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RATIONAL ALGEBRA AND MM FUNCTIONS

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MM functions, formed by finite composition of the operators *min*, *max* and *translation*, represent discrete-event systems involving disjunction, conjunction and delay. The paper shows how they may be formulated as homogeneous rational algebraic functions of degree one, over $(max, +)$ algebra, and reviews the properties of such homogeneous functions, illustrated by some orbit-stability problems.

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1. INTRODUCTION

In the theory of discrete dynamic systems, attention has focused recently on the so-called MM functions, which are those constructible by a finite number of applications of the operators *min*, *max* and *translation*, representing systems involving disjunction, conjunction and delay. This may be see as a development beyond earlier studies of conjunctive systems using the well-known $(max, +)$ algebra (Cuninghame-Green [2]; Baccelli et al [1]), but some sacrifice of the intuitive algebraic properties of $(max, +)$ results when working in a system with three underlying operations.

However, the MM functions may, as discussed below, still be approached within the context of $(max, +)$ by using the fact that the operator *min* is itself expressible rationally in that algebra. In this formulation, MM functions become rational algebraic expressions in which both numerator and denominator are homogeneous, the degree of the numerator exceeding that of the denominator by unity. Manipulation of such expressions closely follows the rules of elementary algebra. Illustrations are given in the context of orbit stability.

2. NOTATION

Denote by $\Re$ the $(max+)$ semiring, see (Cuninghame-Green [2]; Baccelli et al [1]): briefly, the elements of $\Re$ are the real numbers $I\!R$ with the binary operations *max* and $+$ notated as $\oplus$, $\otimes$ respectively. In some contexts, it is useful to augment $\Re$ with the element $-\infty$, though this will not be done here. Iterated use of the 'addition' $\oplus$ and of the 'multiplication' $\otimes$ are notated as $\Sigma^{\oplus}$ and $\prod^{\otimes}$ respectively, and j-fold 'powers' $x \otimes \ldots \otimes x$ by using a bracketed exponent: $x^{(j)}$. Thus $x^{(j)}$ equals the ordinary arithmetical product jx. This can be extended to any real values of j or x, though for clarity all exponents are non-negative integers in examples. E. g.

$$2 \otimes x_1 \otimes x_2^{(2)} \oplus 3 \otimes x_2 \otimes x_3 \oplus x_1^{(3)} \otimes x_3 \tag{1}$$

denotes the function more conventionally written

$$\max(x_1 + 2x_2 + 2,\ x_2 + x_3 + 3,\ 3x_1 + x_3)\,.$$

This notational system was originally developed because it gives many problems of discrete mathematics the familiar character of linear and polynomial algebra and, despite lacking an inverse for its 'addition' operation $\oplus$, it mimics many of the properties of an algebraically complete field. In particular, (Cuninghame-Green and Meijer [3]), any *maxpolynomial* $\Sigma_j^{\oplus}\left(a_j \otimes x^{(j)}\right)$ in one variable possesses a unique resolution into linear factors: $\beta \otimes \Pi_r^{\otimes}(x \oplus \beta_r)$, with *corners* β_r. E. g.,

$$2 \otimes x^{(2)} \oplus 5 \otimes x \oplus 7 = 2 \otimes (x \oplus 2) \otimes (x \oplus 3).$$

Efficient (linear-time) algorithms exist for the $(max, +)$-algebraic composition of such maxpolynomials, including resolution into linear factors, as shown in (Cuninghame-Green [4]).

3. SEVERAL VARIABLES

Ordinary arithmetical multiplication will be denoted by juxtaposition. If $J \subset \Re^N$ is a finite set of (N-tuple) *indices*, the notation $\mathbf{x}^{(\mathbf{j})}$, for $\mathbf{x} = (x_1, \ldots, x_N)$ and $\mathbf{j} \in J$, will denote $x_1^{(j_1)} \otimes \ldots \otimes x_N^{(j_N)}$, which equals the usual real inner product $\langle \mathbf{x}, \mathbf{j} \rangle = j_1 x_1 + \ldots + j_N x_N$. A *term* will mean a monomial $a_{\mathbf{j}} \otimes \mathbf{x}^{(\mathbf{j})}$, where $\{a_{\mathbf{j}} \in \Re \,|\, \mathbf{j} \in J\}$ is a collection of *coefficients* indexed by J. No distinction is made between $j_s = 0$ and the simple absence of x_s from the term. The *degree* of the term is $j_1 + \ldots + j_N$, which may be zero. A *maxpolynomial* (in several variables) is any term, or finite set of terms combined using the associative operation $\oplus$ as in (1), or the operator $\Sigma^{\oplus}$.

4. INESSENTIAL TERMS

A term $a_{\mathbf{k}} \otimes \mathbf{x}^{(\mathbf{k})}$, with $\mathbf{k} \in J$, is *inessential* in the maxpolynomial $\Sigma_{\mathbf{j} \in J}^{\oplus} a_{\mathbf{j}} \otimes \mathbf{x}^{(\mathbf{j})}$ if

$$a_{\mathbf{k}} \otimes \mathbf{x}^{(\mathbf{k})} \le \Sigma_{\mathbf{j} \in J \setminus \mathbf{k}}^{\oplus} a_{\mathbf{j}} \otimes \mathbf{x}^{(\mathbf{j})},\ \forall \mathbf{x},$$

and *strictly inessential* if the foregoing inequality is strict. Evidently, any such term could be deleted from the formal maxpolynomial without changing it as a function. The following result was proved for the case $N = 1$ by Cuninghame-Green and Meijer [3], but the proof adapts easily to the several-variable case.

Theorem 1. The term $a_{\mathbf{k}} \otimes \mathbf{x}^{(\mathbf{k})}$, with $\mathbf{k} \in J$, is inessential (respectively strictly inessential) in the maxpolynomial $\Sigma^{\oplus}_{\mathbf{j}\in J} a_{\mathbf{j}} \otimes \mathbf{x}^{(\mathbf{j})}$ iff $a_{\mathbf{k}}$ lies in (respectively in the interior of) the convex hypograph of the other $\{a_{\mathbf{j}} \,|\, \mathbf{j} \in J \setminus \mathbf{k}\}$.

The removal, if necessary, of inessential terms from a maxpolynomial thus amounts to implementing a convex-hull routine. For e. g. $N = 1$ this can be done in linear time by adapting the algorithm of Graham (Manber [7]). In fact, however, many standard $(max, +)$-algebraic manipulations intrinsically do not generate strictly inessential terms.

5. RATIONAL EXPRESSIONS

The group operation $\otimes$ is invertible. Double fraction bars, reminiscent of division, will denote use of this inverse: $P//Q = P - Q$. Absolute values can thus be notated, e. g.: $|\,a - b\,| = (a//b) \oplus (b//a)$. A *rational expression* is a maxpolynomial P or an expression $P//Q$, where P, Q are maxpolynomials. For example:

$$\left(2 \otimes x_1^{(2)} \oplus 5 \otimes x_2 \otimes x_3\right) // \left(x_1^{(2)} \oplus 3 \otimes x_2^{(3)} \oplus 1 \otimes x_2 \otimes x_3 \oplus 2 \otimes x_1\right) \tag{2}$$

In the obvious way, a rational expression induces a *rational function* from $\Re^N$ to $\Re$. (Function and expression will not be distinguished notationally.)

Such functions admit a straightforward procedure for finding maxima and minima. The following result was proved for $N = 1$ by Cuninghame-Green and Meijer [3], but the proof adapts easily to several variables.

Theorem 2. Given the rational expression

$$R(\mathbf{x}) = P(\mathbf{x})//Q(\mathbf{x}) = \left(\Sigma^{\oplus}_{\mathbf{j}\in J} a_{\mathbf{j}} \otimes \mathbf{x}^{(\mathbf{j})}\right) // \left(\Sigma^{\oplus}_{\mathbf{j}\in K} b_{\mathbf{j}} \otimes \mathbf{x}^{(\mathbf{j})}\right),$$

(i) If $J \subseteq K$ then $\max_{\mathbf{x}} R(\mathbf{x}) \le \max_{\mathbf{j}\in J}(a_{\mathbf{j}}//b_{\mathbf{j}})$, with equality if Q has no strictly inessential terms.

(ii) If $K \subseteq J$ then $\min_{\mathbf{x}} R(\mathbf{x}) \ge \min_{\mathbf{j}\in K}(a_{\mathbf{j}}//b_{\mathbf{j}})$, with equality if P has no strictly inessential terms.

For example, the global maximum of the rational function (2) is $\max(2//0, 5//1) = 4$.

6. HOMOGENEOUS RATIONAL EXPRESSIONS

If all terms in a maxpolynomial P have the same degree r (say), the maxpolynomial is *homogeneous* (of degree $\deg(P) = r$). In particular, a maxpolynomial which is merely a constant is homogeneous of degree zero.

A rational expression $R = P//Q$ is homogeneous if both P and Q are homogeneous, and its *degree* is then $\deg(R) = \deg(P) - \deg(Q)$. Examples of homogeneous rational expressions are:

$$5; \quad 3 \otimes x_1 \oplus 2 \otimes x_3; \qquad \left(x_1 \otimes x_2^{(2)} \oplus x_4^{(3)}\right) // (x_3 \oplus 2 \otimes x_4).$$

7. COMPOSITION OF RATIONAL FUNCTIONS

For given r, a function from $\Re^N$ to $\Re^M$ is *homogeneous rational of degree* r if it is componentwise so. Denote by $\Upsilon(N, M, r)$ the set of such functions. E. g., $\mathbf{F} \in \Upsilon(3,3,1)$, where

$$\mathbf{F} : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} 2 \otimes x_1 \oplus -1 \otimes x_2 \\ \left(3 \otimes x_1 \otimes x_2 \oplus x_2^{(2)}\right) // (4 \otimes x_1 \oplus 3 \otimes x_2) \\ 4 \otimes x_2 \otimes x_3 // (6 \otimes x_2 \oplus x_3) \end{bmatrix}. \tag{3}$$

The following lemma is straightforwardly proved.

Lemma 1. If $\theta \in \Upsilon(N, L, r)$ and $\psi \in \Upsilon(L, M, s)$ then the composition $\psi \circ \theta \in \Upsilon(N, M, rs)$. Thus

(i) $\Upsilon(N, N, r)$ is closed under composition iff r equals 0 or 1.

(ii) Every $\Upsilon(N, M, r)$ is closed under the componentwise action of every $\psi \in \Upsilon(L, 1, 1)$.

8. MM FUNCTIONS

An *MM function* is any function obtained by the composition of a finite number of the functions *max*, *min* and the *translations* $\{m_a : x \mapsto a \otimes x \mid x, a \in \Re\}$; the scalars a will be called *parameters*. MM(N, M) will denote the set of functions from $\Re^N$ to $\Re^M$ in which every component is an MM function. Now, clearly $x \oplus y$ and $a \otimes x$ are homogeneous of degree 1. Moreover, the smaller of two numbers equals their sum less the greater, whence

$$\min(x, y) = x \otimes y // (x \oplus y),$$

which is also homogeneous of degree 1. Hence, using Lemma 1.

Theorem 3. MM$(N, M) \subset \Upsilon(N, M, 1)$, and every $\Upsilon(N, M, r)$ is closed under the componentwise action of any MM function. Hence, every $\Upsilon(N, M, r)$ is a $\oplus$-semimodule.

For example, $\mathbf{F} \in$ MM $(3, 3)$, where

$$\mathbf{F} : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} \max(x_1 + 2, x_2 - 1) \\ \max(\min(x_1, x_2 - 1), x_2 - 3) \\ \min(x_2 + 4, x_3 - 2) \end{bmatrix} \tag{4}$$

Converting this to $(max, +)$ and tidying, produces (3), lying in $\Upsilon(3, 3, 1)$.

9. EIGENVECTORS

$\boldsymbol{\xi} \in \Re^N$ and $\lambda \in \Re$ are respectively *eigenvector* and *eigenvalue* for given $\mathbf{F} : \Re^N \mapsto \Re^N$ iff $\mathbf{F}(\boldsymbol{\xi}) = \lambda \otimes \boldsymbol{\xi}$. They play a classical role in the stability of the *orbits* $\{\mathbf{F}^{[t]}(\mathbf{b}) \mid t = 0, 1, \ldots; \mathbf{b} \in \Re^N\}$ of a discrete dynamic process of which $\mathbf{F}$ is the transfer function, where $\mathbf{F}^{[0]}(\mathbf{b}) = \mathbf{b}$ and $\mathbf{F}^{[t+1]}(\mathbf{b}) = \mathbf{F} \circ \mathbf{F}^{[t]}(\mathbf{b})$, $t = 0, 1, \ldots$ If F_i, ξ_i are the ith components of $\mathbf{F}$, $\boldsymbol{\xi}$ respectively, the eigenvector-eigenvalue relations are thus $F_i(\boldsymbol{\xi})//\xi_i = \lambda$ $(i = 1, \ldots, N)$. Hence $\boldsymbol{\xi}$, and so λ, may in principle be found by solving the set of simultaneous rational equations of the form $F_1(\mathbf{x})//x_1 = \ldots = F_N(\mathbf{x})//x_N$. In fact, these may be reduced to a single equation

$$\Sigma^{\oplus}_{i>1} \left[(x_i \otimes F_1(\mathbf{x})//x_1 \otimes F_i(\mathbf{x})) \oplus (x_1 \otimes F_i(\mathbf{x})//x_i \otimes F_1(\mathbf{x}))\right] = 0,$$

which expresses the condition

$$\max_{i>1} \mid (F_1(\mathbf{x})//x_1) - (F_i(\mathbf{x})//x_i) \mid = 0.$$

Notice that if $\mathbf{F} \in \Upsilon(N, N, 1)$, in particular if $\mathbf{F} \in \mathrm{MM}(N, N)$, all the functions equated in these relations are homogeneous of degree 0. The efficient determination of zeros of homogeneous functions R with $\deg(R) = 0$ is therefore a topic of central interest.

10. ILLUSTRATION 1

To find, if existent, eigenvector and eigenvalue of $\mathbf{F} : (x_1, x_2) \mapsto$

$$\begin{bmatrix} \min(\max(x_1 + 1, x_2 - 1), x_2) \\ \max(\min(x_1 - 2, x_2 + 3), x_1 - 4) \end{bmatrix}$$

$$= \begin{bmatrix} \left(1 \otimes x_1 \otimes x_2 \oplus -1 \otimes x_2^{(2)}\right) // (1 \otimes x_1 \oplus x_2) \\ \left(-6 \otimes x_1^{(2)} \oplus 1 \otimes x_1 \otimes x_2\right) // (-2 \otimes x_1 \oplus 3 \otimes x_2) \end{bmatrix}.$$

On writing u for $x_1//x_2$, the condition $F_1(\mathbf{x})//x_1 = F_2(\mathbf{x})//x_2$ leads to

$$(1 \otimes u \oplus -1) // \left(1 \otimes u^{(2)} \oplus u\right) = \left(-6 \otimes u^{(2)} \oplus 1 \otimes u\right) // (-2 \otimes u \oplus 3).$$

Cross-multiplying and resolving into linear factors, a zero must be found for

$$4 \otimes (u \oplus -2) \otimes (u \oplus 5) // u^{(2)} \otimes (u \oplus -1) \otimes (u \oplus 7).$$

Evaluating this function at its corners as in (Cuninghame-Green, [4, 5]), shows that it equals 4 at $u = -1$ and -8 at $u = 5$. Linear interpolation finds a zero at $u = 1$, giving $\lambda = -1$ and eigenvector $(1, 0)$.

11. CYCLE–TIME VECTOR

For $\mathbf{F} \in \mathrm{MM}(N, N)$ and $\mathbf{x} \in \Re^N$ the limit $\mathbf{a}$, as $t \to \infty$, of $t^{-1}\mathbf{F}^{[t]}(\mathbf{x})$ is independent of $\mathbf{x}$. This *cycle-time vector* (ctv) characterises the asymptotic orbit. If $\mathbf{F}^{\wedge} \in \mathrm{MM}(N, N)$ is derived from $\mathbf{F}$ by setting all parameters to zero, it is not hard to show

Theorem 4. The ctv of $\mathbf{F}$ is a fixed point of $\mathbf{F}^{\wedge}$.

$\mathbf{F}$ may not have an eigenvector, but always has a ctv $\mathbf{a}$ as shown by Gaubert and Gunawardena [6]. For $\mathbf{x} \in \Re^N$, define $\mathbf{x}^{[t]} = \mathbf{F}^{[t]}(\mathbf{x})$. The same authors show that $\mathbf{F}$ always has a *generalised eigenvector* $\boldsymbol{\xi}$ satisfying, in our present notation, $\xi_i^{[t]} = a_i^{(t)} \otimes \xi_i$, $\forall\,(i = 1, \dots, N; t \geq 0)$, giving $F_i\left(\boldsymbol{\xi}^{[t]}\right) // \xi_i^{[t]} = a_i$.

12. ILLUSTRATION 2

To find the ctv of $\mathbf{F}$ in (4), first apply Theorem 2 to (3):

$$F_1(\mathbf{x})//x_1 = (2 \otimes x_1 \oplus -1 \otimes x_2)//x_1 \geq 2.$$

Hence $a_1 \geq 2$ and similarly, $-3 \leq a_2 \leq -1$ and $a_3 \leq -2$. So $a_1 > a_2$, a_3 and $\xi_1^{[t]} \gg \xi_2^{[t]}$, $\xi_3^{[t]}$ for large t. So, substituting in (4):

$$F_1\left(\boldsymbol{\xi}^{[t]}\right) // \xi_1^{[t]} = \left(\xi_1^{[t]} + 2\right) // \xi_1^{[t]} = 2,$$

whence $a_1 = 2$ and similarly $a_2 = -1$. Since thereby $a_3 < a_2$, it follows that $\xi_2^{[t]} \gg \xi_3^{[t]}$ and thence $a_3 = F_3\left(\boldsymbol{\xi}^{[t]}\right) // \xi_3^{[t]} = -2$.

Theorem 4 is now readily verified for this example:

$$\mathbf{F}^{\wedge}(2, -1, -2) = (2, -1, -2).$$

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