KYBERNETIKA — VOLUME *39* (2003), NUMBER 2, PAGES 205–215

CONTINUOUS EXTENSION OF ORDER-PRESERVING HOMOGENEOUS MAPS

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Maps f defined on the interior of the standard non-negative cone K in $\mathbb{R}^N$ which are both homogeneous of degree 1 and order-preserving arise naturally in the study of certain classes of Discrete Event Systems. Such maps are non-expanding in Thompson's part metric and continuous on the interior of the cone. It follows from more general results presented here that all such maps have a homogeneous order-preserving continuous extension to the whole cone. It follows that the extension must have at least one eigenvector in $K - \{0\}$. In the case where the cycle time $\chi(f)$ of the original map does not exist, such eigenvectors must lie in $\partial K - \{0\}$.

Keywords: discrete event systems, order-preserving homogeneous maps
AMS Subject Classification: 93B27, 06F05

1. INTRODUCTION

We study classes of maps motivated originally by applications to Discrete Event Systems. We first consider a general setting, and then apply our results to a particular class of maps.

In Section 2, we consider the general case of maps which are defined on the *interior* of a closed cone K_1, taking values in another closed cone K_2, which are continuous and order-preserving with respect to the usual partial orderings induced by K_1 and K_2. We examine the problem of extending these maps, in a natural way, to the whole of K_1. We give conditions, in considerable generality, (both finite- and infinite-dimensional) under which a natural extension exists and is continuous. (There are also interesting examples in which some of the conditions do not hold and the extension is not everywhere continuous.)

Specifically, we consider order-preserving continuous maps f from the interior of K_1 to K_2, where K_1 satisfies a geometrical condition (which generalizes the notion of a polyhedral cone), K_2 satisfies a weak version of normality, f satisfies a weak version of homogeneity of degree 1. Our results show that all such maps have a continuous extension to the whole of K_1. We state these general results without proofs; the interested reader is referred to [1].

In Section 3, we examine the case of maps f defined on the interior of the standard non-negative cone K in $\mathbb{R}^N$ which are both homogeneous of degree 1 and order-preserving. Our results imply that such maps always have a continuous extension. We examine briefly connections between properties of the extension and existence of the cycle-time vector for such maps.

2. GENERAL RESULTS

We first give some basic definitions.

Definition 2.1. A *closed cone (with vertex at zero)* in a topological vector space X is a closed convex subset $K \subset X$ such that (1) $K \cap (-K) = \{0\}$ and (2) $\lambda K \subseteq K$ for all real $\lambda \geq 0$. The cone structure induces a partial ordering: We write $x \leq_K y$ if $y - x \in K$ (or simply $x \leq y$, if K is obvious from the context). If K has non-empty interior, $\overset{\circ}{K} \neq \emptyset$, we write $x \ll y$ if $y - x \in \overset{\circ}{K}$.

In what follows, we usually let K_1 be a closed cone in a Hausdorff topological vector space X_1 with $\overset{\circ}{K}_1 \neq \emptyset$ and let K_2 be a closed cone in a topological vector space (t.v.s.) X_2, and consider maps $f : \overset{\circ}{K}_1 \to K_2$ that are both (1) *continuous* and (2) *order-preserving* in the sense that, for all $x, y \in \overset{\circ}{K}_1$,

$$x \leq_{K_1} y \quad \text{implies} \quad f(x) \leq_{K_2} f(y).$$

Definition 2.2. When considering the extension of such maps to points $x \in \partial K_1$, on the boundary of K_1, it will be natural to consider sequences $\langle x_n \in \overset{\circ}{K}_1 : n \geq 1 \rangle$ such that $x_n \to x$ as $n \to \infty$ and $x_n \gg x$ for all n. We will call such sequences *allowable.*

Assuming that $\overset{\circ}{K}_1 \neq \emptyset$, if we take any $u \in \overset{\circ}{K}_1$ and define $x_n := x + n^{-1}u$, then $\lim_{n\to\infty} x_n = x$ and $x_n \gg x$ for all $n \geq 1$ (by convexity). So such a sequence always exists.

2.1. Natural extension

Under the assumption (*Condition A*) given below, namely that the images under f of decreasing sequences must converge, it follows that a natural extension exists.

Definition 2.3. Let $f : \overset{\circ}{K}_1 \to K_2$ where K_1, K_2 are closed cones in Hausdorff topological vector spaces X_1, X_2, respectively. Suppose that if $x_1 \geq x_2 \geq \cdots \geq x_k \geq \cdots$ is any decreasing sequence in $\overset{\circ}{K}_1$, then the sequence $\langle f(x_j) : j \geq 1 \rangle$ converges in K_2. If f, K_1, and K_2 satisfy these conditions, then we shall say that *Condition A is satisfied.*

Condition A holds in some useful cases:

Lemma 2.4. Let K_1 be a closed cone in a Hausdorff t.v.s. X_1 with $\overset{\circ}{K_1} \neq \emptyset$. Assume that K_2 is a closed cone in a *finite-dimensional* Hausdorff t.v.s. X_2. Let $f : \overset{\circ}{K_1} \to K_2$ be a continuous order-preserving map. Then Condition A holds.

Note that there are interesting closed cones K which are *not* finite-dimensional, but which have the property that decreasing sequences converge. This motivates the following definition.

Definition 2.5. Let K be a closed cone in a Hausdorff t.v.s. X. We say that K has the *monotone convergence property* if whenever $\langle y_j : j \geq 1\rangle$ is a sequence in K and $y_{j+1} \leq y_j$ for all $j \geq 1$, there exists $y \in K$ with $\lim_{j\to\infty} y_j = y$.

Example 2.6. Any finite-dimensional closed cone has the monotone convergence property.

Example 2.7. Let H be a real Hilbert space and X be the set of bounded self-adjoint linear maps $A : H \to H$. Equip X with the strong operator topology: if $\langle A_j : j \geq 1\rangle$ is a sequence in X, then $A_j \to A$ in the strong operator topology if and only if $||A_j(x) - A(x)|| \to 0$ as $j \to \infty$ for all $x \in H$. Let K be the cone of nonnegative-definite bounded self-adjoint operators in X, so that $\langle Ax, x\rangle \geq 0$ for all $x \in H$. It is a standard result that K has the monotone convergence property (see, for example, [8]).

Example 2.8. Let $(S, \mathcal{M}, \mu)$ be a measure space and let $X = L^1(S, \mathcal{M}, \mu)$ denote the usual Banach space of μ-integrable real-valued maps. Let K denote the closed cone in X of maps which are greater than or equal to zero μ-almost everywhere (two maps in X being identified if they agree μ-almost everywhere). The monotone convergence theorem from real analysis implies that K has the monotone convergence property.

We use the monotone convergence property and the previous remarks to generalize Lemma 2.4 to the case where X_2 need not be finite-dimensional:

Lemma 2.9. Let K_1 be a closed cone in a Hausdorff t.v.s. X_1 with $\overset{\circ}{K_1} \neq \emptyset$. Let K_2 be a closed cone in a Hausdorff t.v.s. X_2 and suppose that K_2 satisfies the monotone convergence property. Let $f : \overset{\circ}{K_1} \to K_2$ be a continuous order-preserving map. Then Condition A is satisfied.

We now state our first extension theorem:

Theorem 2.10. Let K_1 be a closed cone with non-empty interior in a Hausdorff t.v.s. X_1. Let K_2 be a closed cone in a Hausdorff t.v.s. X_2. Assume that $f : \overset{\circ}{K_1} \to K_2$ is order-preserving and continuous and that Condition A is satisfied. Suppose that $x \in K_1$ and that $\langle x_n \in \overset{\circ}{K_1} : n \geq 1 \rangle$ is an allowable sequence with $\lim_{n\to\infty} x_n = x$. Then:

(a) There exists $z = z_x \in K_2$ such that $\lim_{n\to\infty} f(x_n) = z_x$.

(b) Further, if $\langle y_n \in \overset{\circ}{K_1} : n \geq 1 \rangle$ is another allowable sequence such that $\lim_{n\to\infty} y_n = x$, then $\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} f(x_n) = z_x$.

(c) Further, if we define $F(x) := z_x$ for all $x \in K_1$, then $F(x) = f(x)$ for all $x \in \overset{\circ}{K_1}$, i. e., F is an *extension* of f. If $x \in K_1$ and $\langle x_n \in K_1 : n \geq 1 \rangle$ is any sequence such that $\lim_{n\to\infty} x_n = x$ and $x_{n+1} \ll x_n$ for all $n \geq 1$, then $\lim_{n\to\infty} F(x_n) = F(x)$. (In fact, if $\langle y_n \in K_1 : n \geq 1 \rangle$ is any sequence such that $y_n \geq x$ for all $n \geq 1$ and $\lim_{n\to\infty} y_n = x$, it follows that $\lim_{n\to\infty} F(y_n) = F(x)$.)

That the above limits all exist and take the same value, namely $F(x)$ as defined in the theorem, is our intended meaning of the term "natural" extension.

Remark 2.11. The extended map F is also order-preserving since, if $x, y \in K_1$ and $x \leq y$, we may take $u \in \overset{\circ}{K_1}$ and note that

$$\begin{aligned} F(x) &= \lim_{n\to\infty} f(x + n^{-1}u) \\ &\leq \lim_{n\to\infty} f(y + n^{-1}u) = F(y). \end{aligned}$$

Theorem 2.10 also implies that $F : K_1 \to K_2$ is continuous at $0 \in K_1$. However, as we show in detail in [1], the map F need not be everywhere continuous, even if K_1 and K_2 are finite-dimensional closed cones.

2.2. Continuity of the extension

The aim of this section is to give some further conditions on f, K_1, and K_2, in considerable generality, which ensure that $F : K_1 \to K_2$ is continuous.

We begin with a geometrical condition on K_1 which generalizes the polyhedral property of a cone.

Definition 2.12. Let K be a closed cone in a Hausdorff t.v.s. X. If $x \in K$, we shall say that *K satisfies Condition G at x* if, whenever $\langle x_n \in K : n \geq 1 \rangle$ is a sequence in K with $\lim_{n\to\infty} x_n = x$ and $\lambda < 1$, there exists an integer n_* such that $\lambda x \leq x_n$ for all $n \geq n_*$. We shall say simply that *K satisfies Condition G* if K satisfies Condition G at x for all $x \in K$.

In [1], we exhibit a map $f : \overset{\circ}{K_1} \to K_2$ where K_1 is a normal cone for which *Condition G does not hold.* The map is the *harmonic mean* $(A^{-1} + B^{-1})^{-1}$ of two nonnegative-definite symmetric real 2×2 matrices A, B. This is a map from the cone $C \times C = K_1$ to $C = K_2$, where C is the cone of such matrices. Its extension is order-preserving, homogeneous of degree 1 and continuous on the interior of K_1, but is *not* continuous at certain points on the boundary ∂K_1.

Definition 3.12. Recall that a closed cone K in a Hausdorff t.v.s. X is called *polyhedral* if there exist continuous linear functionals $\varphi_j : X \to \mathbb{R}$, $1 \le j \le N < \infty$, such that

$$K = \{x \in X : \varphi_j(x) \ge 0 \text{ for } 1 \le j \le N\}.$$

Lemma 2.14. Let K be a closed cone in a Hausdorff t.v.s. X. It follows that K is polyhedral *if and only if* K is finite-dimensional and satisfies Condition G.

We are grateful to Cormac Walsh, who pointed-out the converse part of the lemma.

The next condition that we will need to ensure continuity of F is a weak form of *homogeneity.*

Definition 2.15. We say that $f : \overset{\circ}{K}_1 \to K_2$ is *homogeneous (of degree* 1*)* if for all $x \in \overset{\circ}{K}$ and $\lambda > 0$,

$$f(\lambda x) = \lambda f(x).$$

Definition 2.16. Let K_1 be a closed cone with $\overset{\circ}{K}_1 \neq \emptyset$ in a Hausdorff t.v.s. X_1 and let K_2 be a closed cone in a Hausdorff t.v.s. X_2. Let $f : \overset{\circ}{K}_1 \to K_2$. We say that f *satisfies Condition WH at* $x \in K_1$ if, for every real α, $0 < \alpha < 1$, there exists $\delta > 0$ and an open neighbourhood V of x in X_1 such that for all $y \in V \cap \overset{\circ}{K}_1$ and for all real $\lambda \in [1 - \delta, 1]$,

$$f(\lambda y) \ge \alpha f(y).$$

If f satisfies Condition WH at x for every $x \in K_1$, then we shall say simply that f *satisfies Condition WH* or that f *is weakly homogeneous.*

Example 2.17. If $f : \overset{\circ}{K}_1 \to K_2$ is continuous, it is easy to prove that f satisfies Condition WH at every $x \in \overset{\circ}{K}_1$ such that $f(x) \in \overset{\circ}{K}_2$.

Example 2.18. If there exists $\delta_* > 0$ and a map $\varphi : [1 - \delta_*, 1] \to (0, \infty)$ with $\lim_{\lambda \to 1-} \varphi(\lambda) = 1$ such that

$$f(\lambda y) \ge \varphi(\lambda) f(y),$$

for all $y \in \overset{\circ}{K}_1$ and $\lambda \in [1 - \delta_*, 1]$, then f satisfies Condition WH.

The final condition that we shall need is a variant of *normality* for the cone K_2. We call this the *weak normality* (WN) or *sandwich* condition.

Definition 2.19. A cone K in a normed linear space, not necessarily finite-dimensional, is said to be *normal* if there exists a constant M such that $||x|| \leq M||y||$ whenever $0 \leq_K x \leq_K y$.

Definition 2.20. Let K_2 be a closed cone in a Hausdorff t.v.s. X_2. We shall say that K_2 *satisfies Condition WN* (or *is weakly normal*) if, *whenever* $\langle x_n \rangle$, $\langle y_n \rangle$, and $\langle z_n \rangle$ are sequences in K_2 with $0 \leq x_n \leq y_n \leq z_n$ for all $n \geq 1$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = y$, for some $y \in K_2$, *it follows that* $\lim_{n\to\infty} y_n$ exists and equals y.

Lemma 2.21. Let K_2 be a closed cone in a normed linear space. Then K_2 is normal if and only if K_2 satisfies Condition WN.

We now state our second extension theorem; we use the above additional conditions to ensure that the extended map F is sequentially continuous:

Theorem 2.22. Let K_1 be a closed cone with $\overset{\circ}{K_1} \neq \emptyset$ in a Hausdorff t.v.s. X_1, K_2 a closed cone in a Hausdorff t.v.s. X_2 and $f : \overset{\circ}{K_1} \to K_2$ a continuous order-preserving map. Assume that Condition A is satisfied, that K_1 satisfies Condition G at some $x \in \partial K_1$, that f satisfies Condition WH at x and that K_2 satisfies Condition WN. Define the extension F as in Theorem 2.10. Then F is sequentially continuous at x.

Corollary 2.23. Let K_1 be a closed polyhedral cone with $\overset{\circ}{K_1} \neq \emptyset$ in a Hausdorff t.v.s. X_1. Let K_2 be a closed cone in a Hausdorff t.v.s. X_2 and assume either that (i) K_2 is finite-dimensional or, more generally, that (ii) K_2 has the monotone convergence property and satisfies Condition WN. Let $f : \overset{\circ}{K_1} \to K_2$ be continuous and order-preserving and satisfy Condition WH on K_1. Then f has a sequentially continuous extension $F : K_1 \to K_2$ that is order-preserving and satisfies Condition WH on K_1.

3. THE POSITIVE CONE

In this section, we focus on maps f defined on the interior of the standard non-negative cone $K = \mathbb{R}_+^N$ in $\mathbb{R}^N$ which are both homogeneous of degree 1 and order-preserving. In this case, the partial ordering induced by the cone is exactly the usual partial product ordering on $\mathbb{R}^N$. These maps arise naturally in the study of certain classes of Discrete Event Systems. Such maps are non-expanding in Thompson's part metric and continuous on the interior of the cone, so that continuity is no longer required as an explicit assumption.

First, we work in a more general setting. Recall the definition of Thompson's part metric:

Definition 3.1. Let v, y denote elements of a cone K. We say that *v and y are comparable* if there exist reals $\alpha > 0$ and $\beta > 0$ with $\alpha y \leq_K v \leq_K \beta y$. The notion of comparability divides K into disjoint equivalence classes called the *components* or *parts* of K; for $v \in K$ we let $K(v)$ denote the set of points that are comparable with v.

Definition 3.2. Let v, y denote comparable elements of a cone K. We define the positive real quantities,

$$\begin{aligned} M(v/y) &:= \inf\{\beta > 0 : v \leq \beta y\}, \\ \bar{d}(v,y) &:= \max\{\log M(v/y), \log M(y/v)\}. \end{aligned}$$

The restriction of $\bar{d}$ to each part $K(v)$ is a metric, called *Thompson's (part) metric.* By defining $\bar{d}(v,y) := +\infty$ if v, y are non-comparable elements of K, we extend $\bar{d}$ to the whole of K (but only the restrictions to each part are actually metrics).

Remark 3.3. Let K_1 be a normal cone with $\overset{\circ}{K}_1 \neq \emptyset$ in a Banach space X_1 and K_2 be a normal cone in a Banach space X_2. Let $\bar{d}_1$ denote the part metric on $\overset{\circ}{K}_1$. It is known that $\bar{d}_1$ gives the same topology on $\overset{\circ}{K}_1$ as the topology induced by the norm on X_1. See Proposition 1.1 in [7] for a proof and references to the literature. See also [9]. If $f : \overset{\circ}{K}_1 \to K_2$ is homogeneous of degree 1 and order-preserving, then there exists $v \in K_2$ such that $f(x)$ is comparable to v in K_2 for all $x \in \overset{\circ}{K}_1$ (i. e., the interior maps to a single part). If $K_2(v)$ is the part corresponding to v and $\bar{d}_2$ is the part metric on $K_2(v)$, then

$$\bar{d}_2(f(x_1), f(x_2)) \leq \bar{d}_1(x_1, x_2),$$

for all $x_1, x_2 \in \overset{\circ}{K}_1$, i. e., f is non-expanding on $\overset{\circ}{K}_1$ in the part metric. It follows automatically that f is continuous as a map from $\overset{\circ}{K}_1$ to K_2. Thus, in this case we do not need to assume explicitly that $f : \overset{\circ}{K}_1 \to K_2$ is continuous.

3.1. Topical maps on the positive cone

Let $K_1 = K_2 = K := \mathbb{R}^N_+$, the standard positive cone in $\mathbb{R}^N$. Maps $f : \overset{\circ}{K} \to \overset{\circ}{K}$ that are order-preserving and homogeneous of degree 1 are a specific example of the general case above; they are non-expanding with respect to the part metric on $\overset{\circ}{K}$. In fact, if homogeneity holds, then being order-preserving is equivalent to being non-expanding, see [3]. Hence these maps are continuous on $\overset{\circ}{K}$. Such maps are sometimes called "topical" in the literature and are of interest for certain classes of discrete event systems. They may be viewed as the image under the bijection (component-wise exponentiation) $\exp : \mathbb{R}^N \to \overset{\circ}{K}$ of maps $g : \mathbb{R}^N \to \mathbb{R}^N$ that are *additively* homogeneous and order-preserving and, consequently, non-expanding in the supremum (ℓ_∞) norm. To each such map g there corresponds a map $f : \overset{\circ}{K} \to \overset{\circ}{K}$ with

$$f(x) = (\exp \circ g \circ \log)(x).$$

With suitable modifications of the proofs to take advantage of full, rather than weak, homogeneity, our results imply the following corollary. This result was proved previously, by more direct means involving the comparison of various componentwise limits, by [2].

Corollary 3.4. All homogeneous order-preserving maps $f : \overset{\circ}{K} \to \overset{\circ}{K}$ with $K = \mathbb{R}^N_+$ have an extension $F : K \to K$ that is homogeneous, order-preserving, and continuous on the whole of K.

Remark 3.5. We can also show, see [2], that the extended map F is non-expanding in the extended part metric, mapping each part of K to a part.

Corollary 3.6. Let $f : \overset{\circ}{K} \to \overset{\circ}{K}$, with $K = \mathbb{R}^N_+$, be an order-preserving homogeneous map with the order-preserving homogeneous (and continuous) extension $F : K \to K$. Then F has at least one eigenvector in $K - \{0\}$.

Proof. Let $\Pi \subset K$ denote the intersection of the positive cone with the surface of the (ℓ_2) unit hyper-sphere,

$$\Pi := \{x \in K : \ ||x||_2 = 1\},$$

and let π denote the projection (normalization),

$$\pi : K - \{0\} \to \Pi, \ x \mapsto \frac{x}{||x||_2}.$$

We have seen that f has a continuous extension F to the whole cone K. If $F(x) = 0$ for some $x \in \Pi$ then, by definition, F has an eigenvector with eigenvalue 0 and we are done. This happens, for example, in the case $f : \overset{\circ}{\mathbb{R}^2_+} \to \overset{\circ}{\mathbb{R}^2_+}$ with $f(x_1, x_2) := \sqrt{x_1 x_2}(1, 1)$, where all $x \in \partial\Pi$ are mapped to the vertex 0. Suppose, on the other hand, that $F(x) \neq 0$ for all $x \in \Pi$, then the projected map $\pi \circ F : \Pi \to \Pi$ is well-defined and continuous on Π. Further, Π is homeomorphic to a compact convex set. Hence, by Brouwer's fixed point theorem, $\pi \circ F$ has at least one fixed point in Π. By homogeneity, it follows that F itself must have at least one eigenvector in $K - \{0\}$. □

In fact, we can prove a stronger result:

Lemma 3.7. Let C be a closed cone with non-empty interior in a finite dimensional Banach space X. Let $F : C \to C$ be a continuous order-preserving homogeneous (of degree one) map that maps the interior of C into itself. Then F has an eigenvector in $C - \{0\}$ *with nonzero eigenvalue.*

Proof. (Outlined.) If $u \in \overset{\circ}{C}$, there exists $c > 0$ such that $F(u) \geq cu$. Take $a > 1$ and define $g(x) := (a/c)F(x)$, so that $g(u) \geq au$ and the sequence $\langle g^m(u) : m \geq 1\rangle$ is unbounded in norm. Theorem 2.1 in [6] then implies that there exists $x \in C$ with $||x|| = 1$ and $t \geq 1$ such that $g(x) = tx$. (The assumption of compactness of the map g, in the statement of that theorem, can be weakened.)

It follows that $F(x) = (tc/a)x$. A simple limiting and compactness argument, letting $a \to 1$, shows that F has an eigenvector with eigenvalue not less than c. In fact, if we define

$$c := \sup\{k : \exists x \in C,\ ||x|| = 1,\ F(x) \geq kx\},$$

then we deduce that there exists an eigenvector of F with eigenvalue c. □

Definition 3.8. From the viewpoint of applications, a natural question is whether the map $f : \overset{\circ}{K} \to \overset{\circ}{K}$ has a *cycle-time vector*, defined formally by

$$\chi(f) := \lim_{k\to\infty} \left(f^k(x)\right)^{1/k}.$$

If this limit exists for some $x \in \overset{\circ}{K}$, then it follows, from the fact that f is non-expanding, that it exists for all $y \in \overset{\circ}{K}$ and takes the same value everywhere. Thus the cycle time is naturally regarded as a property of the map itself.

Existence of an eigenvector $x \in \overset{\circ}{K}$ in the interior with, say, $f(x) = \lambda x$ for some $\lambda > 0$, implies directly the existence of the cycle time with $\chi(f) = \lambda \mathbf{1}$, where $\mathbf{1} := (1, 1, \ldots, 1)$. Thus our above result establishes the following corollary:

Corollary 3.9. If $\chi(f)$ does not exist, for $f : \overset{\circ}{K} \to \overset{\circ}{K}$ homogeneous and order-preserving on the positive cone K, then the extended map F has at least one eigenvector in $\partial K - \{0\}$ with nonzero eigenvalue and there are no eigenvectors in $\overset{\circ}{K}$.

The cycle-time vector is known to exist for certain classes of maps in general dimension N. Specifically, a nonlinear hierarchy of such maps may be built from simple maps by closure under a finite set of operations, see [4]. The cycle-time also exists for all order-preserving homogeneous maps with $N = 1, 2$.

However, $\chi(f)$ need not exist in general for $N \geq 3$, as illustrated by a family of maps introduced by [5]: consider a sequence of reals $\langle a^k \in [0, 1] : k \geq 1\rangle$ and let

$\sigma^k := \sum_{j=1}^k a^k$ with $\sigma^0 := 0$. Then there exists a homogeneous order-preserving map $f : \overset{\circ}{K} \to \overset{\circ}{K}$ with $K := \mathbb{R}^3_+$, such that for all $k \geq 0$,

$$f^k(1,1,1) = \big(1, \exp(\sigma^k), \exp(k)\big)\,.$$

For suitable choices of the sequence $\langle a^k \rangle$, we can arrange that the sequence $\langle \sigma^k/k \rangle$ does not converge and, hence, that $\chi(f)$ does not exist.

Construction of a particular family of such maps, for which there is no cycle time, reveals that the projected map $\pi \circ F$ fixes a continuum of points on one edge of Π.

4. CONCLUSION

We have presented results on the continuous extension of maps defined on the interior of closed cones, firstly in considerable generality and secondly applied to the case of the standard positive cone.

Our general results have applications in other areas, particularly to operator-valued means, where the existence of continuous extensions is a natural question. In this case, it is also relevant to consider maps which are not order-preserving. For further discussions, see [1].

For the positive cone, we have shown that all homogeneous order-preserving maps have a continuous extension to the boundary of the cone and that non-existence of the cycle time restricts the fixed-point set of the extended map. It would be interesting to have a characterization of the possible fixed-point sets and to clarify connections between existence of a cycle time and properties of these sets.

ACKNOWLEDGEMENTS

During this work, A. Burbanks was supported partially by EPSRC GR/L42742, by the research network ALAPEDES, and by BRIMS (Hewlett-Packard Laboratories, Bristol, UK). R. Nussbaum supported partially by NSF DMS 0070829.

(Received April 23, 2002.)

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