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DISTURBANCE DECOUPLING OF NONLINEAR MISO SYSTEMS BY STATIC MEASUREMENT FEEDBACK

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This paper highlights the role of the rank of a differential one-form in the solution of such nonlinear control problems via measurement feedback as disturbance decoupling problem of multi-input single output (MISO) systems. For the later problem, some necessary conditions and sufficient conditions are given.

1. INTRODUCTION

The disturbance decoupling problem by state feedback has been addressed in the literature by several authors (see e. g., the historical accounts of [3, 6, 8]). In general, most of the solutions assume that the state is available for measurement. When this is not the case, two approaches may be followed: the reconstruction of the state by means of an observer or the use of output feedback. Our objective is to investigate solutions with the second kind of controller since it avoids approximations in the estimated state and circumvents the superposition of controller and observer.

The few contributions which deal with the nonlinear disturbance decoupling problem via output or measurement feedback are now briefly mentioned. The disturbance decoupling problem has been solved by a geometric approach in [7]. The same problem has also been considered in [9], and a necessary and sufficient condition has been given using algebraic tools. For the multi-input multi-output (MIMO) case, there just exists a sufficient condition in [1].

The purpose of this paper is to solve the disturbance decoupling problem via measurement feedback for the class of MISO systems with a new approach based on the rank of a differential one-form [4]. In the mean time through these results, it allows to highlight the role of the rank of a differential one-form in the solution of nonlinear control problem. For the disturbance decoupling problem via static measurement feedback of MISO systems, some necessary conditions and sufficient conditions are given. A complete solution is provided for a restricted class of such systems.

The paper is organized as follows. In Section 2, some definitions and technical tools useful are presented. Section 3 gives the main results of this paper and finally, some conclusions are drawn in the last section.

2. DEFINITIONS AND BACKGROUND

Consider,

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + p(x)q \\ y &= h(x) \\ z &= k(x), \end{aligned} \tag{1}$$

where $x \in I\!R^n$ denotes the state, $u \in I\!R^m$ denotes the control, $q \in I\!R^\nu$ denotes the disturbance, $y \in I\!R$ denotes the output-to-be controlled, and $z \in I\!R^\mu$ denotes the measured output. Assume that f, g, p, h and k are meromorphic functions of their arguments.

Let $\mathcal{K}$ denote the field of meromorphic functions of x, u, q, and a finite number of derivatives of u and q and define the vector space $\mathcal{E} = \text{span}_{\mathcal{K}}\{\mathrm{d}\zeta | \zeta \in \mathcal{K}\}$, $\mathcal{X} = \text{span}_{\mathcal{K}}\{\mathrm{d}x\}$, $\mathcal{Z} = \text{span}_{\mathcal{K}}\{\mathrm{d}z,\}$ and $\mathcal{U} = \text{span}_{\mathcal{K}}\{\mathrm{d}u, \mathrm{d}\dot{u}, \ldots, \mathrm{d}u^{(k)}, \ldots\}$.

Definition 1. (Conte et al [3]) The relative degree r of the output y is set to be

$$r := \min\{k \in I\!N | \mathrm{d}y^{(k)} \notin \mathcal{X}\}.$$

If such an integer does not exist, then one sets $r := \infty$.

As in [5], define the subspace Ω by

$$\Omega = \left\{\omega \in \mathcal{X} | \forall k \in I\!N : \omega^{(k)} \in \text{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}y^{(r)}, \ldots, \mathrm{d}y^{(r+k-1)}\}\right\}.$$

The subspace Ω is instrumental for solving the disturbance decoupling problem and may be computed as the limit of the following algorithm:

$$\begin{aligned} \Omega^0 &= \text{span}_{\mathcal{K}}\{\mathrm{d}x\}, \\ \Omega^{k+1} &= \left\{\omega \in \Omega^k | \dot{\omega} \in \Omega^k + \text{span}_{\mathcal{K}}\{\mathrm{d}y^{(r)}\}\right\}, \quad (k \in I\!N). \end{aligned}$$

Problem Statement. (Disturbance decoupling problem by regular static measurement feedback.)
Consider a nonlinear system of the form (1), find, if possible, a nonlinear feedback of the form

$$u = \alpha(z) + \beta(z)v \tag{2}$$

where $\beta(z)$ is invertible and such that the closed loop system satisfies:

$$\begin{aligned} &\text{(i)} \quad \mathrm{d}y^{(k)} \in \text{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}v, \ldots, \mathrm{d}v^{(k-r)}\}, \quad \forall\, k \geq r \\ &\text{(ii)} \quad \mathrm{d}y^{(r)} \notin \text{span}_{\mathcal{K}}\{\mathrm{d}x\}. \end{aligned}$$

Condition (i) represents the noninteraction with the disturbances whereas condition (ii) represents the output controllability. Invertibility of $\beta(z)$ implies the existence of an inverse matrix $\beta^{-1}(z)$ whose entries belong to $\mathcal{K}$. Both $\beta(z)$ and $\beta^{-1}(z)$ may have singularities in $\mathbb{R}^n$.

The following notations are borrowed from [4]. The originality of this paper is mainly due to the fact that this definition is used for solving the disturbance decoupling problem by static measurement feedback.

Let $(\mathrm{d}\omega)^\gamma = \mathrm{d}\omega \wedge \ldots \wedge \mathrm{d}\omega$ be a γ-fold product.

Definition 2. The rank of an one-form ω is γ if $\omega \wedge (\mathrm{d}\omega)^{(\gamma)} \neq 0$ but $\omega \wedge (\mathrm{d}\omega)^{(\gamma+1)} = 0$.

From [4], $\gamma + 1$ is the dimension of the smallest integrable space containing a given one-form, i. e., $\gamma + 1$ is the minimal number of exact one forms $\mathrm{d}\alpha_1, \ldots, \mathrm{d}\alpha_{\gamma+1}$ such that

$$\omega = \sum_{i=1}^{\gamma+1} \xi_i \mathrm{d}\alpha_i,$$

for some $\xi_i \in \mathcal{K}$. In the sequel, we will call the set $\{\mathrm{d}\alpha_1, \ldots, \mathrm{d}\alpha_{\gamma+1}\}$ a basis of ω.

As shown in [2, 4], the Pfaff–Darboux Theorem can be used to construct such a basis. Another matrix based construction is described in [4].

The rank of a differential form will be extensively used to give necessary and sufficient conditions for the disturbance decoupling problem with measurement feedback.

3. MAIN RESULTS

A direct extension of the result in [9] allows to give a sufficient condition for the solvability of the problem.

Theorem 1. The disturbance decoupling problem is solvable by static measurement feedback if:

(i) $\mathrm{d}y^{(r)} \in \Omega + \mathcal{Z} + \mathcal{U}$.

(ii) There exists a one-form $\omega \in \mathcal{Z} + \mathcal{U}$ s.t.$\mathrm{d}y^{(r)} - \omega \in \Omega$ and s.t. rank $(\omega) = \gamma \leq m - 1$.

(iii) For any basis $\mathrm{span}_{\mathcal{K}}\{\mathrm{d}\alpha_1, \ldots, \mathrm{d}\alpha_{\gamma+1}\}$ of Ω, i. e., $\omega = \beta_1 \mathrm{d}\alpha_1(z,u) + \beta_2 \mathrm{d}\alpha_2(z,u) + \beta_3 \mathrm{d}\alpha_3(z,u) + \ldots + \beta_{\gamma+1} \mathrm{d}\alpha_{\gamma+1}(z+u)$,

$$\operatorname{rank} \frac{\partial}{\partial u} [\alpha_i(z,u)] = \gamma + 1, \quad (i = 1, \ldots, \gamma + 1). \tag{3}$$

Proof. Assume that condition (i) is fulfilled. Then there exists a one-form ω such that $dy^{(r)} - \omega \in \Omega$. From [4], ω can be written as follows

$$\begin{aligned} \omega &= \beta_1 d\alpha_1(z,u) + \beta_2 d\alpha_2(z,u) \\ &\quad + \beta_3 d\alpha_3(z,u) + \ldots + \beta_{\gamma+1} d\alpha_{\gamma+1}(z,u). \end{aligned} \tag{4}$$

When conditions (ii) and (iii) are satisfied, the $\gamma+1$ one-forms $d\alpha_i$ are independent in u. Since $\gamma + 1 \leq m$, the following definitions

$$v_i = \alpha_i(z,u),$$

for $i = 1, \ldots, \gamma+1$ can be extended to define a static measurement feedback. Under this feedback,

$$dy^{(r)} \in \Omega \oplus \text{span}_{\mathcal{K}}\{dv\}.$$ □

Remark 1. In the special case where $\dim \mathcal{Z} = 1$, if $\Omega \cap \mathcal{Z} \neq 0$ then the disturbance decoupling problem has a solution if and only if the system is already decoupled.

Example 1. This example illustrates Theorem 1. The considered system admits a static measurement feedback solution of the disturbance decoupling problem.

Consider,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1(\sin x_3)u_1 + x_2(\cos x_3)u_2 + x_5 \\ \dot{x}_3 &= f(x) + q \\ \dot{x}_4 &= u_2 \\ \dot{x}_5 &= x_1(\sin x_3)u_1 + x_2(\cos x_3)u_2 + x_5 - x_1^2 \\ y &= x_1 \\ z &= x_3. \end{aligned} \tag{5}$$

Since $\Omega = \text{span}_{\mathcal{K}}\{dx_1, dx_2, dx_4, dx_5\}$, choosing $\omega = x_1 d(\sin x_3)u_1 + x_2 d(\cos x_3)u_2$, one verifies that the rank of ω is one, and a basis for Ω is given by $\{d(\sin x_3 u_1), d(\cos x_3 u_2)\}$. Condition (iii) is easily verified. A disturbance decoupling feedback is $v_1 = (\sin z)u_1$ and $v_2 = (\cos z)u_2$.

Example 2. This example show that for the MISO case condition (i) of Theorem 1 is not necessary. Indeed, this system can be disturbance decoupled by static measurement feedback whereas condition (i) is not fulfilled.

Consider,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1(\sin x_3)u_1 + x_2(\cos x_3)u_2 + x_4 \\ \dot{x}_3 &= f(x) + q \\ \dot{x}_4 &= (\cos x_3)u_2 \\ y &= x_1 \\ z &= x_3. \end{aligned} \tag{6}$$

Since $\Omega = \text{span}_{\mathcal{K}}\{\mathrm{d}x_1, \mathrm{d}x_2\}$, one has $\mathrm{d}\ddot{y} \notin \Omega + \mathcal{Z} + \mathcal{U}$. Despite this fact, the disturbance decoupling problem is solvable via the following measurement feedback

$$\begin{aligned} u_1 &= v_1/\sin x_3 \\ u_2 &= v_2/\cos x_3. \end{aligned}$$

Theorem 1 becomes a necessary and sufficient condition for the class of SISO systems. Moreover for this special class of system condition (iii) is always fulfilled. Then, the following corollary is equivalent to the result in [9].

Corollary 1. The disturbance decoupling problem is solvable by static measurement feedback for SISO nonlinear systems if and only if:

(i) $\mathrm{d}y^{(r)} \in \Omega + \mathcal{Z} + \mathcal{U}$.

(ii) There exists a one-form $\omega \in \mathcal{Z} + \mathcal{U}$ s.t.$\mathrm{d}y^{(r)} - \omega \in \Omega$ and s.t. rank $(\omega) = 0$.

Proof. (necessity) Assume that system (1) is decoupled by static measurement feedback. Since one has, by Lemma 1 of [9],

$$\mathrm{d}y^{(r)} \in \Omega + \text{span}_{\mathcal{K}}\{\mathrm{d}v\}, \tag{7}$$

this implies that $\mathrm{d}y^{(r)} = \omega_0 + \omega$, where $\omega_0 \in \Omega$ and $\omega \in \text{span}_{\mathcal{K}}\{\mathrm{d}v\}$.

Consider the following regular static measurement feedback which is solution of the problem

$$u = F(z, v), \quad v = F^{-1}(z, u). \tag{8}$$

Then (7) and (8) imply condition (i). Since $\omega = \xi \mathrm{d}(F^{-1}(z, u))$, condition (ii) is also fulfilled.

(sufficiency) Assume that condition (i) holds. Then this implies

$$\mathrm{d}y^{(r)} \in \Omega \oplus \text{span}_{\mathcal{K}}\{\mathrm{d}z, \mathrm{d}u\}. \tag{9}$$

Since r is the relative degree of the output y, when condition (ii) is fulfilled, one can set $\lambda \mathrm{d}v = \omega$. Thus one obtain,

$$\mathrm{d}y^{(r)} \in \Omega \oplus \text{span}_{\mathcal{K}}\{\mathrm{d}v\}. \tag{10}$$

The system is the decoupled. □

On the other hand, one can see that for the class of MISO systems, which fulfills the assumptions such that $\Omega \cap \mathcal{Z} = 0$ and $\mathrm{d}y^{(r)} \in \Omega \oplus \mathcal{Z} + \mathcal{U}$, one has a necessary and sufficient condition which can be written as follows:

Theorem 2. Under the assumptions that $\Omega \cap \mathcal{Z} = 0$, and $\mathrm{d}y^{(r)} \in \Omega \oplus \mathcal{Z} + \mathcal{U}$, the disturbance decoupling problem is solvable by static measurement feedback if and only if:

(i) There exists a one-form $\omega \in \mathcal{Z} + \mathcal{U}$ s.t. $\mathrm{d}y^{(r)} - \omega \in \Omega$ and s.t. rank $(\omega) = \gamma \leq m-1$.

(ii) For any basis $\mathrm{span}_{\mathcal{K}}\{\mathrm{d}\alpha_1, \ldots, \mathrm{d}\alpha_{\gamma+1}\}$ of ω, i.e., $\omega = \beta_1 \mathrm{d}\alpha_1 + \beta_2 \mathrm{d}\alpha_2 + \ldots + \beta_{\gamma+1}\mathrm{d}\alpha_{\gamma+1}$

$$\text{rank}\ \frac{\partial}{\partial u}[\alpha_i] = \gamma + 1 (i = 1, \ldots, \gamma+1). \tag{11}$$

Proof. (necessity) Assume that system (1) is disturbance decouplable by static measurement feedback defined by $v = \alpha(z,u)$.

Thus by definition,

$$\mathrm{d}y^{(r)} \in \mathcal{X} + \mathcal{V},$$

in which $\mathcal{V} = \mathrm{span}\{\mathrm{d}v_1, \ldots, \mathrm{d}v_m\}$. Thus, there exist a differential one form $\tilde{\omega} \in \mathcal{X}$ and some coefficients $\xi \in \mathcal{K}$ such that

$$\mathrm{d}y^{(r)} = \tilde{\omega} + \xi \mathrm{d}\alpha(z,u).$$

Since one has $\mathrm{d}y^{(r)} \in \Omega \oplus \mathcal{Z} + \mathcal{U}$, this implies that $\tilde{\omega} \in \Omega + \mathcal{Z}$. Assume that $\tilde{\omega} = \tilde{\omega}_0 + \tilde{\omega}_z$ for some $\tilde{\omega}_0 \in \Omega$ and $\tilde{\omega}_z \in \mathcal{Z}$, then from the fact that

$$\mathrm{d}y^{(r+k)} \in \mathcal{X} + \mathrm{span}\{\mathrm{d}v, \ldots, \mathrm{d}v^{(k)}\},$$

one can prove, by mathematical induction, that

$$\tilde{\omega}_z^{(k)} \in \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x, \mathrm{d}y^{(r)}, \ldots, \mathrm{d}y^{(r+k-1)}\}.$$

That is, $\tilde{\omega}_z \in \Omega$, by the definition of Ω. Thus $\tilde{\omega}_z = 0$, due to the assumption that $\Omega \cap \mathcal{Z} = 0$.

Then define $\omega = \xi \mathrm{d}\alpha(z,u)$, the necessity of (i) is fulfilled.

From [4] ω can be described as

$$\begin{aligned} \omega &= \beta_1 \mathrm{d}\alpha_1(z,u) + \beta_2 \mathrm{d}\alpha_2(z,u) \\ &\quad + \beta_3 \mathrm{d}\alpha_3(z,u) + \ldots + \beta_{\gamma+1}\mathrm{d}\alpha_{\gamma+1}(z,u). \end{aligned} \tag{12}$$

We show that for any such a choice, condition (ii) is necessarily fulfilled.

If not, there will be a linear combination

$$\xi_1 \mathrm{d}\alpha_1 + \ldots + \xi_{\gamma+1}\mathrm{d}\alpha_{\gamma+1} \in \mathcal{Z}.$$

Assume without loss of generality that $\xi_1 \neq 0$, then ω can be decomposed into

$$\omega = \tilde{\omega}_z + \eta_2 \mathrm{d}\alpha_2 + \ldots + \eta_{\gamma+1}\mathrm{d}\alpha_{\gamma+1}$$

in which

$$\tilde{\omega}_z = \frac{\beta_1}{\xi_1}(\xi_1 \mathrm{d}\alpha_1 + \ldots + \xi_{\gamma+1}\mathrm{d}\alpha_{\gamma+1}) \in \mathcal{Z}$$

and

$$\eta_i = \beta_i - \frac{\beta_1}{\xi_1}(\xi_i),$$

for $i = 2, \ldots, \gamma + 1$.

Analogous to the proof of (i), we can show that $\tilde{\omega}_z = 0$, a contradiction. Thus (ii) holds.

(sufficiency) This proof follows the proof of Theorem 1. □

Example 3. Consider,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3(\sin x_2) + (\cos x_2)u \\ \dot{x}_3 &= f(x) + q \\ y &= x_1 \\ z &= x_3. \end{aligned} \tag{13}$$

Since $\Omega = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_1, \mathrm{d}x_2\}$, one has $\omega = (\sin x_2)\mathrm{d}x_3 + (\cos x_2)\mathrm{d}u$ ($\omega \wedge (\mathrm{d}\omega) \neq 0$, $\omega \wedge (\mathrm{d}\omega)^{(2)} = 0$).

For this example, there is no solution and condition (ii) is not fulfilled. Indeed, since the rank of ω is equal to 1, two one-forms are needed to construct an integrable space containing ω. Or system (13) has just one input.

Example 4. Consider,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1(\sin x_3)u_1 + x_2(\cos x_3)u_2 \\ \dot{x}_3 &= f(x) + q \\ y &= x_1 \\ z &= x_3. \end{aligned} \tag{14}$$

This system admits the following solution $v_1 = (\sin x_3)u_1$, $v_2 = (\cos x_3)u_2$, since $\Omega = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_1, \mathrm{d}x_2\}$, and $\omega = x_1\mathrm{d}(\sin x_3)u_1 + x_2\mathrm{d}(\cos x_3)u_2$ ($\omega \wedge (\mathrm{d}\omega) \neq 0$, $\omega \wedge (\mathrm{d}\omega)^{(2)} = 0$).

Here, the rank of ω is equal to 1, and then we need two one-forms to construct our basis. Since the considered system has two outputs, we are able to define them (see [4] for a constructive way). We note $\mathrm{span}_{\mathcal{K}}\{\mathrm{d}\alpha_1, \mathrm{d}\alpha_2\} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}(\sin x_3)u_1, \mathrm{d}(\cos x_3)u_2\}$.

4. CONCLUSION

The notion of the rank of a one-form is used to solve the nonlinear disturbance decoupling problem for MISO systems. At this moment, this new approach gives a complete solution for a special class of MISO nonlinear systems. Further research will consider a more general class of system. Moreover, note that this notion of rank of one-form can be used for some other nonlinear control problem.

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