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POLYNOMIAL CONTROLLER DESIGN BASED ON FLATNESS

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By the use of flatness the problem of pole placement, which consists in imposing closed loop system dynamics can be related to tracking. Polynomial controllers for finite-dimensional linear systems can then be designed with very natural choices for high level parameters design. This design leads to a Bezout equation which is independent of the closed loop dynamics but depends only on the system model.

1. INTRODUCTION

For finite-dimensional linear systems, a well-known control design technique is constituted by polynomial two-degrees-of-freedom controllers [2, 11, 15], which have been introduced forty years ago by [13]. Whatever the chosen design method, this powerful method is based on pole placement and presents one deficiency: it needs to know *a priori* where to place all the poles of the closed loop system. Following [1]: "the key issue is to choose the closed loop poles. This choice requires considerable insight ... ". This can be done, for instance, through LQR design, but the problem is then replaced by the correct choice of the weighting matrices in the cost functions.

In order to overcome the drawback of this design technique, it will be seen, in the following, that the use of a new method for system control, namely with a flatness point of view, enlightens on the choice of the high level parameters and brings physical meanings to obtain a clear guideline for polynomial pole placement design. Following [7, 8], flatness is a very interesting property of processes to design a control, specially for trajectory planning and tracking for nonlinear systems.

The paper is organized as follows. Section 2 is devoted to survey very quickly, the design of polynomial controllers. Section 3 resumes the flatness property and the control design implied for a flat system. At the end of Section 3, a methodology for the control of flat systems is proposed. The implication of method on finite-dimensional linear systems is given in Section 4. This point of view leads to propose a flatness-based two-degrees-of-freedom controller which is realized in Section 5. Section 6 is devoted to the rejection of a static perturbation which can be seen as a complement to the previously designed control. In Section 7, an example of a RST controller applied to a second order linear system is presented.

In the following, for $n \in \mathbb{N}$, the following notations will be used, $u^{(n)}(t) = \frac{d^n u(t)}{dt^n} = p^n u(t)$, where p denotes the differential operator, and the paper will be developed, for the sake of shortness, for SISO linear systems, but all the results can be adapted to MIMO linear systems.

2. POLYNOMIAL CONTROLLERS

This section offers a short description of the design principles of the polynomial two-degrees-of-freedom controllers for linear systems. More details are given in [2, 11, 15] and the references therein, and in the following these controllers will be denoted as RST controllers [16].

Consider the finite-dimensional SISO linear system described by the input-output model:

$$Ay = Bu, \tag{1}$$

where y and u are the output and control signals, A is monic and A and B are coprime polynomials.

For (1), the RST (two-degrees-of-freedom) controller [2] is given by:

$$Ru = -Sy + Tr, \tag{2}$$

where r is the reference to track, and R, S and T are polynomials in the considered operator. These polynomials are given by the following rules: R and S are solutions of the Bezout equation:

$$P = AR + BS, \tag{3}$$

where the roots of the polynomial P are constituted by the desired closed loop and observer poles, and S and R are monic; T is given by the desired closed loop transfer such that:

$$PB_m = TBA_m. \tag{4}$$

When all these conditions are fulfilled, the closed loop behavior is obtained:

$$A_m y = B_m r. \tag{5}$$

Some remarks for the design:

(i) It has been used, for the choice of T, the point of view developed in [2], where (B_m, A_m) was a model-to-follow, but it can be also chosen the proposed one in [16], where r is given by:

$$A_m r = B_m, \tag{6}$$

where (B_m, A_m) defines a trajectory-to-follow or a trajectory generator of $r(t)$. In this last point of view, T is designed such that:

$$TB = P. \tag{7}$$

(ii) For the implementation, the RST controller (2) must be written in the proper operator (p^{-1}) which leads to write the RST control as:

$$R^*(p^{-1})\,u(t) = -S^*(p^{-1})\,y(t) + T^*(p^{-1})\,r(t), \tag{8}$$

with $R^*(0) = 1$.

As the major point is to choose the desired poles, we will see that the flatness point of view can fruitfully help us.

3. SHORT SURVEY ON FLATNESS

The flat property, which has been introduced recently [5, 6, 7] for continuous-time nonlinear systems, leads to interesting points of view for control design. In the following, a short review about flatness of systems and the application of this property to design a controller will be given. The interested reader may find more details in the quoted literature and the references therein.

A system described by:

$$x^{(1)} = f(x, u), \tag{9}$$

where x is the state vector of dimension n, and u is the control signal, possesses the flatness property (or is flat) if there exist a variable z:

$$z = h(x, u, u^{(1)}, \dots, u^{(\alpha)}), \tag{10}$$

where $\alpha \in N$, two functions $A(\cdot)$and $B(\cdot)$, and an integer β such that:

$$\begin{aligned} x &= \mathbf{A}(z, \dots, z^{(\beta)}), \\ u &= \mathbf{B}(z, \dots, z^{(\beta+1)}). \end{aligned} \tag{11}$$

The selected output z is called a flat output and, obviously, there is no uniqueness. But, as it has been observed on numerous examples, the flat output has a simple and physical meaning.

Roughly speaking, the implications of flatness are of very importance in several ways for control. For motion planning, by imposing a desired trajectory on the flat output, the necessary control to generate the trajectory, can be obtained explicitly (without any integration of the differential equations). The desired trajectory, $z_d(t)$, must be $(\beta + 1)$-times continuously differentiable. For feedback control which only ensures a good stabilization around the desired motion $z_d(t)$.

All these points, which have been formalized through the Lie–Bäcklund equivalence of systems in [6, 8], lead to propose a nonlinear feedback which ensures a stabilized tracking of a desired motion for the flat output. This methodology has been applied on many industrial processes as it has been shown previously, for instance, on magnetics bearings [18], chemical reactors [25], cranes or flight control [19] or turning process [22, 23], among many other examples.

The main objective of the flatness based controller is to obtain the asymptotic tracking of a desired trajectory and this can be ensured through the following steps:

(i) *Motion planning*: it consists in the design of a trajectory defined by $z_d(t)$, which must be differentiable at the order $(\beta+1)$.

(ii) *Motion tracking*: by the control:

$$v = z_d^{(\beta+1)}(t) + \sum_{i=0}^{\beta} k_i(z_d^{(i)}(t) - z^{(i)}(t)), \tag{12}$$

where the k_i ensure that the polynomial $K(p) = p^{\beta+1} + \sum_{i=0}^{\beta} k_i p^i$is Hurwitz, the complete control is then as follows:

$$\begin{aligned} u &= \mathbf{B}(z, \ldots, z^{(\beta)}, z_d^{(\beta+1)}(t) + \sum_{i=0}^{\beta} k_i(z_d^{(i)}(t) - z^{(i)}(t))) \\ &= \Phi(z, \ldots, z^{(\beta)}, K(p)\, z_d(t)), \end{aligned} \tag{13}$$

which leads to the asymptotic tracking of the desired trajectory.

Notice, in the one hand, that the information needed by this control can be obtained through observers, and a major advantage of this controller with respect to other nonlinear strategies is that it overcomes the problems generated by non stable zeros dynamics [12, 21]. In the other hand, if the output of (9) is given by $y = g(x, u)$ then from (11) it can be related to the flat output by:

$$y = \Gamma(z, \ldots, z^{(\beta+1)}). \tag{14}$$

This relationship leads to a trajectory for the output deduced from the designed trajectory for the flat output, namely, $y_d = \Gamma(z_d, \ldots, z_d^{(\beta+1)})$. If this trajectory is not admissible for the output, the key is to design a piecewise trajectory where some conditions for smoothness are verified on the cutting points, but keeping in mind that the relationship (14) is available between these points.

We will see in the next part that the design of the flat output trajectory will be a guideline, in a linear framework, for the poles choice of a RST controller.

4. IMPLICATION FOR LINEAR SYSTEMS: TOWARDS RST CONTROLLERS

Despite the fact that flatness has been firstly developed for nonlinear systems, it has been applied to finite-dimensional linear systems [3, 10] and extended for infinite-dimensional ones [9]. It will be seen, in this section, that applying the guideline induced by a flatness based control to a linear system leads to express it in a natural RST form.

The previous methodology will be applied now to a linear lumped parameter SISO system defined by the transfer:

$$A(p)\, y(t) = B(p)\, u(t), \tag{15}$$

where the notations have been previously defined but with:

$$A(p) = p^n + \sum_{i=0}^{n-1} a_i p^i = p^n + A^*(p),\ B(p) = \sum_{i=0}^{n-1} b_i p^i. \tag{16}$$

From coprimeness, it has been shown in [3], [8], that this system is flat with a flat output defined by:

$$z(t) = N(p)\, y(t) + D(p)\, u(t), \tag{17}$$

where $N(p)$ and $D(p)$ are the polynomial solutions of the following Bezout equation:

$$N(p)\, B(p) + D(p)\, A(p) = 1. \tag{18}$$

Due to coprimeness, existence of $N(p)$ and $D(p)$ are guaranteed and the minimum degree solution is, for $n > 1$, such that $\deg N = n - 1$ and $\deg D = n - 2$.

The explicit expressions of the output $y(t)$ and the control $u(t)$ are given by:

$$u(t) = A(p)\, z(t),\ y(t) = B(p)\, z(t), \tag{19}$$

which allows to relate the flat output of a linear system to the partial state defined by [14].

Following the step (ii) of the methodology, the control is given by:

$$u(t) = v(t) + A^*(p)\, z(t), \tag{20}$$

where:

$$v(t) = z_d^{(n)}(t) + \sum_{i=0}^{n-1} k_i (z_d^{(i)}(t) - z^{(i)}(t)), \tag{21}$$

and by introducing the polynomials:

$$K(p) = p^n + \sum_{i=0}^{n-1} k_i p^i = p^n + K^*(p), \tag{22}$$

the control $u(t)$ is given by:

$$u(t) = K(p)\, z_d(t) + [A^*(p) - K^*(p)]\, z(t). \tag{23}$$

Taking into account that $z(t) = N(p)\, y(t) + D(p)\, u(t)$, then it can be written:

$$u(t) = K(p)\, z_d(t) + [A^*(p) - K^*(p)]\, [N(p)\, y(t) + D(p)\, u(t)]\,, \tag{24}$$

which leads to:

$$[1 - [A^*(p) - K^*(p)]\, D(p)]\, u(t) = K(p)\, z_d(t) + [A^*(p) - K^*(p)]\, N(p)\, y(t). \tag{25}$$

This appears as a RST controller form with:

$$\begin{aligned} R(p) &= 1 - [A^*(p) - K^*(p)]\, D(p), & (26)\\ S(p) &= -\,[A^*(p) - K^*(p)]\, N(p), & (27)\end{aligned}$$

with the difference that here the trajectory to follow is directly integrated to the controller with the term $K(p)\, z_d(t)$. An important property of this controller can be also deduced, due to the fact that $P = AR + BS$. From the previous definitions of $R(p)$ and $S(p)$, and with the help of $N(p)\, B(p) + D(p)\, A(p) = 1$, and $A^*(p) - K^*(p) = A(p) - K(p)$, it follows that:

$$A(p)\, R(p) + B(p) S(p) = K(p). \tag{28}$$

From (28), it is then obtained that the closed loop poles for the proposed RST controller are those designed for the tracking of the desired flat output trajectory. The choice of these poles is then enlightened. But as:

$$\deg\left(1 - [A^* - K^*]\, D\right) = \deg\left([A^* - K^*]\, N\right) - 1, \tag{29}$$

it is not realizable. The realization of this controller will be the subject of the next part.

5. REALIZATION

To implement the control (23), it can be used an observer of the vector $Z = \begin{bmatrix} z(t) & \dots & z^{(n-1)}(t) \end{bmatrix}^T$ which is the state vector of the controllable Luenberger realization of $u(t) = A(p)\, z(t)$, $y(t) = B(p)\, z(t)$, namely:

$$\begin{aligned} Z^{(1)} &= AZ + Bu,\\ y &= CZ, \end{aligned} \tag{30}$$

where:

$$A = \begin{bmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tag{31}$$

$$C = \begin{bmatrix} b_0 & b_1 & \cdots & b_{n-1} \end{bmatrix}.$$

A full-order observer of Z is given by:

$$\hat{Z}^{(1)} = (A - \Gamma C)\hat{Z} + Bu + \Gamma y, \tag{32}$$

where Γ is chosen such that the eigenvalues of $F = A - \Gamma C$ are with negative real part. This leads to:

$$\hat{Z} = (pI - F)^{-1}(Bu + \Gamma y). \tag{33}$$

By introducing $a = \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} \end{bmatrix}$ and $k = \begin{bmatrix} k_0 & k_1 & \cdots & k_{n-1} \end{bmatrix}$, the control (23) is implemented by:

$$u(t) = K(p)\, z_d(t) + (a - k)\hat{Z}(t), \tag{34}$$

as in [10]. But, in this solution the difficulty is the choice of the observer poles. To overcome this point the enlightening ideas suggested in [4] and applied in [20] can be used. In the one hand, from [14]:

$$Y = O_{(A,C)} Z + M_{(A,B,C)} U, \tag{35}$$

where $Y = \begin{bmatrix} y & y^{(1)} & \cdots & y^{(n-1)} \end{bmatrix}^T$, $U = \begin{bmatrix} u & u^{(1)} & \cdots & u^{(n-2)} \end{bmatrix}^T$, and $O_{(A,C)}$ is the observability matrix:

$$O_{(A,C)} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{36}$$

and $M_{(A,B,C)}$ is given by:

$$M_{(A,B,C)} = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ CB & \ddots & & \vdots \\ CAB & CB & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ CA^{n-2}B & \cdots & CAB & CB \end{bmatrix}. \tag{37}$$

From this equation, and due to the fact that $A(p)$ and $B(p)$ are coprime, thus $\text{rank} O_{(A,C)} = n$, it becomes:

$$Z = O^{-1}_{(A,C)} \left\{ Y - M_{(A,B,C)} U \right\}. \tag{38}$$

As the first component of Z is $z(t)$, it can be seen that the first line gives the flat output expressed in terms of the derivatives of $y(t)$ and $u(t)$. Namely, a solution of the Bezout identity (18) is obtained, with $h = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$:

$$N(p) = hO^{-1}_{(A,C)} \operatorname{diag}\left\{1, p, \ldots, p^{n-1}\right\}, \tag{39}$$

$$D(p) = -hO_{(A,C)}^{-1} M_{(A,B,C)} \operatorname{diag}\{1, p, \dots, p^{n-2}\}. \tag{40}$$

In the other hand, from [4]:

$$\forall \mu \in N,\ Z = A^{\mu} p^{-\mu} Z + \sum_{i=1}^{\mu} A^{i-1} B p^{-i} u, \tag{41}$$

where p^{-1} stands for the integration operator:

$$p^{-1} x(t) = \int_{-\infty}^{t} x(\tau)\, \mathrm{d}\tau, \tag{42}$$

with $x(-\infty) = 0$. This last hypothesis ensures commutativity between p and p^{-1}. As a particular case, it comes that for $\mu = n - 1$:

$$Z = A^{n-1} p^{-(n-1)} Z + \sum_{i=1}^{n-1} A^{i-1} B p^{-i} u, \tag{43}$$

then, by combining (38) and (43), it follows:

$$\begin{aligned} Z &= A^{n-1} O_{(A,C)}^{-1} p^{-(n-1)} Y - A^{n-1} O_{(A,C)}^{-1} M_{(A,B,C)} p^{-(n-1)} U \\ &\quad + \sum_{i=1}^{n-1} A^{i-1} B p^{-i} u. \end{aligned} \tag{44}$$

By replacing this expression in the control (23), it follows the control:

$$u(t) = K(p)\, z_d(t) - S^*(p^{-1})\, y(t) - Q^*(p^{-1})\, u(t), \tag{45}$$

where:

$$S^*(p^{-1}) = [k - a]\, A^{n-1} O_{(A,C)}^{-1} \Pi, \tag{46}$$

$$\begin{aligned} Q^*(p^{-1}) &= [a - k] \times \\ &\quad \times \left\{ \begin{array}{l} A^{n-1} O_{(A,C)}^{-1} M_{(A,B,C)} \\ \qquad - \left[\ A^{n-2} B \ \ \cdots \ \ B\ \right] \end{array} \right\} \Pi^*, \end{aligned} \tag{47}$$

with:

$$\Pi = \left[\ p^{-(n-1)} \ \ p^{-(n-2)} \ \ \cdots \ \ p^{-1} \ \ 1\ \right]^T, \tag{48}$$

$$\Pi^* = \left[\ p^{-(n-1)} \ \ p^{-(n-2)} \ \ \cdots \ \ p^{-1}\ \right]^T. \tag{49}$$

By denoting $R^*(p^{-1}) = 1 + Q^*(p^{-1})$, this control can be written in the RST form:

$$R^*(p^{-1})\,u(t) = K(p)\,z_d(t) - S^*(p^{-1})\,y(t). \tag{50}$$

As a remark, from (50) and (19), we get:

$$R^*(p^{-1})\,A(p)\,z(t) = K(p)\,z_d(t) - S^*(p^{-1})\,B(p)\,z(t). \tag{51}$$

After some manipulations, we deduce $R^*(p^{-1})\,A(p)+S^*(p^{-1})\,B(p) = K(p)$, and if we notice that $A(p)$ and $K(p)$ are of the same degree, the expression $A(p^{-1})\,R^*(p^{-1}) + B(p^{-1})S^*(p^{-1})$, can be written as:

$$A(p^{-1})\,R^*(p^{-1}) + B(p^{-1})S^*(p^{-1}) = p^{-n}K(p), \tag{52}$$

which gives the relationship of the poles of the RST controller with the tracking dynamics.

A second remark can be done here. Namely, it follows also:

$$z(t) = h\left\{A^{n-1}p^{-(n-1)}Z + \sum_{i=1}^{n-1} A^{i-1}Bp^{-i}u\right\}, \tag{53}$$

where h is previously defined. Thus:

$$\begin{aligned} z(t) &= h\left\{ A^{n-1}O^{-1}_{(A,C)} \begin{bmatrix} p^{-(n-1)}y \\ p^{-(n-2)}y \\ \vdots \\ y \end{bmatrix} + \sum_{i=1}^{n-1} A^{i-1}Bp^{-i}u \right. \\ &\qquad \left. -A^{n-1}O^{-1}_{(A,C)}M_{(A,B,C)} \begin{bmatrix} p^{-(n-1)}u \\ p^{-(n-2)}u \\ \vdots \\ p^{-1}u \end{bmatrix} \right\}, \\ &= N^*(p^{-1})\,y(t) + D^*(p^{-1})\,u(t), \end{aligned} \tag{54}$$

which defines the flat output in terms of the proper operator p^{-1}.

6. DISTURBANCE REJECTION

In order to reject a static perturbation, an integral action must be added to the model. The proposed methodology is then applied to the following augmented model:

$$\begin{aligned} \tilde{Z}^{(1)} &= \tilde{A}\tilde{Z} + \tilde{B}\tilde{u}, \\ y &= \tilde{C}\tilde{Z}, \end{aligned} \tag{55}$$

where $\tilde{Z} = \begin{pmatrix} z & \dots & z^{(n)} \end{pmatrix}^T$ and $\tilde{u}(t) = pu(t)$. The matrices $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ are given by:

$$\tilde{A} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ 0 & -a_0 & \cdots & -a_{n-1} \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tag{56}$$

$$\tilde{C} = \begin{bmatrix} b_0 & b_1 & \cdots & b_{n-1} & 0 \end{bmatrix}.$$

The corresponding RST controller is obtained from equations (47) and (46). However, as the degree of the polynomial S^* becomes in this case $n-1$ and the degree of R^* is n, it then follows a simplification of the operator p with the integrator. This will have as a consequence that the controller will not be able to reject a constant disturbance added to the output of the model.

To overcome the simplification problem which appears in the case of the augmented model, it is proposed to replace in the equation (44) the operator p by a new operator $p+a$ which is denoted π $(a > 0)$. By applying the same methodology as before, the following S_a^* and Q_a^* expressions are obtained:

$$S_a^*(\pi^{-1}) = \left[\tilde{k} - \tilde{a}\right] \tilde{A}_a^n O_a^{-1} \tilde{\Pi}, \tag{57}$$

$$Q_a^*(\pi^{-1}) = \left[\tilde{a} - \tilde{k}\right] \left\{ \begin{array}{c} \tilde{A}_a^n O_a^{-1} M_a - \\ \begin{bmatrix} \tilde{A}_a^{n-1}\tilde{B} & \cdots & \tilde{B} \end{bmatrix} \end{array} \right\} \tilde{\Pi}^*, \tag{58}$$

where $\tilde{A}_a$ represents the matrix $\tilde{A} + aI_n$ and $\tilde{\Pi} = \begin{bmatrix} \pi^{-n} & \cdots & \pi^{-1} & 1 \end{bmatrix}^T$ and $\tilde{\Pi}^* = \begin{bmatrix} \pi^{-n} & \cdots & \pi^{-1} \end{bmatrix}^T$. The control $\tilde{u}(t)$ is then obtained from:

$$R_a^*(\pi^{-1})\tilde{u}(t) = \tilde{K}(p)\, z_d(t) - S_a^*(\pi^{-1})\, y(t), \tag{59}$$

where $R_a^*(\pi^{-1}) = 1 + Q_a^*(\pi^{-1})$. The a parameter is chosen with respect to the bandwidth of the considered model.

This strategy is illustrated in the next section on a second order continuous-time model.

7. EXAMPLE

Let be considered the system defined by $A(p) = p^2 + p + 1$ and $B(p) = p + 1$, then the Bezout equation (18) admits the least degree solution:

$$N(p) = -p, \quad D(p) = 1, \tag{60}$$

which leads to the flat output $z(t) = u(t) - py(t)$. The control is then:

$$u(t) = K(p)\, z_d(t) + [(1 - k_1)p + 1 - k_0]\, z(t), \tag{61}$$

where $K(p) = p^2 + k_1 p + k_0$ defines the error dynamics. From $u(t) = \left(p^2 + p + 1\right) z(t)$ and $y(t) = (p+1)\, z(t)$, we have:

$$\begin{aligned}\dot{Z}(t) &= \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} Z(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \\ y(t) &= \begin{bmatrix} 1 & 1 \end{bmatrix} Z(t).\end{aligned} \tag{62}$$

The chosen closed loop poles are $-4.2 \pm 4.28j$ and lead to the following RST controller in the realizable form (50):

$$\begin{aligned}R^*(p^{-1}) &= 1 - 27.6p^{-1}, \\ S^*(p^{-1}) &= 35 + 27.6p^{-1}, \\ K(p) &= p^2 + 8.4p + 36.\end{aligned} \tag{63}$$

In order to cope with a static perturbation, added to the system output, an integral action must be integrated into the model and one obtains the augmented model:

$$y(t) = \frac{p+1}{p\,(p^2 + p + 1)} \tilde{u}(t), \tag{64}$$

where $\tilde{u}(t) = pu(t)$ and $\tilde{A}(p) = pA(p)$. The RST controller corresponding to this model is given by:

$$\begin{aligned}\tilde{R}^*(p^{-1}) &= 1 - 22.4p^{-1} + 379p^{-2}, \\ \tilde{S}^*(p^{-1}) &= 517.6 + 379p^{-1}, \\ \tilde{K}(p) &= p^3 + 23.4p^2 + 162p + 540,\end{aligned} \tag{65}$$

where the chosen closed loop poles are: $-4.2 \pm 4.28i$, -15. The trajectory to follow by the output is $y_d(t) = 1$. To achieve this purpose, the desired flat output must have the following expression:

$$z_d(t) = 1 - \gamma \exp(-t), \tag{66}$$

which satisfies the derivability condition for any constant γ. It can be noted here that the obtained controller does not reject the added constant perturbation as shown in Figure 1 (for $\gamma = 1$).

By applying the same methodology replacing the operator p by the operator π as proposed previously, the state representation becomes:

$$\begin{aligned}\pi \tilde{Z}(t) &= \tilde{A}_a \tilde{Z}(t) + \tilde{B}\tilde{u}(t), \\ y(t) &= \tilde{C}\tilde{Z}(t),\end{aligned} \tag{67}$$

where $\tilde{Z}(t) = \begin{pmatrix} z(t) & z^{(1)}(t) & z^{(2)}(t) \end{pmatrix}^T$ and $z(t)$ is the flat output of the augmented model. $\tilde{A}_a$ represents the matrix $\tilde{A} + aI_3$. The same methodology is applied and the following RST expression is obtained:

$$\tilde{R}_a^*(\pi^{-1})\tilde{u}(t) = \tilde{K}(p)\, z_d(t) - \tilde{S}_a^*(\pi^{-1})\, y(t), \tag{68}$$

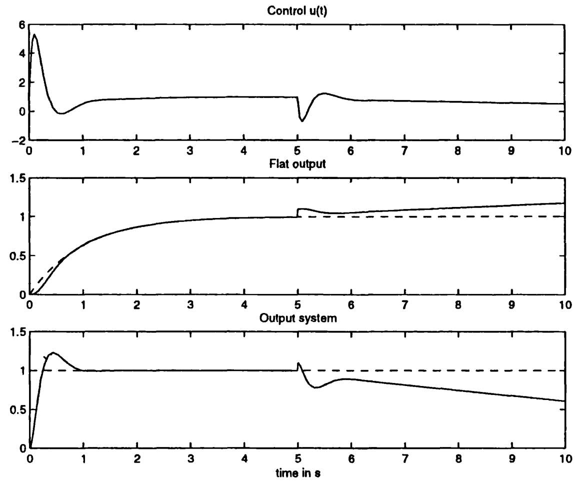

Fig. 1. Tracking in the case of $a = 0$.

with:

$$\tilde{S}_a^*(\pi^{-1}) = \begin{bmatrix} k_0 & k_1 - 1 & k_2 - 1 \end{bmatrix} \tilde{A}_a^2 O_a^{-1} \begin{pmatrix} \pi^{-2} \\ \pi^{-1} \\ 1 \end{pmatrix}, \tag{69}$$

$$\begin{aligned} Q_a^*(\pi^{-1}) &= \begin{bmatrix} -k_0 & 1 - k_1 & 1 - k_2 \end{bmatrix} \times \\ &\times \left(\begin{bmatrix} \tilde{A}_a \tilde{B} & \tilde{B} \end{bmatrix} - \tilde{A}_a^2 O_a^{-1} M_a \right) \begin{pmatrix} \pi^{-2} \\ \pi^{-1} \end{pmatrix}. \end{aligned} \tag{70}$$

The matrices O_a and M_a are given by:

$$O_a = \begin{pmatrix} \tilde{C} \\ \tilde{C}\tilde{A}_a \\ \tilde{C}\tilde{A}_a^2 \end{pmatrix}, \; M_a = \begin{pmatrix} 0 & 0 \\ \tilde{C}\tilde{B} & 0 \\ \tilde{C}\tilde{A}_a\tilde{B} & \tilde{C}\tilde{B} \end{pmatrix}. \tag{71}$$

Also notice that the following relation remains valid:

$$\begin{aligned} &\tilde{A}(p^{-1})\tilde{R}_a^*(\pi^{-1}) + B(p^{-1})\tilde{S}_a^*(\pi^{-1}) \\ = \; &1 + k_2 p^{-2} + k_1 p^{-1} + k_0 p^{-2} = p^{-3}\tilde{K}(p). \end{aligned} \tag{72}$$

It can be noted that the rejection dynamics depend on the choice of the a parameter. A good disturbance rejection has been obtained for $a = 1$. The numerical expressions for the polynomials $\tilde{R}_a^*$ and $\tilde{S}_a^*$ are:

$$\begin{aligned} \tilde{R}_a^*(\pi^{-1}) &= 1 - 22.4\pi^{-1}, \\ \tilde{S}_a^*(\pi^{-1}) &= 161 + 356.6\pi^{-1} + 22.4\pi^{-2}, \end{aligned} \tag{73}$$

which leads to a realizable RST which does not simplify the integral action.

The simulation results are displayed in Figure 2.

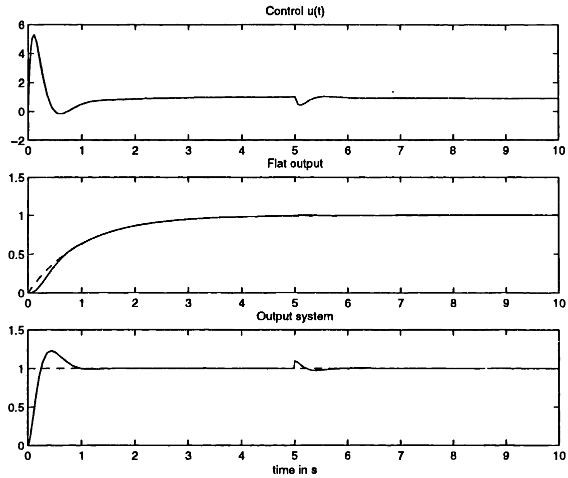

Fig. 2. Tracking in the case of $a = 1$.

8. CONCLUSION

This paper showed that the use of a flatness point of view allows a simplification in the design of high level parameters of RST controllers. The main feature of the flatness approach for RST controller design is to avoid the problem of the closed loop poles choice which are constituted of the observer poles and those obtained with a state feedback [2]. Now the design is focused in the choice of the trajectory z_d to follow and the tracking dynamics with $K(p)$.

In the case where a constant output perturbation, for instance, is to be rejected, an integral action must be added in R. This can be achieved by forcing the presence of an integrator in the open loop transfer function as shown in the equation (64). Then the proposed method can be applied again using a new operator $\pi = p + a$ instead of the usual operator p.

These developments were done in a continuous-time framework, but are transposable for discrete-time systems [24]. In this case, the robustness of the proposed digital controller based on flatness, by introducing fixed polynomials H_R and H_S as proposed in [17], were treated for the flat discrete-time systems.

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