KYBERNETIKA — VOLUME 38 (2002), NUMBER 5, PAGES 503–520

CHARACTERIZATION OF GENERIC PROPERTIES OF LINEAR STRUCTURED SYSTEMS FOR EFFICIENT COMPUTATIONS

CHRISTIAN COMMAULT, JEAN-MICHEL DION AND JACOB W. VAN DER WOUDE

In this paper we investigate some of the computational aspects of generic properties of linear structured systems. In such systems only the zero/nonzero pattern of the system matrices is assumed to be known. For structured systems a number of characterizations of so-called generic properties have been obtained in the literature. The characterizations often have been presented by means of the graph associated to a linear structured system and are then expressed in terms of the maximal or minimal number of certain type of vertices contained in a combination of specific paths. In this paper we give new graph theoretic characterizations of structural invariants of structured systems. It turns out that these new characterizations allow to compute these invariants via standard and efficient algorithms from combinatorial optimization.

1. INTRODUCTION

In this paper we consider linear structured systems in state space form, which represent a large class of parameter dependent linear systems, see [15, 18]. Many properties of linear systems can be phrased in terms of invariants, which contain essential information stated in condensed form. We investigate here some of the computational aspects of generic properties of linear structured systems, in particular of their invariants. In such systems only the zero/nonzero pattern of the system matrices is assumed to be known. For structured systems a number of characterizations of generic invariants, like the finite or infinite zero structure, and generic properties, like decoupling and disturbance rejection, have been obtained in the literature [1, 16, 23, 24, 25]. The characterizations often have been presented by means of the graph associated to a linear structured system and are then expressed in terms of the maximal or minimal number of certain type of vertices contained in a combination of specific paths. In this paper we start from the graph theoretic characterizations and explain how the graph theoretic computations verifying these characterizations can actually be done.

Some interesting results to compute the rank and the infinite structure of a structured system have been given in [11, 12, 13, 17, 28]. In this paper we present some new ideas concerning the computation of the invariant zero structure and the infinite structure together with some applications. We present a unified approach using a bipartite graph associated to the structured system and show that the considered computations reduce to standard optimal assignment problems for which efficient algorithms exist.

The outline of the paper is as follows. In Section 2 we give the mathematical description of linear structured systems and introduce the associated graphs. In Section 3 we recall some known graph theoretic characterizations of generic properties of structured systems. Section 4 is dedicated to the representation of structured system by bipartite graphs and to well-known combinatorial problems. In Section 5 we give new characterizations for controllability, finite and infinite structure, and the solvability of control problems in terms of maximal matching and optimal assignment in the associated bipartite graph. In Section 6 we conclude with some remarks.

2. MATHEMATICAL DESCRIPTION

2.1. Structured systems and genericity

We study linear time-invariant systems of the following form

$$\Sigma : \begin{array}{rcrcl} \dot{x}(t) & = & Ax(t) & + & Bu(t) \\ y(t) & = & Cx(t) & + & Du(t) \end{array} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ denotes the state of the system, $u(t) \in \mathbb{R}^m$ the input and $y(t) \in \mathbb{R}^p$ the output. As indicated, the vectors x, u and y all depend on the time t. The matrices A, B, C and D are real valued constant matrices of suitable dimensions.

In this paper we assume that we only know the zero/nonzero pattern of the matrices A, B, C and D, i.e. which entries in the matrices are fixed to zero, and consequently which entries are not fixed to zero. In the remainder the latter nonzero entries are referred to as the *nonzeros*. Besides the zero/nonzero information, we assume that the nonzeros can attain any real value, including possibly even zero. We can therefore parametrize each nonzero by means of a real scalar parameter. Then, if the system has f nonzeros, it can be parametrized by means of a parameter vector $\lambda \in \Lambda = \mathbb{R}^f$. The set of parametrized systems thus obtained is referred to as a structured system and is denoted by

$$\Sigma_\lambda : \begin{array}{rcrcl} \dot{x}(t) & = & A_\lambda x(t) & + & B_\lambda u(t) \\ y(t) & = & C_\lambda x(t) & + & D_\lambda u(t) \end{array} \tag{2}$$

with $\lambda \in \Lambda$. For each value of λ the system (2) is completely known. In this paper we refer to such a completely known system as a system of type (1), whereas a structured system will be denoted as a system of type (2) with λ unspecified. An example of a structured system is given in subsection 2.3.

By choosing $\lambda \in \Lambda$ the system (2) becomes completely known and can be written as a system of the form (1). Hence, for each value of $\lambda \in \Lambda$ system theoretic properties can be studied in the normal way. However, it is clear that the properties do depend on the chosen parameter value. We will study system theoretic properties (or invariants) which hold generically i. e. for almost all parameter values. Here "for almost all parameter values" is to be understood as "for all parameter values except for those in some proper algebraic variety in the parameter space Λ". The proper algebraic variety for which a property is not true is the zero set of some nontrivial polynomial with real coefficients in the f parameters of the system. The polynomial can be written down explicitly, i. e. we can precisely describe when a property fails to be true. A proper algebraic variety has Lebesgue measure zero. Therefore, a property which is true for almost all parameter values is often also said to be true *generically.*

2.2. Graphs, paths, linkings and circuit families

Structured systems can be represented elegantly by means of directed graphs. Using such type of representation it is possible to study well-known system theoretic properties from a graph theoretic point of view. The results of these studies are conditions that only depend on the structure or the graph of the system and therefore, besides exceptional cases, do not depend on the numerical values of the parameters of the system, i. e. the values of the nonzeros in the matrices describing the system.

The graph $G = (V, E)$ of a structured system of type (2) is defined by a vertex set V and an edge set E. The vertex set V is given by $U \cup X \cup Y$ with $U = \{u_1, \dots, u_m\}$ the set of input vertices, $X = \{x_1, \dots, x_n\}$ the set of state vertices and $Y = \{y_1, \dots, y_p\}$ the set of output vertices. Denoting (v, v') for a directed edge from the vertex $v \in V$ to the vertex $v' \in V$, the edge set E is described by $E_A \cup E_B \cup E_C \cup E_D$ with $E_A = \{(x_j, x_i) | A_{\lambda,ij} \neq 0\}$, $E_B = \{(u_j, x_i) | B_{\lambda,ij} \neq 0\}$, $E_C = \{(x_j, y_i) | C_{\lambda,ij} \neq 0\}$ and $E_D = \{(u_j, y_i) | D_{\lambda,ij} \neq 0\}$. In the latter, for instance $A_{\lambda,ij} \neq 0$ means that the (i, j)th entry of the matrix A_λ is a parameter (a nonzero).

Let W, W' be two nonempty subsets of the vertex set V of the graph G. We say that there exists a *path* from W to W' if there is an integer t and there are vertices $w_0, w_1, \dots, w_t \in V$ such that $w_0 \in W$, $w_t \in W'$ and $(w_{i-1}, w_i) \in E$ for $i = 1, 2, \dots, t$. We call the vertex w_0 the *begin vertex* and w_t the *end vertex* of the path. We say that the path consists of the vertices $w_0, w_1, \dots, w_t$, where it may happen that some of the vertices occur more than once. We also say that each of the vertices in $w_0, w_1, \dots, w_t$ is *contained* in the path. We call the path *simple* if every vertex on the path occurs only once. Occasionally, we denote a path as above, containing the vertices $w_0, w_1, \dots, w_t$, as the sequence of edges it consists of, i. e. $(w_0, w_1), (w_1, w_2), \dots, (w_{t-1}, w_t)$, or also simply as $w_0 \to w_1 \to \dots \to w_t$.

We say that two paths from W to W' are *disjoint* if they consist of disjoint sets of vertices. We call l paths from W to W' disjoint if they are mutually disjoint, i. e. each two of them are disjoint. We call a set of l *disjoint* and *simple* paths from W to W' a *linking from W to W' of size l.* Since there are only a finite number of linkings, there obviously exist linkings consisting of a maximal number of disjoint paths. We call such linkings *maximal (size) linkings.*

We call a simple path a *U-rooted path* if the path has its begin vertex in U. A number of mutually disjoint U-rooted paths is called a *U-rooted path family.*

Similarly, we call a simple path a *Y-topped path* if the path has its end vertex in Y. A number of mutually disjoint Y-topped paths is called a *Y-topped path family.* We call a *closed* and simple path in X a *circuit*, i. e. a circuit is a path in X of the form $(w_0, w_1), (w_1, w_2), \dots, (w_{t-1}, w_0)$, consisting of distinct vertices $w_0, w_1, \dots, w_{t-1}$. We say that two circuits are *disjoint* if they consist of disjoint sets of vertices. We call a set of l circuits disjoint if they are mutually disjoint. We call such a set of l disjoint circuits a *circuit family of size l.* We say that the union of a combination of a linking from U to Y, a U-rooted path family, a Y-topped path family and/or a circuit family in X is *disjoint* if they mutually have no vertices in common. If such a union contains all vertices of X it is also said to *cover* X.

2.3. An example

We consider a linear structured system as in (2) with 2 inputs, 5 states, 2 outputs and defined by the matrices:

$$A_\lambda = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_7 & 0 \\ 0 & \lambda_4 & \lambda_5 & 0 & 0 \\ 0 & 0 & \lambda_6 & 0 & 0 \\ 0 & \lambda_8 & 0 & 0 & 0 \end{bmatrix}, \quad B_\lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_3 \\ 0 & 0 \\ 0 & 0 \\ \lambda_2 & 0 \end{bmatrix}, \tag{3}$$

$$C_\lambda = \begin{bmatrix} \lambda_9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_{10} \end{bmatrix}, \quad D_\lambda = 0.$$

The graph corresponding to this system is depicted in Figure 1.

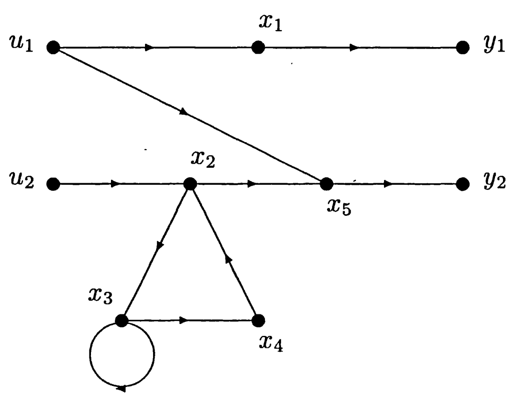

Fig. 1. Graph of the system (3).

3. GENERIC RESULTS

In this section we recall some of the well-known generic results for structured linear systems of type (2). The results concern several generic zero structures and applications towards the generic solvability of the structural versions of several classical control problems.

3.1. Generic controllability

The notion of controllability for completely specified systems of type (1) is well-known. For this we refer to the many textbooks on system theory that are available. As for each choice of $\lambda \in \Lambda$ a system of type (2) is completely known, it consequently can be checked for controllability for each $\lambda \in \Lambda$. It turns out that once a structured system is controllable for one choice of $\lambda \in \Lambda$, it is controllable for almost all $\lambda \in \Lambda$, in which case the structured system then will be said to be *generically controllable*.

For structured systems of type (2) the following result has been proved, see [18].

Theorem 1. A structured linear system of type (2) with a graph G is generically controllable if and only if the following two conditions are satisfied.

1. In G every vertex in X is the end vertex of a U-rooted path.
2. There exists a disjoint union of a U-rooted path family and a circuit family in X that covers all vertices in X.

Theorem 1 allows us to check the generic controllability of the system on the associated graph. For other graph theoretic descriptions we refer to [6, 15, 21]. Note further that a similar graph theoretic result holds for generic observability.

3.2. Structure at infinity

The *rank* of a known system of type (1) is equal to the so-called *normal rank* of its transfer matrix $T(s)$, i.e. the rank of $C(sI - A)^{-1}B + D$ for almost all s. The *orders of the zero at infinity* of such a system are given by $t_1, t_2, \ldots, t_r$, with $r =$ rank $T(s)$, being the degrees of the denominator polynomials of the nonzero entries on the diagonal of the Smith–McMillan form at infinity of $T(s)$, see [3]. We write n_{infz} for the *sum of the orders of the zero at infinity.* In the previous notation, $n_{\text{infz}} = \sum_{i=1}^{r} t_i$.

For structured systems of type (2) the above definitions of the rank and orders of the zero at infinity make sense for each $\lambda \in \Lambda$. It turns out that the obtained rank and orders will have the same values for almost all $\lambda \in \Lambda$. It is therefore possible to define the *generic rank* of a structured system of type (2) to be the normal-rank that its transfer matrix $T_\lambda(s)$ has for almost all parameter values $\lambda \in \Lambda$, where $T_\lambda(s) = C_\lambda(sI - A_\lambda)^{-1}B_\lambda + D_\lambda$. In the same way, the *generic orders of the zero at infinity* of a structured system of type (2) are given as the orders of the zero at infinity that the system has for almost all parameter values $\lambda \in \Lambda$. In line with the previous notation, we write g-rank $T(s)$ for the generic rank of $T_\lambda(s)$ and g-n_{infz} for the sum of the generic orders of the zero at infinity, which themselves are denoted by g-t_i, $i = 1, 2, \ldots, r$, where $r =$ g-rank $T(s)$. So, g-$n_{\text{infz}} = \sum_{i=1}^{r}$ g-t_i.

For structured systems of type (2) the following results have been proved, see [1, 22, 23, 24].

Theorem 2. Consider a linear structured system of type (2) with graph G and transfer matrix $T_\lambda(s)$.

1. g-rank $T(s)$ is equal to the maximal size of a linking in G from U to Y.
2. Let g-rank $T(s) = r$. Then g-n_{infz} is equal to the minimal number of vertices in X contained in a size r linking in G from U to Y.
3. Let g-rank $T(s) = r$ and let α_i be the minimal number of vertices in X contained in a size i linking in G from U to Y for $i = 1, 2, \ldots, r$. Then g-t_i is equal to $\alpha_i - \alpha_{i-1}$, for $i = 1, 2, \ldots, r$, with $\alpha_0 = 0$.

3.3. Finite structure

In system theory the notion of *invariant zero* plays an important role. Invariant zeros are the zeros of the nonzero polynomials on the diagonal of the Smith form of the *system pencil*

$$P(s) = \begin{pmatrix} A - sI & B \\ C & D \end{pmatrix}.$$

We write n_{invz} for the *number of invariant zeros* where we count the multiplicity.

For structured systems of type (2) the above makes sense for each individual $\lambda \in \Lambda$. In fact, we can even talk about the *generic number of invariant zeros* of a structured system as the number of invariant zeros that the structured system has for almost all $\lambda \in \Lambda$. It turns out that we can make a distinction between invariant zeros that are located *at* $s = 0$ and invariant zeros that are located *outside* $s = 0$. It can be shown that the latter are mutually distinct, while the former can occur with one or more orders. This is analogous to the zero at infinity which also can occur with one or more orders. We write g-n_{invz} for the generic number of all invariant zeros and g-$n_{\text{invz}}(0)$ for the generic number of the invariant zeros at $s = 0$, where in the latter case we count the multiplicity.

3.3.1. Number of invariant zeros

We start by presenting graph theoretic characterizations of the generic number of invariant zeros of a structured system for some special cases. In the theorem below g-rank $P(s)$ denotes the generic rank of $P_\lambda(s)$, i.e. the rank that the system pencil corresponding to a linear structured system of type (2) has for almost all $\lambda \in \Lambda$. For proofs we refer to [25].

Theorem 3. Consider a linear structured system of type (2) with graph G and transfer matrix $T_\lambda(s)$.

1. Let $m = p$ and g-rank $P(s) = n + p$ (system Σ_λ is square and generically invertible). Then g-n_{invz} is equal to n minus the minimal number of vertices in X contained in a size p linking in G from U to Y, i.e. g-$n_{\text{invz}} = n - g - n_{\text{infz}}$.
2. Let g-rank $P(s) = n + p$, even after the deletion of an arbitrary column from $P_\lambda(s)$. Then generically the invariant zeros of system Σ_λ are all located at $s = 0$ and their number, g-$n_{\text{invz}}(0)$, is equal to n minus the maximal number

of vertices in X contained in the disjoint union of a size p linking from U to Y, a circuit family in X and a U-rooted path family.

In the general case, under the mild assumption that every vertex of X is contained on a path from U to Y, it is possible by means of a decomposition to transform the system matrix pencil $P(s)$ into an upper block triangular form

$$\begin{pmatrix} P_1(s) & * & * \\ 0 & P_2(s) & * \\ 0 & 0 & P_3(s) \end{pmatrix}$$

where the $*$'s denote subpencils/submatrices of suitable dimensions that are not relevant in the present context. The subpencils $P_1(s), P_2(s)$ and $P_3(s)$ correspond to linear structured subsystems of the same type as (2) with specific properties. In particular, the subpencil $P_2(s)$ satisfies the requirements mentioned in part 1 of Theorem 3, i.e. it is square and generically invertible. The subpencils $P_1(s)$ and $P_3^T(s)$ satisfy each the requirements in part 2 of Theorem 3, i.e. they each generically have full row rank, even after the deletion of an arbitrary column. Hence, with the results in Theorems 2 and 3 the generic number of invariant zeros can also be established in the general case. The above decomposition is based on computing the set of vertices that are contained in any maximum size linking from U to Y. For details on the decomposition we refer to [25].

3.3.2. Generic structure at $s = 0$

It has been shown in [26] that the invariant zeros not located at $s = 0$ are mutually distinct whereas the invariant zeros at $s = 0$ can occur with more than one order. The next theorem is inspired by the results in [10, 19, 27]. It enables us to determine the structure of the invariant zero at $s = 0$. With Theorems 2 and 3 and the above decomposition then also the number of invariant zeros outside $s = 0$ can be determined (only coming from the subpencil $P_2(s)$). For an alternative approach towards the second part of the theorem we refer to [26].

Theorem 4. Consider a linear structured system of type (2) with graph G.

1. Let g-rank $T(s) = r$. Then g-$n_{\text{invz}}(0)$ is equal to n minus the maximal number of vertices in X contained in the disjoint union of a size r linking from U to Y, a U-rooted path family, a Y-topped path family and a circuit family in X in the graph G.

2. Let g-rank $T(s) = r$ and let β_i be the largest possible maximal number of vertices in X contained in the disjoint union of a size r linking from U to Y, a U-rooted path family, a Y-topped path family and a circuit family in X in *the graph G to which i extra edges have been added*, where $i = 0, 1, \dots, t$, with $t = n + r -$g-rank $P(0)$. Then generically the orders of the invariant zeros at $s = 0$, denoted g-n_i, are equal to $\beta_i - \beta_{i-1}$ for $i = 1, 2, \dots, t$.

3.4. Applications

3.4.1. State feedback decoupling

The *decoupling or noninteracting control problem* is one of the most famous problems of control theory. Besides its practical interest it has also led to a number of fundamental results in system theory.

We consider a system of type (1), where we assume that the system is square, i.e. $m = p$. We look for a state feedback $u = Fx + Jv$, with J nonsingular, such that the closed loop system transfer matrix

$$T_{F,J}(s) = (C + DF)(sI - A - BF)^{-1}BJ + DJ \tag{4}$$

is diagonal and nonsingular. It was shown in [2] that this problem has a solution if and only if the infinite structure of the system coincides with the union of the infinite structures of the p subsystems obtained by focusing on each output component individually as the output of a subsystem. In the framework of linear structured systems the formulation of the generic problem and the previous result can be combined in a natural way. After some simplification the result can be stated as in the following theorem [4, 16].

Theorem 5. Consider a structured system of type (2) with graph G and with $m = p$. The state feedback decoupling problem is generically solvable if and only if the following two conditions are satisfied.

1. There exists a size m linking in G from U to Y.
2. $l = \sum_{i=1}^{m} l^{(i)}$, where l is the minimal number of vertices in X contained in a size m linking in G from U to Y, and $l^{(i)}$ is the minimal number of vertices in X contained in a size 1 linking (a path) in G from U to the output vertex set $\{y_i\}$.

3.4.2. State feedback disturbance rejection

We first recall the statement of the well-known disturbance rejection problem. Therefore, we consider a known system of type (1) with an additional input $q(t) \in \mathbb{R}^d$ which is called disturbance and which we would like to have no effect on the output

$$\Sigma_q : \quad \begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bu(t) + Eq(t) \\ y(t) & = & Cx(t) + Du(t). \end{array} \tag{5}$$

We look for a state feedback $u = Fx + Jq$ such that the closed loop system transfer matrix from disturbance to output is equal to zero, i.e.

$$T_{F,J}(s) = (C + DF)(sI - A - BF)^{-1}(BJ + E) + DJ = 0. \tag{6}$$

This problem is called the *disturbance rejection problem by state feedback and disturbance measurement*, where the phrase "disturbance measurement" indicates that the disturbances can and will be used for control purposes.

It can be shown that the problem has a solution if and only if the system considering the control inputs alone and the system considering control and disturbance inputs together have the same infinite structure. The latter includes that the rank of both systems is equal.

The problem can also be stated in the structured system framework, where we have to deal with a linear structured system $\Sigma_{q\lambda}$ described as follows.

$$\Sigma_{q\lambda}: \quad \begin{array}{rcl} \dot{x}(t) & = & A_\lambda x(t) + B_\lambda u(t) + E_\lambda q(t) \\ y(t) & = & C_\lambda x(t) + D_\lambda u(t). \end{array} \tag{7}$$

We associate a graph G_q to the linear structured system $\Sigma_{q\lambda}$ by adding to the graph G of the original structured system a set of vertices Q corresponding to the disturbances and a set of edges corresponding to the parameters (the nonzeros) in E_λ. In addition to G_q, we also will consider the graph G of the original system without disturbances, which can be seen as a subgraph of G_q. The conditions for disturbance rejection can now be stated in terms of the two graphs G and G_q as follows (see [1, 24]).

Theorem 6. Consider a linear structured system of type (7) with graph G_q. The disturbance rejection problem by state feedback and disturbance measurement is generically solvable if and only if the following two conditions are satisfied.

1. The maximum size of a linking in G from U to Y is equal to the maximum size of a linking in G_q from $U \cup Q$ to Y, say r.
2. $l_r = l_{q,r}$, where l_r is the minimal number of vertices in X contained in a size r linking in G from U to Y, and $l_{q,r}$ is the minimal number of vertices in X contained in a size r linking in G_q from $U \cup Q$ to Y.

In case the disturbances are not available for control purposes, so that we have to take $J = 0$, we have a similar result but on a slightly modified graph.

Further, the problem of simultaneous disturbance rejection and decoupling by state feedback can be solved using similar techniques. It turns out that the combined problem has a solution if and only if both decoupling and disturbance rejection have a solution. A condensed condition that is necessary and sufficient for the generic solvability of the combined problem is given in [5].

4. BIPARTITE GRAPH, MAXIMAL MATCHING AND OPTIMAL ASSIGNMENT

In subsection 2.2 we have presented a graph which can be naturally associated with a structured system. This graph gives a visual representation of the internal structure and the solvability of several structural problems can be stated in a very pedagogical way in terms of this graph. However, it seems that another representation in terms of a bipartite graph, although probably less appealing in terms of visualization, is better adapted for efficient computations. We will now introduce this graph.

4.1. Bipartite graph of a system

We consider a linear structured system of type (2) as previously. The bipartite graph of this system is defined as $B = (V, V', E)$ as follows, where we give a new meaning to V and E. The sets V and V' are two disjoint vertex sets and E is the edge set. The vertex set V is given by $U \cup X^1$, the vertex set V' is given by $X^2 \cup Y$, with $U = \{u_1, \dots, u_m\}$ the set of input vertices, $X^1 = \{x_1^1, \dots, x_n^1\}$ the first set of state vertices, $X^2 = \{x_1^2, \dots, x_n^2\}$ the second set of state vertices and $Y = \{y_1, \dots, y_p\}$ the set of output vertices. Notice that here we have split each state vertex x_i of G in subsection 2.2 into two vertices x_i^1 and x_i^2. Denoting (v, v') for an edge from the vertex $v \in V$ to the vertex $v' \in V'$, the edge set E is newly described by $E_A \cup E_B \cup E_C \cup E_D$ with $E_A = \{(x_j^1, x_i^2) | A_{\lambda,ij} \neq 0\}$, $E_B = \{(u_j, x_i^2) | B_{\lambda,ij} \neq 0\}$, $E_C = \{(x_j^1, y_i) | C_{\lambda,ij} \neq 0\}$ and $E_D = \{(u_j, y_i) | D_{\lambda,ij} \neq 0\}$. In the latter, for instance $A_{\lambda,ij} \neq 0$ means that the (i,j)th entry of the matrix A_λ is a parameter (a nonzero). As an illustration we give in Figure 2 the bipartite graph associated with the example of subsection 2.3.

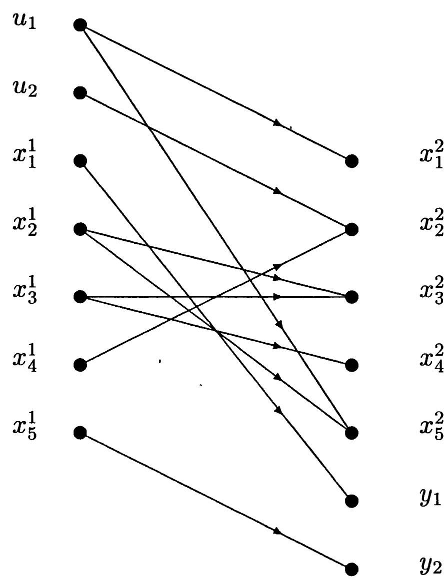

Fig. 2. Bipartite graph B of the system (3).

4.2. Maximal matching in a bipartite graph

In the two following subsections we consider a general bipartite graph $B = (V, V', E)$ as follows. The sets V, V' are two disjoint vertex sets and E is the edge set, where all edges have the form (v, v') with $v \in V$ and $v' \in V'$. These graphs received a considerable attention in the literature on combinatorics. We will consider here the *maximal matching problem* and the *optimal assignment problem* that will have a direct application in the study of structured systems. A *matching* in a bipartite graph $B = (V, V', E)$ is an edge set $M \subseteq E$ such that the edges in M have no common vertex. The cardinality of a matching, i.e. the number of edges it consists of, is also called its size. The maximal matching problem is the problem of just finding a matching of maximal cardinality. This problem can be solved using very efficient algorithms based on alternate augmenting chains or ideas of maximum flow theory [8].

As an application of the previous consider a $p \times m$ matrix N_λ, structured in the sense defined previously. To this structured matrix one can associate the bipartite graph $B_N = (V, V', E)$ with V having m vertices $\{v_1, \dots, v_m\}$, V' having p vertices $\{v'_1, \dots, v'_p\}$ and where the edge (v_j, v'_i) is an element of E if and only if $N_{\lambda,ij}$ is a parameter (a nonzero). Let the generic rank of N be the rank that N_λ will have for almost all λ. The following result is well known [17].

Lemma 1. The generic rank of N is equal to the size of a maximal matching in B_N.

4.3. The optimal assignment problem

Let us now consider a *weighted* bipartite graph, i.e. a bipartite graph $B = (V, V', E)$, for which a real number $w(e)$ is associated to each edge $e \in E$. The weight of a matching $M \subseteq E$ is defined as $w(M) = \sum_{e \in M} w(e)$. The *optimal k-matching problem* consists of finding a matching M of size k such that $w(M)$ is maximal (or minimal). In case that both V and V' consist of q vertices and there exists a matching of size q (known as a perfect matching) the optimal q-matching problem is called the *optimal assignment problem.* Again, there exist a lot of efficient algorithms to solve these problems, among them there is the famous Hungarian method [7, 14]. All the classical software packages in operations research contain optimized versions of these algorithms. We will prove in the following that the verification of the characterizations related to structured systems reduces to particular matching and assignment problems.

5. COMPUTATIONS FOR STRUCTURED SYSTEMS

5.1. Connectedness and controllability

In condition 1 of Theorem 1 the condition is that every vertex in X is the end vertex of a U-rooted path. In [18] an algorithm is presented, using a finite polynomially bounded number of Boolean operations, to determine a decomposition of the graph G based on its connectability properties. See also [20]. In fact, in the decomposition, the system matrix A_λ of a structured system like (2) is brought into an upper block triangular form. In this form every two vertices corresponding to the same block are connected to each other by means of a path, and the block themselves are possibly connected to each other by means of edges, not creating additional circuits through various blocks. With the decomposition, it is easy to verify whether or not a vertex is the end vertex of a U-rooted path. For more details we refer to [18]. Condition 2 of Theorem 1 is a way to express that the matrix $[A_\lambda, B_\lambda]$ has generic rank n, see [9]. Since $[A_\lambda, B_\lambda]$ is a structured matrix this can also be checked using Lemma 1. For this we introduce the bipartite graph associated to the matrix $[A_\lambda, B_\lambda]$ that we denote $B_{[A,B]}$ and which is obtained as in subsection 4.1.

Lemma 2. There is a disjoint union of a U-rooted path family and a circuit family in X in the graph G which covers all vertices of X if and only if the size of a maximal matching in $B_{[A,B]}$ is n.

In our example the bipartite graph $B_{[A,B]}$ can be obtained from Figure 2 by leaving out the output vertices and the edges incident to the output vertices. It can be easily seen that a maximal matching in $B_{[A,B]}$ has only size 4, therefore the linear structured system given in (3) is not controllable.

5.2. A simple technical result

We now present a technical result that will be crucial in establishing properties later on. Therefore, we introduce the *new* bipartite graph $\hat{B} = (V, V', E)$, which is the same as $B = (V, V', E)$ described in subsection 4.1, except that there are edges (x_i^1, x_i^2), for $i = 1, 2, \ldots, n$, even if $A_{\lambda,ii} = 0$, i.e. even if the (i,i)th entry of A is a fixed zero. In the following we will call the edges of the form (x_i^1, x_i^2) horizontal edges and we will sometimes refer to the newly introduced horizontal edges as fictitious edges. For our example the bipartite graph $\hat{B}$ is represented in Figure 3. We can then state the following lemma (see also [17]).

Lemma 3. A maximal matching in $\hat{B}$, say of size $n + r$, is made of a union of a subset of the horizontal edges and a subset of the other edges that is in one to one correspondence with a disjoint union of the following type of path families in G:

— a linking from U to Y of size r,

— a U-rooted path family,

— a Y-topped path family,

— a circuit family in X.

Proof. Consider a disjoint union H of a linking from U to Y of size r, a U-rooted path family, a Y-topped path family and a circuit family in X. Suppose that the disjoint union H contains γ vertices in X. Then it consists of $\gamma + r$ edges. Indeed, each U-rooted path, Y-topped path or circuit family in X consists of as many edges as it contains vertices in X. Each simple path in a linking from U to Y of size r consists of one more edge than it contains edges in X. The edges in the bipartite graph $\hat{B}$ seen in the disjoint union H have no vertices in common and therefore form a matching of size $\gamma + r$. The $n - \gamma$ vertices in X not present in the disjoint union H can be associated in the bipartite graph $\hat{B}$ to $n - \gamma$ horizontal edges, disjoint from the previous matching and together forming a matching of size $n + r$ in $\hat{B}$.

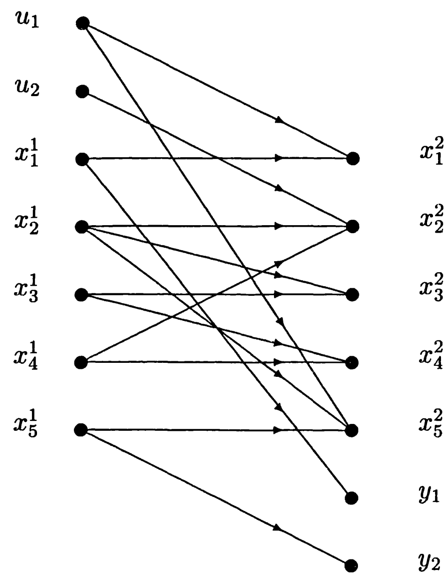

Fig. 3. Bipartite graph $\hat{B}$ of the system (3).

Conversely, consider a maximal matching M in the bipartite graph $\hat{B}$. The size of a maximal matching is at least n because of the set of edges (x_i^1, x_i^2), $i = 1, 2, \ldots, n$, forming a matching of size n. Therefore, assume that the size of M is $n + r$ with $r \geq 0$. Now consider the edges in M. Seen as edges in G it follows that every vertex of G is the begin vertex of at most one edge in M. Likewise, every vertex of G is the end vertex of at most one edge in M. This implies that the edges of M seen as edges in G form a collection of just simple paths that are disjoint to each other. Now consider one such paths in its full length, say its length is τ. By the maximality of M it follows that the path can not begin at one vertex of X and end at an other vertex of X. Indeed, if this would be the case the τ edges of the path can in the matching M be replaced by $\tau + 1$ horizontal edges forming a new matching with a size larger than $n + r$. The only possibility to have that a simple path in G completely contained in X is that its begin vertex and its end vertex are the same, i. e. the path is actually a circuit. It is however possible to have paths in G that begin in U, that end in Y, or that do both. Think respectively of U-rooted paths, Y-topped paths and paths from U to Y. Note that the number of vertices in X they contain and the number of edges they consist of is the same for U- rooted paths, Y-topped paths and circuits in X. A simple path from U to Y contains one more edge than it contains vertices in X. Counting the number of vertices in X on the simple paths in G made up from the edges in M it follows that there must be r disjoint simple paths from U to Y, together forming a linking of size r. The other simple paths form a U-rooted path family, a Y-topped path family and a circuit family in X. Notice that the horizontal edges of the form (x_i^1, x_i^2) of $\hat{B}$ correspond either to a self loop in G or to a fictitious edge. □

5.3. Rank and infinite structure

In subsection 3.2 we presented a graph theoretic characterization of the rank and the structure at infinity of a linear structured system. The characterizations were given in terms of linkings in the graph G associated to the linear structured system. We will prove in this subsection that these characterizations can equivalently be obtained in terms of matchings and optimal assignments in a suitably defined weighted bipartite graph.

To a given structured system we associate the bipartite graph with the same vertex set and edge set as $\hat{B}$ and where all the edges are given a weight 1, except for the edges (x_i^1, x_i^2), for $i = 1, 2, \ldots, n$, which have weight 0. We will denote this weighted graph by B_{inf}. We can state the next result.

Lemma 4. Consider a linear structured system of type (2) with weighted bipartite graph B_{inf} and transfer matrix $T_\lambda(s)$.

1. g-rank $T(s)$ is equal to the size of a maximal matching in B_{inf} minus n.
2. Let g-rank $T(s) = r$. Then g-n_{infz} is equal to the minimal weight of a size $n + r$ matching in B_{inf} minus r.
3. Let g-rank $T(s) = r$ and let α_i be the minimal weight of a size $n + i$ matching in B_{inf} minus i for $i = 1, 2, \ldots, r$. Then g-t_i is equal to $\alpha_i - \alpha_{i-1}$, for $i = 1, 2, \ldots, r$, with $\alpha_0 = 0$.

Proof.

1. The result follows from Theorem 2 and Lemma 3.
2. From Lemma 3 it is clear that a minimal weight matching of size $n + r$ corresponds to a minimal linking of size r completed by a set of zero cost edges of the form (x_i^1, x_i^2). Indeed the U-rooted paths, the Y-topped paths and circuit families in X cannot belong to a minimal weight matching . If such paths would exist they could be replaced by lower weight horizontal edges.
3. In the case of matchings of size $n + i$ with $i < r$, Lemma 3 does not apply directly. Such matchings may also contain state-state paths besides the paths in G of Lemma 3. But as in point 2, these state-state paths cannot belong to a minimal weight matching. Then the considered size $n + i$ matching will correspond to a size i linking completed by horizontal edges as in point 2 and the result follows. □

Notice that in [11] the generic infinite structure computation is transformed in a minimal cost maximal flow problem on a modified graph, and is performed via an efficient primal-dual algorithm. In our example the size of a maximal matching in B_{inf} is 7, consider for example the matching $M_1 = \{(u_1, x_1^2), (u_2, x_2^2), (x_1^1, y_1), (x_2^1, x_5^2), (x_3^1, x_3^2), (x_4^1, x_4^2), (x_5^1, y_2)\}$. Then g-rank of $T(s) = r = 2$. It is easy to verify that this size 7 matching is of minimal weight in B_{inf}. As $w(M_1) = 5$ it follows that g-$n_{\text{infz}} = \alpha_2 = 5 - 2 = 3$. Furthermore it turns out that the minimal weight of a size

6 matching is 2, consider for example the matching $M_2 = \{(u_1, x_5^2), (x_5^1, y_2), (x_1^1, x_1^2), (x_2^1, x_2^2), (x_3^1, x_3^2), (x_4^1, x_4^2)\}$, then $\alpha_1 = 2 - 1 = 1$. It follows that the generic infinite zero orders are g-$t_2 = 2$, g-$t_1 = 1$.

5.4. Finite structure

To study the finite structure we will use also a weighted bipartite graph. This bipartite graph is the same as $\hat{B}$ but now with weight 0 for all edges, except for fictitious edges (x_i^1, x_i^2) when $A_{\lambda,ii} = 0$ for $i = 1, 2, \ldots, n$, which have weight 1. We denote this new weighted graph by B_{fin}. The result can be stated as follows.

Lemma 5. Consider a linear structured system of type (2) with weighted bipartite graph B_{fin} and transfer matrix $T_\lambda(s)$.

1. Let g-rank $T(s) = r$. Then g-$n_{\text{inv}}(0)$ is equal to the minimal weight of a size $n + r$ matching in B_{fin}.
2. Let g-rank $T(s) = r$ and let β_i be the minimal weight of a matching of size $n + r - i$ in the graph B_{fin}, where $i = 0, 1, \ldots, t$, with $t = n + r -$g-rank $P(0)$. Then generically the orders of the invariant zeros at $s = 0$, denoted g-n_i, are equal to $\beta_{i-1} - \beta_i$ for $i = 1, 2, \ldots, t$.

Proof.

1. From the decomposition of a maximal matching in Lemma 3 and from the weights in B_{fin} it follows that in a minimal weight matching the fictitious edges (x_i^1, x_i^2) have to be avoided as much as possible. The number of such edges will be the number of vertices in G which are not covered by a maximal linking from U to Y, a U-rooted path family, a Y-topped path family and a circuit family in X. The result follows.
2. Inspired by parts 1 and 2 of Theorem 4, and by point 1 above, we have to determine the minimal weight, denoted γ_i say, of a matching of a size $n + r$ in the bipartite graph B_{fin} to which i extra edges with zero cost have been added, where $i = 0, 1, \ldots, t$ with $t = n + r -$g-rank $P(0)$. Then generically the orders of the invariant zeros at $s = 0$ are equal to $\gamma_{i-1} - \gamma_i$ for $i = 1, 2, \ldots, t$.

   The edges that are added to the bipartite graph B_{fin} have zero cost and will be used to replace the edges that have cost one in a minimal weight matching of size $n + r$. However, instead of adding zero cost edges and looking at size $n + r$ matchings, it is also possible to look at matchings of smaller size and to avoid the edges that have weight one. This then precisely gives the formulation of part 2 of the present theorem. For more on this see [26]. Finally, notice that

$$P(0) = \begin{pmatrix} A_\lambda & B_\lambda \\ C_\lambda & D_\lambda \end{pmatrix}. \tag{8}$$

   Therefore, its rank is simply obtained as the size of the maximal matching in the bipartite graph B introduced in subsection 4.1. □

Consider again our example. The minimal weight of a size 7 matching in B_{fin} is $\beta_0 = 1$. For example consider the matching M_1 of the previous subsection but on B_{fin}. Then g-$n_{\text{inv}}(0) = 1$. Leaving out an edge with cost one, i.e. the only one (x_4^1, x_4^2), it follows that the minimal weight of a size 6 matching is $\beta_1 = 0$. Then g-$n_1 = \beta_0 - \beta_1 = 1$. From Theorem 3, part 1, the total number of invariant zeros is g-$n_{\text{inv}} = 5 - 3 = 2$. We have one invariant zero at $s = 0$ and the other invariant zero is located outside $s = 0$, in fact at $s = \lambda_5$. This can be seen from the determinant of the system pencil of the system defined in (3):

$$\det P_\lambda(s) = \lambda_1\lambda_3\lambda_8\lambda_9\lambda_{10}s(s-\lambda_5).$$

5.5. Control applications

The results which have have been obtained before concerning the structure at infinity can easily be used to check the conditions for decoupling and disturbance rejection. We will consider state feedback decoupling and disturbance rejection problems, and give solvability conditions expressed as easy-to-check conditions on some bipartite graphs. These results will be given without proof since they easily follow from the previous subsections.

Let us consider first the decoupling problem and introduce the new graph $B_{\text{inf}}^{(i)}$ which is obtained as B_{inf} but considering only output i. It means that $B_{\text{inf}}^{(i)}$ is obtained from B_{inf} by deleting all output vertices except the ith one and all edges incident to these vertices. The result of Theorem 5 becomes.

Lemma 6. Consider a structured system of type (2) with $m = p$ and associated bipartite graphs B_{inf} and $B_{\text{inf}}^{(i)}$ for $i = 1, \ldots, m$. The state feedback decoupling problem is generically solvable if and only if the following two conditions are satisfied.

1. There exists a size $(n+m)$ matching in B_{inf}.
2. $W = \sum_{i=1}^{m} w^{(i)}$, where W is the minimal weight of a matching of size $(n+m)$ in B_{inf}, and $w^{(i)}$ is the minimal weight of a matching of size $(n+1)$ in $B_{\text{inf}}^{(i)}$.

Concerning our example we already noticed that there exists a size $(n+m)$ matching in B_{inf} with minimal weight 5. It can be checked that a minimal weight matching of size $(n+1) = 6$ in $B_{\text{inf}}^{(1)}$ has weight 2, for example $M_3 = \{(u_1, x_1^2), (x_1^1, y_1), (x_2^1, x_2^2), (x_3^1, x_3^2), (x_4^1, x_4^2), (x_5^1, x_5^2)\}$, then $w^{(1)} = 2$. Similarly we get $w^{(2)} = 2$, therefore from the second condition this system is not decouplable.

In the same way we can study the disturbance rejection problem, this needs the introduction of a new bipartite graph, B_{inf}^q which is built as B_{inf} but taking into account all inputs including controls and disturbances. This leads to a formulation which is equivalent to Theorem 6.

Lemma 7. Consider a linear structured system of type (7) with associated bipartite graphs B_{inf} and B_{inf}^q. The disturbance rejection problem by state feedback and disturbance measurement is generically solvable if and only if the following two conditions are satisfied.

1. The maximum size of a matching in B_{inf} is equal to the maximum size of a matching in B^q_{inf}, say $(n+r)$.
2. $L_r = L^q_r$, where L_r is the minimal weight of a size $(n+r)$ matching in B_{inf}, and L^q_r is the minimal weight of a size $(n+r)$ matching in B^q_{inf}.

6. CONCLUDING REMARKS

In this paper we have recalled some of the generic properties and feedback invariants of linear structured systems. Further we have presented an efficient and new method to determine the finite and infinite zero structure of a linear structured system. The method is based on the use of a weighted bipartite graph associated to the structured system. We have shown that the computation of generic invariants and properties of structured systems reduces to study particular matchings and optimal assignment problems. The solution of such problems can be implemented in a very efficient way by using well-known approaches.

(Received March 27, 2002.)

REFERENCES

[1] C. Commault, J. M. Dion, and A. Perez: Disturbance rejection for structured systems. IEEE Trans. Automat. Control *AC-36* (1991), 884–887.

[2] J. Descusse and J. M. Dion: On the structure at infinity of linear square decouplable systems. IEEE Trans. Automat. Control *AC-27* (1982), 971–974.

[3] J. M. Dion and C. Commault: Smith–McMillan factorisations at infinity of rational matrix functions and their control interpretation. Systems Control Lett. *1* (1982), 312–320.

[4] J. M. Dion and C. Commault: Feedback decoupling of structured systems. IEEE Trans. Automat. Control *AC-38* (1993), 1132–1135.

[5] J. M. Dion, C. Commault, and J. Montoya: Simultaneous decoupling and disturbance rejection – a structural approach. Internat. J. Control *59* (1994), 1325–1344.

[6] K. Glover and L. M. Silverman: Characterization of structural controllability. IEEE Trans. Automat. Control *AC-21* (1976), 534–537.

[7] M. Gondran and M. Minoux: Graphs and Algorithms. Wiley, New York 1984.

[8] J. E. Hopcroft and R. M. Karp: An $n^{5/2}$ algorithm for maximum matchings in bipartite graphs. SIAM J. Comput. *2* (1973), 225–231.

[9] S. Hosoe: Determination of generic dimensions of controllable subspaces and applications. IEEE Trans. Automat. Control *AC-25* (1980), 1192–1196.

[10] V. Hovelaque: Analyse Structurelle, Géométrique, et Graphique des Systèmes Linéaires Structurés, Thèse de Doctorat. Inst. Nat. Polytechnique de Grenoble 1997.

[11] V. Hovelaque, C. Commault, and J. M. Dion: Analysis of linear structured systems using a primal-dual algorithm. Systems Control Lett. *27* (1996), 73–85.

[12] V. Hovelaque, C. Commault, and J. M. Dion: Disturbance decoupling for linear structured systems via a primal-dual algorithm. Comp. Engrg. Syst. Appl. IMACS Lille (1996), 455–459.

[13] V. Hovelaque, N. Djidi, C. Commault, and J. M. Dion: Decoupling problem for structured systems via a primal-dual algorithm. In: Proc. European Control Conference (ECC97), Brussels 1997.

[14] H. W. Kuhn: The Hungarian method for the assignment problem. Nav. Res. Log. Quat. *2* (1955), 83–97.
[15] C. T. Lin: Structural controllability. IEEE Trans. Automat. Control *AC-19* (1974), 201–208.
[16] A. Linnemann: Decoupling of structured systems. Systems Control Lett. *1* (1981), 79–86.
[17] K. Murota: System analysis by graphs and matroids, Algorithms and Combinatorics. Springer–Verlag, New York 1987.
[18] K. J. Reinschke: Multivariable Control: A Graph–heoretic Approach. Springer–Verlag, New York 1988.
[19] K. Röbenack and K. J. Reinschke: Digraph based determination of Jordan block size structure of singular matrix pencils. Linear Algebra Appl. *275–276* (1998), 495–507.
[20] C. Schizas and F. J. Evans: A graph theoretic approach to multivariable control system design. Automatica *17* (1981), 371–377.
[21] R. W. Shields and J. B. Pearson: Structural controllability of multi-input linear systems. IEEE Trans. Automat. Control *AC-21* (1976), 203–212.
[22] W. Söte: Eine graphische Methode zur Ermittlung der Nullstellen in Mehrgrössensystemen. Reglungstechnik *28* (1980), 346–348.
[23] N. Suda, B. Wan, and I. Ueno: The orders of infinite zeros of structured systems. Trans. Soc. Instr. Control Engineers *25* (1989), 346–348.
[24] J. W. van der Woude: On the structure at infinity of a structured system. Linear Algebra Appl. *148* (1991), 145–169.
[25] J. W. van der Woude: The generic number of invariant zeros of a structured linear system. SIAM J. Control Optim. *38* (2000), 1, 1–21.
[26] J. W. van der Woude: The generic canonical form of a regular structured matrix pencil. Linear Algebra Appl. *353* (2002), 267–288.
[27] J. W. van der Woude, C. Commault, and J. M. Dion: Invariants for linear structured systems. Internal report of the Laboratoire d'Automatique de Grenoble 2000.
[28] T. Yamada: A network flow algorithm to find an elementary I/O matching. Networks *18* (1988), 105–109.

Prof. Dr. Christian Commault and Prof. Dr. Jean-Michel Dion, Laboratoire d'Automatique de Grenoble, ENSIEG, INPG, BP 46, 38402 Saint-Martin-d'Heres. France.
e-mails: Christian.Commault,Jean-Michel.Dion@inpg.fr

Dr. Jacob W. van der Woude, Delft University of Technology, Faculty ITS, Mekelweg 4, 2628 CD Delft. The Netherlands.
e-mail: j.w.vanderwoude@its.tudelft.nl