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ON THE g–ENTROPY AND ITS HUDETZ CORRECTION[1]

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The Hudetz correction of the fuzzy entropy is applied to the g-entropy. The new invariant is expressed by the Hudetz correction of fuzzy entropy.

1. INTRODUCTION

The fuzzy entropy $h(T)$ of a dynamical system has been introduced in [5] (see also [1, 3, 8, 10]). Generalizing the notion of a fuzzy partition Mesiar and Rybárik have studied the g-entropy (see [7, 10, 11]) based on the Pap g-calculus ([9]). The notion is based on an increasing bijective function $g : [0, \infty] \to [0, \infty]$, such that $g(0) = 0$ and $g(1) = 1$. The choice $g(x) = x$ leads to the fuzzy entropy. The corresponding theorem states that to any g-decomposable measure there exists a fuzzy measure such that the g-entropy can be expressed by the fuzzy entropy.

Of course, the fuzzy entropy depends on a family $\mathcal{F}$ of fuzzy sets. If $\mathcal{F}$contains all constant functions, then the fuzzy entropy equals infinity. This defect has been corrected by Hudetz ([4]) by introducing a correcting member in the definition of the entropy of a fuzzy partition.

The aim of this paper is a study of an analogous correction in the case of g-entropy. Similarly as Mesiar and Rybárik in [7] we prove the corresponding representation theorem. We construct also an example demonstrating that the Hudetz modification of g-entropy can be used although the usual g entropy is not available.

2. g–ENTROPY

Let $(\Omega, \mathcal{S}, P, T)$ be the classical dynamical system, i.e. $(\Omega, \mathcal{S}, P)$ is a probability space and $T : \Omega \to \Omega$ is a measure preserving transformation, i.e. $A \in \mathcal{S}$ implies $T^{-1}(A) \in \mathcal{S}$ and $P(T^{-1}(A)) = P(A)$.

We shall consider a σ-algebra $\mathcal{F}$ of $\mathcal{S}$-measurable fuzzy subsets of Ω, i.e. functions $f : \Omega \to [0, 1]$ satisfying the following conditions:

(i) $1_\Omega \in \mathcal{F}$;

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(ii) if $f_1, f_2 \in \mathcal{F}$, then $(f_1 - f_2)^+ \in F$;

(iii) if $f_n \in \mathcal{F}, n = 1, 2, \ldots$, then $\bigvee_{n=1}^{\infty} f_n \in \mathcal{F}$;

(iv) if $f_1, f_2 \in \mathcal{F}$, then $f_1 \cdot f_2 \in \mathcal{F}$.

Consider further a $\oplus$-decomposable (with respect to a function g mentioned above) measure on $\mathcal{F}$, i.e. a mapping $m : \mathcal{F} \to [0,1]$ such that $m(1_\Omega) = 1$, $m(0_\Omega) = 0$, and

$$m\left(g^{-1}\left(\sum_{n=1}^{\infty} g(f_n)\right)\right) = g^{-1}\left(\sum_{n=1}^{\infty} g(m(f_n))\right)$$

whenever $f_n \in \mathcal{F}$ $(n = 1, 2, \ldots)$ are such that $\sum_{n=1}^{\infty} g \circ f_n \le 1$. (Recall that by [6] the function $g \circ f_n \in \mathcal{F}$). If m satisfies the above condition, then $\mu = g \circ m \circ g^{-1} : \mathcal{F} \to [0,1]$ is a fuzzy measure, i.e.

$$\mu\left(\sum_{n=1}^{\infty} f_n\right) = \sum_{n=1}^{\infty} \mu(f_n)$$

whenever $f_n \in \mathcal{F}$ $(n = 1, 2, \ldots)$ and $\sum_{n=1}^{\infty} f_n \le 1$.

A family $\mathcal{A} = \{f_1, \ldots, f_k\} \subset \mathcal{F}$ is a g-fuzzy partition of Ω, if $\sum_{i=1}^{k} g(f_i(\omega)) = 1$ for any $\omega \in \Omega$. The g-entropy $H_g(\mathcal{A}))$ of the g-fuzzy partition $\mathcal{A}$ is defined by the formula

$$H_g(\mathcal{A}) = g^{-1}\left(\sum_{i=1}^{k} g(\Phi(m(f_i)))\right),$$

where $\Phi = g^{-1} \circ \varphi \circ g, \varphi(x) = -x \log x$ for $x > 0, \varphi(0) = 0$, hence

$$H_g(\mathcal{A}) = g^{-1}\left(\sum_{i=1}^{k} \varphi(\mu(g(f_i)))\right).$$

If $\mathcal{A} = \{f_1, \ldots, f_k\}$ and $\mathcal{B} = \{h_1, \ldots, h_t\}$ are two g-fuzzy partitions, then their common refinement $\mathcal{A} \vee \mathcal{B}$ is given by the formula

$$\mathcal{A} \vee \mathcal{B} = \{g^{-1}((g \circ f_i) \cdot (g \circ h_j));\ i = 1, \ldots, k,\ j = 1, \ldots, t\}.$$

It is possible to show the existence of the limit

$$h_g(\mathcal{A}, T) = \lim_{n \to \infty} g^{-1}\left(\frac{1}{n} g(H_g\left(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{A}\right)\right),$$

where $T^{-i}(\mathcal{A}) = \{f_1 \circ T^i, \ldots, f_k \circ T^i\}$. The entropy of T is defined by the formula

$$h_g(T) = \sup\{h_g(\mathcal{A}, T); \mathcal{A} \text{ is a } g\text{-fuzzy partition}\}.$$

As we have already mentioned, the fuzzy entropy $h(T)$ can be obtained putting $g(u) = u$, $u \in [0,1]$. In the following proposition the symbols $H_g(\mathcal{A})$, $h_g(\mathcal{A},T)$, $h_g(T)$ are taken with respect to the given g-decomposable measure m, the symbols $H(\mathcal{B})$, $h(\mathcal{B},T)$, $h(T)$ with respect to the induced fuzzy measure $\mu = g \circ m \circ g^{-1}$.

Recall that if $\mathcal{A} = \{f_1, \dots, f_k\}$ is a g-fuzzy partition and $h_i = f_i \circ g (i = 1, 2, \dots, k)$, then $g(\mathcal{A}) = \{h_1, \dots, h_k\}$ is a fuzzy partition, i. e. $\sum_{i=1}^k h_i = 1$.

Proposition. For any dynamical system $(\Omega, \mathcal{S}, P, T)$, any g and any g-partition $\mathcal{A}$ there holds:

(i) $H_g(\mathcal{A}) = g^{-1}(H(g(\mathcal{A})))$,

(ii) $h_g(\mathcal{A},T) = g^{-1}(h(g(\mathcal{A}),T))$,

(iii) $h_g(T) = g^{-1}(h(T))$.

Proof. [10], Proposition 10.6.6. □

3. HUDETZ CORRECTION

Let us start with a dynamical system $(\Omega, \mathcal{S}, P, T)$. Define μ on the family of all integrable functions by the formula $\mu(f) = \int_\Omega f \,\mathrm{d}P$. Let $m = g^{-1} \circ \mu \circ g$. The Hudetz correction instead of entropy of a fuzzy partition $\mathcal{B} = \{h_1, \dots, h_k\}$

$$H(\mathcal{B}) = \sum_{i=1}^{k} \varphi(\mu(h_i))$$

uses the difference

$$H^\flat(\mathcal{B}) = H(\mathcal{B}) - F(\mathcal{B})$$

where

$$F(\mathcal{B}) = \mu\left(\sum_{i=1}^{k} \varphi(h_i)\right).$$

Mention that the sum $\sum_{i=1}^k \varphi(h_i))$ need not belong to $\mathcal{F}$, of course μ is defined on the family of all integrable functions on Ω. We want to define a g-analogy of the value $F(\mathcal{B})$. Recall that in g-calculus

$$a \oplus b = g^{-1}(g(a) + g(b))$$

($\oplus$ is a partial operation on [0,1], $a \oplus b$ is defined if $g(a) + g(b) \leq 1$). Therefore the entropy $H_g(\mathcal{A})$ can be reformulated as

$$H_g(\mathcal{A}) = \bigoplus_{i=1}^{k} \Phi(m(f_i)).$$

Similarly

$$a \odot b = g^{-1}(g(a) \cdot g(b)),$$

whence

$$\mathcal{A} \vee \mathcal{B} = \{f_i \odot h_j;\ i = 1, \dots, k,\ j = 1, \dots, t\}.$$

Analogously $a \ominus b$ could be defined by the formula

$$a \ominus b = g^{-1}(g(a) - g(b)),$$

of course, only if $g(b) \leq g(a)$, i.e. $b \leq a$. Since we want to define

$$F_g(\mathcal{A}) = m\left(\bigoplus_{i=1}^{k} \Phi(f_i)\right),$$

and

$$H^\flat(\mathcal{A}) = H_g(\mathcal{A}) \ominus F_g(\mathcal{A})$$

we must to prove the inequality $F_g(\mathcal{A}) \leq H_g(\mathcal{A})$.

Lemma 1. $F_g(\mathcal{A}) = g^{-1}(F(g(\mathcal{A})))$ for any g-fuzzy partition $\mathcal{A}$.

Proof. We have $m = g^{-1} \circ \mu \circ g$, $\bigoplus_{i=1}^{k} a_i = g^{-1}\left(\sum_{i=1}^{k} g(a_i)\right)$, $\Phi = g^{-1} \circ \varphi \circ g$, $g(\mathcal{A}) = \{g \circ f_1, \dots, g \circ f_k\}$. Therefore

$$\begin{aligned} m\left(\bigoplus_{i=1}^{k} \Phi(f_i)\right) &= g^{-1} \circ \mu \circ g \circ g^{-1}\left(\sum_{i=1}^{k} g(g^{-1} \circ \varphi \circ g)(f_i)\right) \\ &= g^{-1}\left(\mu\left(\sum_{i=1}^{k} \varphi(g \circ f_i))\right) = g^{-1}(F(g(\mathcal{A})\right)\right). \end{aligned}$$

□

Lemma 2. $F_g(\mathcal{A}) \leq H_g(\mathcal{A})$ for any g-fuzzy partition $\mathcal{A}$.

Proof. By Proposition we have $H_g(\mathcal{A}) = g^{-1}(H(g(\mathcal{A})))$, by Lemma 1 we have $F_g(\mathcal{A}) = g^{-1}(F(g(\mathcal{A}))$. Since φ is concave, we have

$$\mu(\varphi(h_i)) = \int_\Omega \varphi(h_i)\, \mathrm{d}P \leq \varphi\left(\int_\Omega h_i\, \mathrm{d}P\right) = \varphi(\mu(h_i)),$$

hence

$$F(g(\mathcal{A})) = \mu\left(\sum_{i=1}^{k} \varphi(h_i)\right) = \sum_{i=1}^{k} \mu(\varphi(h_i)) \leq \sum_{i=1}^{k} \varphi(\mu(h_i)) = H(g(\mathcal{A})),$$

and

$$F_g(\mathcal{A}) = g^{-1}(F(g(\mathcal{A}))) \leq g^{-1}(H(g(\mathcal{A}))) = H_g(\mathcal{A}).$$

□

Definition. For any g-fuzzy partition $\mathcal{A} = \{f_1, \ldots, f_k\}$ we define

$$H_g^\flat(\mathcal{A}) = H_g(\mathcal{A}) \ominus F_g(\mathcal{A}).$$

Theorem 1. $H_g^\flat(\mathcal{A}) = g^{-1}(H^\flat(g(\mathcal{A})))$ for any g-fuzzy partition $\mathcal{A}$.

Proof. By the definition of the operation $\ominus$, Proposition and Lemma 1 we obtain

$$\begin{aligned} H_g^\flat(\mathcal{A}) &= H_g(\mathcal{A}) \ominus F_g(\mathcal{A}) \\ &= g^{-1}(g((H_g A)) - g(F_g(\mathcal{A}))) \\ &= g^{-1}(g(g^{-1}(H(g(\mathcal{A}))) - g(g^{-1}(F(g(\mathcal{A})))) \\ &= g^{-1}(H(g(\mathcal{A})) - F(g(\mathcal{A}))) \\ &= g^{-1}(H^\flat(g(\mathcal{A}))). \end{aligned}$$ □

Theorem 2. $H_g^\flat\left(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{A}\right) = g^{-1}\left(H\left(\bigvee_{i=0}^{n-1} T^{-i}(g(\mathcal{A}))\right)\right)$ for any g-fuzzy partition $\mathcal{A}$.

Proof. We have $T^{-i}(\mathcal{A}) = \{f_1 \circ T^i, \ldots, f_k \circ T^i\}, T^{-i}(g(\mathcal{A})) = \{g \circ f_1 \circ T^i, \ldots, g \circ f_k \circ T^i\}$. Of course, recall the definition of the refinement of g-fuzzy partitions: $\bigvee_{i=0}^{n-1} T^{-i}(\mathcal{A})$ consists of all $\odot$-products

$$\begin{aligned} & f_{i_1} \odot (f_{i_2} \circ T) \odot \ldots \odot (f_{i_n} \circ T^{n-1}) \\ = \; & g^{-1}((g \circ f_{i_1}) \cdot ((g \circ f_{i_2}) \circ T) \cdot \ldots \cdot ((g \circ f_{i_n} \circ T^{n-1})) \end{aligned}$$

i.e. of all functions $g^{-1} \circ h$, where $h \in \bigvee_{i=0}^{n-1} T^{-i}(g(\mathcal{A}))$. Therefore

$$g\left(\bigvee_{i=0}^{n-1} T^{-i}(\mathcal{A})\right) = \bigvee_{i=0}^{n-1} T^{-i}(g(\mathcal{A})),$$

and

$$\begin{aligned} H_g^\flat\left(\bigvee_{i=0}^{n-1} T^{-i}(\mathcal{A})\right) &= g^{-1}\left(H\left(g\left(\bigvee_{i=0}^{n-1} T^{-i}(\mathcal{A}\right)\right)\right) \\ &= g^{-1}\left(H\left(\bigvee_{i=0}^{n-1} T^{-i}(g(\mathcal{A})\right)\right). \end{aligned}$$ □

Theorem 3. For any g-fuzzy partition $\mathcal{A}$ there exists

$$h_g^\flat(\mathcal{A}, T) := \lim_{n\to\infty} g^{-1}\left(\frac{1}{n}\right) \odot H_g^\flat\left(\bigvee_{i=0}^{n-1} T^{-i}(\mathcal{A})\right),$$

and there holds

$$h_g^\flat(\mathcal{A}, T) = g^{-1}(h^\flat(g(\mathcal{A}), T)).$$

Proof. By the definition of $\odot$ and Theorem 2 we have

$$\begin{aligned} & g^{-1}\left(\frac{1}{n}\right) \odot H_g^\flat\left(\bigvee_{i=0}^{n-1} T^{-i}(\mathcal{A})\right) \\ = \; & g^{-1}\left(g\left(g^{-1}\left(\frac{1}{n}\right)\right) g\left(H_g^\flat\left(\bigvee_{i=0}^{n-1} T^{-i}(\mathcal{A})\right)\right)\right) \\ = \; & g^{-1}\left(\frac{1}{n} H\left(\bigvee_{i=0}^{n-1} T^{-i}(g(\mathcal{A})\right)\right). \end{aligned}$$

Of course,

$$\lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} T^{-i}(g(\mathcal{A}))\right) = h^\flat(g(\mathcal{A}), T).$$

Since g^{-1} is continuous,

$$\begin{aligned} g^{-1}(h^\flat(g(\mathcal{A}), T)) &= \lim_{n\to\infty} g^{-1}\left(\frac{1}{n} H\left(\bigvee_{n=0}^{n-1} T^{-i}(g(\mathcal{A})\right)\right) \\ &= \lim_{n\to\infty} g^{-1}\left(\frac{1}{n}\right) \odot H_g^\flat\left(\bigvee_{i=0}^{n-1} T^{-i}(\mathcal{A})\right) \\ &= h_g^\flat(\mathcal{A}, T). \end{aligned}$$

□

Definition. Hudetz g-entropy $h_g^\flat(T)$ is defined by the formula

$$h_g^\flat(T) = \sup\{h_g^\flat(\mathcal{A}, T); \mathcal{A} \text{ is a } g\text{-fuzzy partition}\}.$$

Theorem 4. $h_g^\flat(T) = g^{-1}(h^\flat(T)$.

Proof. By Theorem 3

$$h_g^\flat(\mathcal{A}, T) = g^{-1}(h^\flat(g(\mathcal{A}), T) \le g^{-1}(h^\flat(T))$$

for any g-fuzzy partition $\mathcal{A}$. Therefore

$$h_g^\flat(T) = \sup\{h_g^\flat(\mathcal{A}, T); \mathcal{A}\} \le g^{-1}(h^\flat(T)).$$

Now let $\mathcal{B} = \{h_1, \dots, h_k\}$ be any fuzzy partition, i.e. $\sum_{i=1}^k h_i = 1$. Then $\mathcal{A} = \{g^{-1} \circ h_1, \dots, g^{-1} \circ h_k\}$ is a g-fuzzy partition, and $g(\mathcal{A}) = \mathcal{B}$. Therefore

$$h_g^\flat(\mathcal{A}, T) \le h_g^\flat(T).$$

But

$$h_g^\flat(\mathcal{A}, T) = g^{-1}(h^\flat(g(\mathcal{A}), T)) = g^{-1}(h^\flat(\mathcal{B}, T)).$$

We have obtained

$$h^\flat(\mathcal{B},T) = g(h_g^\flat(\mathcal{A},T)) \le g(h_g^\flat(T))$$

for any fuzzy partition $\mathcal{B}$. Therefore

$$h^\flat(T) = \sup h^\flat(\mathcal{B},T) \le g(h_g^\flat(T)).$$

Example. Let $\Omega = [0,1), \mathcal{S} = \mathcal{B}([0,1))$ be the σ-algebra of Borel subsets of $[0,1)$, $P = \lambda$ be the Lebesgue measure, $T : \Omega \to \Omega, T(x) = 2x (\mathrm{mod}\, 1)$, i.e. $T(x) = 2x$, if $x < 1/2$, $T(x) = 2x - 1$, if $x \ge 1/2$, $\mathcal{F}$ be the family of all $\mathcal{S}$-measurable functions $f : \Omega \to [0,1]$, $g(x) = x^2$. Let

$$\mathcal{A} = \{f_1, \dots, f_{k^2}\},$$

where $f_i = k^{-2}, i = 1, 2, \dots, k^2$. Then $h_g(\mathcal{A},T) = (\log k^2)^{1/2}$, whence

$$h_g(T) = \infty.$$

Put now $\mathcal{B} = \{\chi_{<0,1/2)}, \chi_{<1/2,1)}\}$. Then $\mathcal{B}$ is generating partition, whence

$$h^\flat(T) = h(\mathcal{B},T) = \log 2$$

by [10] Theorem 10.3.16. Now

$$h_g^\flat(T) = (\log 2)^{1/2}$$

by Theorem 4.

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