, namely, the *core* of a game. To define the core of an NTU game, we first introduce the following two auxiliary notions.

Let x and y be points in $\mathbf{R}^n$ and let K be a coalition. We say that x dominates y via K, and write $x \operatorname{dom}_K y$, if

— $x_i \geq y_i$ for all $i \in K$, and

— $x_j > y_j$ for some $j \in K$.

Using this binary relation of domination, we define, for each coalition K, the subset $V^*(K)$ of $\mathbf{R}^n$ as the set of those points in $\mathbf{R}^n$ that are not dominated via K by points from $V(K)$. It follows that $y \in V^*(K)$ if and only if, for each $x \in V(K)$, either $y_i > x_i$ for some $i \in K$, or $y_i = x_i$ for all $i \in K$.

The set $V^*(K)$ is called the superoptimum of coalition K, and the intersection

$$P(K) = V(K) \cap V^*(K)$$

is called the Pareto optimum of K. Notice that the superoptimum of each coalition is nonempty and that $V^*(\emptyset) = \mathbf{R}^n$.

Definition 2.3. Let (I, V) be an NTU coalitional game. The set C_V defined by

$$C_V = V(I) \cap \left(\bigcap_{K \subset I} V^*(K) \right)$$

is called the core of game (I, V).

In other words, a point $y \in \mathbf{R}^n$ belongs to the core of game (I, V) if and only if it is a $V(I)$-imputation and such that, for each coalition K, no $V(K)$-imputation dominates y via K.

In the following sections we denote by $\mathcal{V}(K)$ for every $K \subset I$, the class of all sets $V(K)$ fulfilling (2.1), (2.2), (2.3).

3. FUZZY NTU COALITIONAL GAMES

As indicated in the Introduction, the concept of deterministic NTU coalitional games can be fuzzified in various ways.

One of the possibilities is to substitute the sets $\mathcal{V}(K)$ by fuzzy subsets of $\mathbf{R}^n$. This approach was used, e.g., in [4, 5, 6], and it is also mentioned in Section 7 of this paper. In this section we introduce an alternative concept of fuzzy NTU games that seems to be quite natural, and which, to our best knowledge, has not been considered in the literature.

Definition 3.1. Let $\mathcal{W}$ be a mapping assigning to each coalition K a fuzzy subset $\mathcal{W}(K)$ of $\mathcal{V}(K)$. The ordered pair $(I, \mathcal{W})$ is said to be a proper fuzzy NTU coalitional game (fuzzy NTU game for short). The membership function of $\mathcal{W}(K)$ is denoted by $\pi_K^{\mathcal{W}}$.

Note that since $\mathcal{V}(\emptyset) = \{\mathbf{R}^n\}$, the domain of definition of $\pi_\emptyset^{\mathcal{W}}$ has only one element. Throughout this paper we consider only games $(I, \mathcal{W})$ with $\pi_\emptyset^{W}(\mathbf{R}^n) = 1$. Also note that the set of deterministic NTU coalitional games can naturally be embedded into the set of fuzzy NTU coalitional games as follows.

Let (I, V) be a deterministic NTU game. For each coalition $K \subset I$, define $\sigma_K^V : \mathcal{V}(K) \to [0,1]$ by

$$\sigma_K^V(A) = \begin{cases} 1 & \text{if } A = V(K), \\ 0 & \text{if } A \in \mathcal{V}(K) \setminus \{V(K)\}. \end{cases}$$

Then the game (I, V) can be identified with game $(I, \mathcal{W})$ for which

$$\pi_K^{\mathcal{W}} = \sigma_K^V \quad \text{for each } K \subset I.$$

Definition 3.2. Let (I, V) be a deterministic NTU game. Every fuzzy NTU game $(I, \mathcal{W})$ with the property

$$\pi_K^{\mathcal{W}}(V(K)) = 1 \quad \text{for each } K \subset I$$

is called a proper fuzzy extension of game (I, V).

In this section, we have formalized the main principle of fuzzification of NTU coalitional principle considered in this paper. We do not substitute the deterministic sets $V(K)$ by fuzzy subsets of $\mathbf{R}^n$, as it was done, e. g., in [4, 5, 6]. Such method leads to formal complications whenever the model is to be processed (e. g., the superoptima $V^*(\cdot)$ can be defined and handled with significant difficulties, only). Here, we are to manage the fuzzy classes of possible deterministic sets $V(K)$ for every coalition K. It can be simply verified that the processing of such model is much easier. Moreover, it can be easily and naturally extended into fuzzy class of complete NTU games (where each of them is deterministic), as it is shown in the next section and used in the following parts of this paper.

4. FUZZY CLASSES OF NTU COALITIONAL GAMES

Let $\mathcal{D}(I)$ be the set of all deterministic NTU games over I and let $(I, \mathcal{W})$ be a proper fuzzy NTU coalitional game. Furthermore, let $\rho_{\mathcal{W}} : \mathcal{D}(I) \to [0,1]$ be a function defined on $\mathcal{D}(I)$ by

$$\rho_{\mathcal{W}}(I, V) = \min_{K \subset I} \pi_K^{\mathcal{W}}(V(K)). \tag{4.1}$$

We denote the fuzzy subset of $\mathcal{D}(I)$ whose membership function is $\rho_{\mathcal{W}}$ by $G(\mathcal{W})$, and say that $G(\mathcal{W})$ is generated by game $(I, \mathcal{W})$.

The following example shows that two different games $(I, \mathcal{W}_1)$ and $(I, \mathcal{W}_2)$ can generate the same fuzzy subset of $\mathcal{D}(I)$, that is, $G(\mathcal{W}_1)$ may be equal to $G(\mathcal{W}_2)$.

Example 4.1. Let $I = \{1,2\}$ and let (I, V_0) be a game from $\mathcal{D}(I)$. Moreover, let $(I, \mathcal{W}_1)$ and $(I, \mathcal{W}_2)$ be fuzzy NTU games such that

$$\pi_{\emptyset}^{\mathcal{W}_1}(V_0(\{\emptyset\})) = 1,\ \pi_{\{1\}}^{\mathcal{W}_1}(V_0(\{1\})) = \frac{1}{2},\ \pi_{\{2\}}^{\mathcal{W}_1}(V_0(\{2\})) = 1,\ \pi_{\{1,2\}}^{\mathcal{W}_1}(V_0(\{1,2\})) = \frac{1}{10},$$

$$\pi_K^{\mathcal{W}_1}(A) = 0 \quad \text{for all } K \subset I \text{ and } A \in \mathcal{V}(K) \text{ such that } A \neq V_0(K);$$

$$\pi_{\emptyset}^{\mathcal{W}_2}(V_0(\{\emptyset\})) = 1,\ \pi_{\{1\}}^{\mathcal{W}_2}(V_0(\{1\})) = 1,\ \pi_{\{2\}}^{\mathcal{W}_2}(V_0(\{2\})) = 1,\ \pi_{\{1,2\}}^{\mathcal{W}_2}(V_0(\{1,2\})) = \frac{1}{10},$$

$$\pi_K^{\mathcal{W}_2}(A) = 0 \quad \text{for all } K \subset I \text{ and } A \in \mathcal{V}(K) \text{ such that } A \neq V_0(K).$$

Simple calculations show that

$$\rho_{\mathcal{W}_1}(I, V_0) = \rho_{\mathcal{W}_2}(I, V_0) = \frac{1}{10},$$

$$\rho_{\mathcal{W}_1}(I, V) = \rho_{\mathcal{W}_2}(I, V) = 0 \quad \text{for } (I, V) \text{ with } V \neq V_0.$$

Therefore $G(\mathcal{W}_1) = G(\mathcal{W}_2)$, although $\mathcal{W}_1 \neq \mathcal{W}_2$.

Observe that the games $(I, \mathcal{W}_1)$ and $(I, \mathcal{W}_2)$ in the above example are not proper fuzzy extensions of any game in $\mathcal{D}(I)$, because the values $\rho_{\mathcal{W}_1}(I, V)$ and $\rho_{\mathcal{W}_2}(I, V)$ are different from 1 for every game $(I, V) \in \mathcal{D}(I)$. In contrast to the example, the following lemma shows that the fuzzy subsets $G(\mathcal{W}_1)$ and $G(\mathcal{W}_2)$ are different if they are generated by different proper fuzzy extensions of a deterministic NTU game.

Lemma 4.1. If $(I, \mathcal{W}_1)$ and $(I, \mathcal{W}_2)$ are two different proper fuzzy extensions of a deterministic NTU game, then $\rho_{\mathcal{W}_1} \neq \rho_{\mathcal{W}_2}$.

Proof. Let $(I, \mathcal{W}_1)$ and $(I, \mathcal{W}_2)$ be proper fuzzy extensions of a game $(I, V_0) \in \mathcal{D}(I)$ such that $\mathcal{W}_1 \neq \mathcal{W}_2$. We wish to show that there is a game $(I, V) \in \mathcal{D}(I)$ such that $\rho_{\mathcal{W}_1}(I, V) \neq \rho_{\mathcal{W}_2}(I, V)$. From the assumption that $\mathcal{W}_1 \neq \mathcal{W}_2$, we have

$$\pi_L^{\mathcal{W}_1}(A) \neq \pi_L^{\mathcal{W}_2}(A) \quad \text{for some } L \subset I \text{ and } A \in \mathcal{V}(L).$$

Let V be defined by

$$V(K) = \begin{cases} V_0(K) & \text{for } K \neq L, \\ A & \text{for } K = L. \end{cases}$$

Then

$$\rho_{\mathcal{W}_1}(I, V) = \min_{K \subset I} \pi_K^{\mathcal{W}_1}(V(K)) = \pi_L^{\mathcal{W}_1}(V(L)) = \pi_L^{\mathcal{W}_1}(A),$$

$$\rho_{\mathcal{W}_2}(I, V) = \min_{K \subset I} \pi_K^{\mathcal{W}_2}(V(K)) = \pi_L^{\mathcal{W}_2}(V(L)) = \pi_L^{\mathcal{W}_2}(A).$$

Since $\pi_L^{\mathcal{W}_1}(A) \neq \pi_L^{\mathcal{W}_2}(A)$, we conclude that $\rho_{\mathcal{W}_1} \neq \rho_{\mathcal{W}_2}$. □

The above results show that a fuzzy class of (deterministic) NTU games can be derived from proper fuzzy NTU coalitional games.

5. SUPERADDITIVITY

Let (I, V_0) be a deterministic NTU coalitional game, and let $\mathcal{P}_0$ be the set of all proper fuzzy extensions of (I, V_0). Every $(I, \mathcal{W})$ game from $\mathcal{P}_0$ generates a fuzzy subset $G(\mathcal{W})$ of $\mathcal{D}(I)$ whose membership function $\rho_{\mathcal{W}}$ is given by (4.1). Using these membership functions, we associate with each game $(I, V_0) \in \mathcal{D}(I)$ a fuzzy subset of $\mathcal{P}_0$ whose membership function $\sigma_{V_0}(I, \mathcal{W}) : \mathcal{P}_0 \to [0,1]$ is defined by

$$\sigma_{V_0}(I, \mathcal{W}) = \sup \rho_{\mathcal{W}}(I, V)$$

where the supremum is taken over all superadditive games from $\mathcal{D}(I)$.

Evidently, the membership function $\sigma_{V_0}(W)$ defines a fuzzy subset of proper fuzzy coalitional games which can be, with some possibility, considered for superadditive.

Theorem 5.1. If $(I, \mathcal{W})$ is a proper extension of a deterministic NTU coalitional game (I, V_0) and if (I, V_0) is superadditive, then $\sigma(I, \mathcal{W}) = 1$.

Proof. From the definition of proper fuzzy extensions, we have

$$\pi_K^{\mathcal{W}}(V_0(K)) = 1 \quad \text{for each } K \subset I.$$

Therefore

$$\min_{K \subset I} \pi_K^{\mathcal{W}}(V_0(K)) = 1.$$

As (I, V_0) is superadditive, we obtain

$$\sigma(I, \mathcal{W}) = \sup\{\rho_{\mathcal{W}}(I, V) | (I, V) \text{ is superadditive}\} = 1. \qquad \square$$

Theorem 5.2. Let $(I, \mathcal{W}_1)$ and $(I, \mathcal{W}_2)$ be proper fuzzy coalitional games. If $\pi_K^{\mathcal{W}_1}(A) \geq \pi_K^{\mathcal{W}_2}(A)$ for all $K \subset I$ and $A \in \mathcal{V}(K)$, then $\sigma(I, \mathcal{W}_1) \geq \sigma(I, \mathcal{W}_2)$.

Proof. From the assumption on the membership functions of $\mathcal{W}_1(K)$ and $\mathcal{W}_2(K)$, we have

$$\rho_{\mathcal{W}_1}(I, V) = \min_{K \subset I} \pi_K^{\mathcal{W}_1}(V(K)) \geq \min_{K \subset I} \pi_K^{\mathcal{W}_2}(V(K)) = \rho_{\mathcal{W}_2}(I, V)$$

for each $(I, V) \in \mathcal{D}(I)$. It follows

$$\sup\{\rho_{\mathcal{W}_1}(I, V) | (I, V) \text{ is superadditive}\} \geq \sup\{\rho_{\mathcal{W}_2}(I, V) | (I, V) \text{ is superadditive}\}.$$

$\square$

The approach to the fuzziness of a coalitional game, used in this section, is natural and evident. Instead of "translating" the definition of some property of a game (in our case it was the superadditivity) in the language of fuzzy set theory, we rather find the possibility with which the fuzzy class of NTU games extending the original game contains also some superadditive elements. This strategy simplifies the formal processing of the conception of fuzzification of coalitional games, and we can use it also in the case of other game theoretical concepts.

6. CORES OF PROPER FUZZY GAMES

The further concept which we try to define and process by means of the fuzzy subclass of NTU games representing the vagueness of the results of negotiation, is the concept of core. Let us consider a proper fuzzy NTU coalitional game W which is connected, in the sense of (4.1) with a fuzzy class of (deterministic) NTU games $G(W)$. Now, the fuzzy core of the proper fuzzy game W can be defined as a fuzzy subset of $\mathbf{R}^n$. Each imputation x belongs to the core with the possibility with which a game whose core contains x belongs to the fuzzy set $G(W)$. More formally, this approach to the fuzziness of core can be described in the following way.

Let $(I, \mathcal{W})$ be a proper fuzzy NTU coalitional game, and let $\bar{\gamma}_{\mathcal{W}} : \mathbf{R}^n \times \mathcal{D}(I) \to [0,1]$ be defined by

$$\bar{\gamma}_{\mathcal{W}}(x, V) = \begin{cases} \rho_{\mathcal{W}}(I, V) & \text{if } x \in \mathcal{C}_V, \\ 0 & \text{if } x \notin \mathcal{C}_V. \end{cases}$$

Definition 6.1. The core of a fuzzy NTU game $(I, \mathcal{W})$ is the fuzzy subset of $\mathbf{R}^n$ whose membership function is the function $\gamma_{\mathcal{W}} : \mathbf{R}^n \to [0,1]$ defined by

$$\gamma_{\mathcal{W}}(x) = \sup\{\bar{\gamma}_{\mathcal{W}}(x, V) | (I, V) \in \mathcal{D}(I)\}.$$

Theorem 6.1. If $(I, \mathcal{W})$ is a proper fuzzy extension of a deterministic NTU coalitional game (I, V_0) and if the core of (I, V_0) is nonempty, then

$$\gamma_{\mathcal{W}}(x) = 1 \quad \text{for } x \in \mathcal{C}_{V_0}.$$

Proof. From the definition of $\gamma_{\mathcal{W}}$, we have $1 \geq \gamma_{\mathcal{W}}(x) \geq \bar{\gamma}_{\mathcal{W}}(x, V_0)$. If $x \in \mathcal{C}_{V_0}$, then $\bar{\gamma}_{\mathcal{W}}(x, V_0) = \rho_{\mathcal{W}}(I, V_0)$. Since $(I, \mathcal{W})$ is a proper fuzzy extension of (I, V_0), we obtain $\rho_{\mathcal{W}}(I, V_0) = 1$. Consequently $\gamma_{\mathcal{W}}(x) = 1$. □

Theorem 6.2. Let $(I, \mathcal{W}_1)$ and $(I, \mathcal{W}_2)$ be proper fuzzy coalitional games. If $\pi_K^{\mathcal{W}_1}(A) \geq \pi_K^{\mathcal{W}_2}(A)$ for all $K \subset I$ and $A \in \mathcal{V}(K)$, then

$$\gamma_{\mathcal{W}_1}(x) \geq \gamma_{\mathcal{W}_2}(x) \quad \text{for all } x \in \mathbf{R}^n.$$

Proof. From the assumption it follows that $\rho_{\mathcal{W}_1}(I, V_0) \geq \rho_{\mathcal{W}_2}(I, V_0)$ for each $(I, V) \in \mathcal{D}(I)$. Therefore $\bar{\gamma}_{\mathcal{W}_1}(x) \geq \bar{\gamma}_{\mathcal{W}_2}(x)$ for every $x \in \mathbf{R}^n$ and $(I, V) \in \mathcal{D}(I)$. Consequently,

$$\gamma_{\mathcal{W}_1}(x) = \sup\{\bar{\gamma}_{\mathcal{W}_1}(x, V) | (I, V) \in \mathcal{D}(I)\} \geq \sup\{\bar{\gamma}_{\mathcal{W}_2}(x, V) | (I, V) \in \mathcal{D}(I)\} = \gamma_{\mathcal{W}_2}(x).$$

□

Observe that as corollaries of Theorem 6.1 and 6.2 we have, respectively,

$$\sup_{x \in \mathbf{R}^n} \gamma_{\mathcal{W}}(x) = 1$$

and

$$\sup_{x \in \mathbf{R}^n} \gamma_{\mathcal{W}_1}(x) \geq \sup_{x \in \mathbf{R}^n} \gamma_{\mathcal{W}_2}(x).$$

7. MARGINAL NOTE ON FUZZY PAY-OFF SETS

We have attempted to fuzzify the NTU coalitional games in some of our previous papers [4, 5, 6]. The approach of these papers differs significantly from that of this paper. For comparison, we briefly recall basic principles of our previous attempt.

If (I, V) is a deterministic NTU coalitional game then its fuzzy extension (I, W) is defined in [4] as a pair in which the mapping W assigns to every coalition $K \subset I$ a fuzzy subset $W(K)$ of $\mathbf{R}^n$ with membership function $\kappa_K : \mathbf{R}^n \to [0, 1]$ such that

(7.1) $\{x \in \mathbf{R}^n | \kappa_K(x) = 1\} \supset V(K)$,

(7.2) $\{x \in \mathbf{R}^n | \kappa_K(x) = 1\}$ is closed,

(7.3) if $x, y \in \mathbf{R}^n$, $x \operatorname{dom}_K y$ then $\kappa_K(x) \leq \kappa_K(y)$,

(7.4) if $K \neq \emptyset$ then there always exists $x \in \mathbf{R}^n$ such that $\kappa_K(x) = 0$.

The properties of this model of fuzzification of an NTU game are summarized in [4], and its relations of the fuzzy coalitional games with transferable utility are formulated in [5, 6]. Regarding the superadditivity and core, which are in the presented paper formalized even for the proper fuzzy extensions $(I, \mathcal{W})$, their properties are different from those presented in the previous sections.

There exists a relation between both fuzzifications of the NTU coalition games. Namely, each proper fuzzy coalitional game $(I, \mathcal{W})$ can be transferred into the fuzzy coalitional game (I, W) of the type considered in this section by means of the relation

$$\forall K \subset I, \forall x \in \mathbf{R}^n, \quad \kappa_K(x) = \sup \left\{ \pi_K^W(V(K)) | V(K) \in \mathcal{V}(K), x \in V(K) \right\}. \quad (7.5)$$

It is not difficult to verify the following statement.

Theorem 7.1. If $(I, \mathcal{W})$ is a proper fuzzy extension of (I, V_0) and if (I, W) is derived from it by means of (7.5) then (I, W) fulfills (7.1) – (7.4).

Proof. Condition (7.1) follows directly from Definition 3.2 and (7.5), and (7.2) is an immediate consequence of (2.1). Property (7.3) is fulfilled due to (2.3) and (7.5), and (7.4) follows from (2.2), the definition of mapping $\mathcal{V}$, and (7.5). □

Generally and rather heuristically, it is possible to say that the approach suggested in this paper appears more natural and its reflection of the intuitively expected behavior of the fuzzification of a coalitional game is more immediate. For example, the superadditivity of (I, W) is defined, analogously to Definition 2.2; that is, by means of the fuzzy set theoretical inclusion and intersection

$$W(K \cup L) \supset W(K) \cap W(L) \quad (7.6)$$

for any pair of disjoint coalitions. In fact, relation (7.6) regards fuzzy sets but it is a crisp relation quite suppressing the fuzzy character of the game model. In [4, 5, 6], some additional concepts based on the method of α-cuts had to be introduced to bring the fuzziness into the model of superadditivity of (I, W). On the contrary, the superadditivity of $(I, \mathcal{W})$ presented above and following from the properties of games forming a fuzzy class, is naturally a fuzzy property.

Similarly, also the concept of the core of (I, W) must be supported by rather artificial concept of so called essential domination to achieve intuitively natural properties as shown in [4]. Even then the fuzzy superoptimum gains some incomfortable properties (e. g., the degree of membership of its elements need not be increasing nor non-decreasing). The fuzzy core defined in Section 6 of this paper gains quite natural properties directly thanks to the method of its construction.

The facts mentioned here show that it could be useful to pay attention to the fuzzification of the coalitional games which is of the type discussed in the previous sections.

8. CONCLUSION

Probably the most significant general conclusion following from the previous sections is formulated in Section 7. The methodology based on the processing of fuzzy classes of complete NTU games, and based on the fuzzification of the classes $\mathcal{V}(K)$ seems to be natural and better manageable than other methods of the fuzzification of the pay-offs of NTU (and, perhaps, not only NTU) coalitional games.

For the processing of the fuzzified games, this method offers the well managed apparatus of the deterministic games which can be fully applied and then turned into its fuzzy counterpart by means of its extension to the class (and its fuzzy subclass) of games realized over the same set of players.

The fuzzification method presented here can be probably modified also for the transferable utility coalitional games which are treated in [4, 5, 6] by means of fuzzy quantities theory. Even in this case, probably, the results achieved by the processing of fuzzy subclasses of essentially deterministic games can be different from those presented in the referred publications. Anyhow, even the comparison of both groups of results and analysis of their differences can bring an interesting information about the essential structures of the vagueness in the coalitional games. Further analysis of the model is needed for finding whether it promises new contribution to our knowledge in general cooperative situations.

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