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A DETERMINISTIC LQ TRACKING PROBLEM: PARAMETRISATION OF THE CONTROLLER

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The article discusses an optimal Linear Quadratic (LQ) deterministic control problem when the Youla–Kučera parametrisation of controller is used. We provide a computational procedure for computing a deterministic optimal single-input single-output (SISO) controller from any stabilising controller. This approach allows us to calculate a new optimal LQ deterministic controller from a previous one when the plant has changed. The design based on the Youla–Kučera parametrisation approach is compared to the classical LQ design.

1. INTRODUCTION

Optimal control design, based on LQ performance criterion has been derived historically first in terms of the state space approach. By this method Riccati equations have to be solved (e. g. [5]). Progress in polynomial algebra and the algebraic polynomial approach to the synthesis of control loops presented e. g. by [3, 4], have offered new tools for the tracking LQ control problem. Algebraic methods have been well developed for a wide class of both deterministic and stochastic (LQG control) systems.

In [1], a non-conventional deterministic LQ tracking problem is discussed. This deterministic problem follows from some features of control of real technological processes. For the most part of theoretical works reference signal is assumed to be from a class of stochastic functions. However, in technological practice, references belong always to a class of deterministic functions. Moreover, practical needs of control show, that it is not always sufficient to restrict the output and control signals only. Very often, the control signal derivatives should be restricted as well. The solution of such a control problem represents then a non-conventional LQ problem. This paper introduces the non-conventional problem of optimal tracking based on minimisation of a modified quadratic performance criterion.

The aim of this paper is to present an alternative to the classical LQ tracking problem. It is based on the Youla–Kučera parametrisation approach. We provide a computational procedure for computing a deterministic optimal controller from any nominal (stabilising) controller. This approach allows us to calculate a new

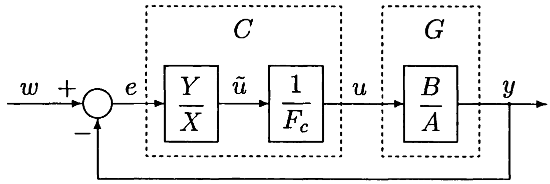

Fig. 1. Block diagram of the closed-loop system.

optimal LQ deterministic controller from a previous one when the plant has changed (supposing that the previous controller is stabilising for the new plant, too). The nominal controller is based on algebraic approach developed by Kučera. The control design is performed in input-output formulation leading to Diophantine and spectral factorisation equations.

1.1. Notation

All systems in this work are assumed to be SISO and continuous-time. The systems are described by means of fractions of polynomials in complex argument s, used in $\mathcal{L}$-transform. $\mathcal{RH}_\infty$ denote the set of stable proper rational transfer functions and $\mathcal{S}$ denote the set of stable polynomials.

For simplicity, the arguments of polynomials are omitted whenever possible – a polynomial $X(s)$ is denoted by X. We denote $X^*(s) = X(-s)$ for any function $X(s)$.

2. CLOSED–LOOP SYSTEM

2.1. System description

Consider the closed-loop system illustrated in Figure 1. A continuous-time linear time-invariant input-output representation of the plant to be controlled is considered

$$Ay = Bu \tag{1}$$

where y, u are process output and controller output, respectively. A and B are polynomials that describe the input-output properties of the plant.

We assume that the condition deg $B \leq$ deg A holds (i.e. transfer function of the plant is proper).

The reference w is considered to be from a class of functions expressed in the form

$$Fw = H \tag{2}$$

where H, F are coprime polynomials and $\deg H \leq \deg F$.

The controller is described by the equation

$$X\tilde{u} = Ye \tag{3}$$

where the pair X, Y are coprime polynomials and $X(0)$ is nonzero. The precompensator is described by the equation

$$F_c u = \tilde{u}. \tag{4}$$

Evidently, the precompensator is only the component of the feedback controller. We suppose here that AF_c and B are coprime polynomials.

Asymptotic tracking of the reference w is ensured for an arbitrary F just when F in (4) divides F_c. This claim will be fulfilled always for $F_c = F$. By substituting this relation to (4), this equation can be expressed in the form

$$Fu = \tilde{u}. \tag{5}$$

See [1] for details.

Remark 1. When considering the most common case of references – step changes then $H = 1, F = s$ in (2) and the precompensator is given as $1/s$. However, if the controlled plant has a pole $s = 0$ (on the stability boundary), then the precompensator can be removed. In general, the precompensator is not necessary if F divides A, which is unfortunately not true for the majority of the plants.

2.2. Nominal controller

Consider the nominal plant and the nominal controller transfer functions in the fractional representations

$$G = \frac{N_G}{D_G}, \quad C = \frac{N_C}{D_C}, \tag{6}$$

where

$$N_G = \frac{B}{M_1}, \quad D_G = \frac{A}{M_1} \tag{7}$$

$$N_C = \frac{Y}{M_2}, \quad D_C = \frac{FX}{M_2} \tag{8}$$

and M_1, $M_2 \in \mathcal{S}$ with degrees $\deg M_1 = \deg A$ and $\deg M_2 = \deg FX$, D_G, N_G, D_C and $N_C \in \mathcal{RH}_\infty$.

Stabilising nominal controllers are then given by solution of Diophantine equation

$$D_G D_C + N_G N_C = 1. \tag{9}$$

Substituting equations (7) and (8) into (9), the condition of stability in $\mathcal{S}$ takes the form

$$AFX + BY = M_1 M_2 = D \tag{10}$$

3. DETERMINISTIC NON-CONVENTIONAL LQ TRACKING PROBLEM

In this section two approaches to LQ tracking problem will be compared. The first one is more classical and it is based on the determination of optimal closed-loop poles that minimise the LQ cost function. The second approach follows more modern ideas of the Youla–Kučera parametrisation of all stable controllers.

The general conditions required to govern the control system properties are

- stability of the control system
- asymptotic tracking of the reference.

The goal of optimal deterministic LQ tracking is to design a controller that enables the control system to satisfy the above basic requirements and in addition the control law that minimises the cost function

$$J = \int_0^\infty \left(\varphi\tilde{u}^2(t) + \psi e^2(t)\right) dt \tag{11}$$

where $e = w - y$ denotes the control error and $\varphi > 0$, $\psi \geq 0$ are weighting coefficients. The cost function (11) can be rewritten using Parseval's theorem, to obtain an expression in the complex domain

$$J = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \left(\tilde{u}^*(s)\varphi\tilde{u}(s) + e^*(s)\psi e(s)\right) ds. \tag{12}$$

3.1. Classical LQ problem

Theorem 1. Define stable polynomials D_c and D_f resulting from spectral factorisations

$$D_c^* D_c = \varphi A^* F^* A F + \psi B^* B \tag{13}$$
$$D_f^* D_f = A^* A H^* H \tag{14}$$

then internal stability and solution of the deterministic LQ problem (11) is given by the controller polynomials X_c, Y_c calculated from a pair of Diophantine equations. The solution exists if AF and B have no unstable common factors and is unique.

The feedback part of the controller results as a solution of the coupled bilateral Diophantine equations:

$$\psi B^* D_f - AFV^* = D_c^* Y_c \tag{15}$$
$$\varphi A^* F^* D_f + BV^* = D_c^* X_c. \tag{16}$$

Proof. See [1]. □

Corollary 1. If polynomials AF and B are coprime then the pair of Diophantine equations (13), (14) is reduced to the implied Diophantine equation

$$AFX_c + BY_c = D_c D_f. \tag{17}$$

Proof. See [2, 3]. □

3.2. LQ problem: Youla–Kučera parametrisation

Let us now follow another approach. Suppose that a stabilising controller that gives rise to the closed-loop polynomial D (not necessarily LQ optimal or minimum degree) has been found and let us study the use of the Youla–Kučera parametrisation.

There are infinitely many solutions of (10) that stabilise the plant. The nominal solution (X, Y) will serve only as a starting point. It is possible to search among general solutions to minimise the cost (11). In our case, all such controllers (cf. Figure 2) are given by the following theorem:

Theorem 2. Let the nominal model plant $G = N_G/D_G$, with N_G and D_G coprime over $\mathcal{RH}_\infty$, be stabilised by a controller $C = N_C/D_C$, with N_C and D_C coprime over $\mathcal{RH}_\infty$. Then the set of all stabilising controllers for the plant G is given by

$$C(\bar{S}) = \frac{N_s}{D_s} = \frac{N_C + D_G\bar{S}}{D_C - N_G\bar{S}}, \tag{18}$$

where

$$\bar{S} \in \mathcal{RH}_\infty. \tag{19}$$

Proof. See [6]. □

Corollary 2. Let the nominal model plant $G = N_G/D_G = B/A$, with N_G, D_G, B and A defined by (7), be stabilised by a controller $C = N_C/D_C = Y/FX$, with N_C, D_C, Y and FX defined by (8). Then the set of all stabilising controllers for the plant G is given by

$$C(\bar{S}) \equiv C(S) = \frac{Y_s}{FX_s} = \frac{Y_m + A_mFS}{FX_m - B_mFS} = \frac{Y_m + A_mFS}{X_m - B_mS}\frac{1}{F}, \tag{20}$$

where

$$\bar{S} = FS \in \mathcal{RH}_\infty,\ A_m = AM_2,\ B_m = BM_2,\ X_m = XM_1 \text{ and } Y_m = YM_1. \tag{21}$$

Remark 2. Asymptotic tracking can be assured only if the denominator in (20) is divisible by F. Therefore, $\bar{S} = FS$ is chosen. The term $1/F$ represents the precompensator that forms a part of the controller. From aspects of some following procedures one may be formally separated from the controller.

We now present a solution to the deterministic LQ controller design problem in the Youla–Kučera parametrisation framework starting from the plant model B/A and any stabilising controller Y/FX, using the set of all stabilising controllers for the plant, i. e. we show how to compute optimal $\bar{S}$ that minimises (11).

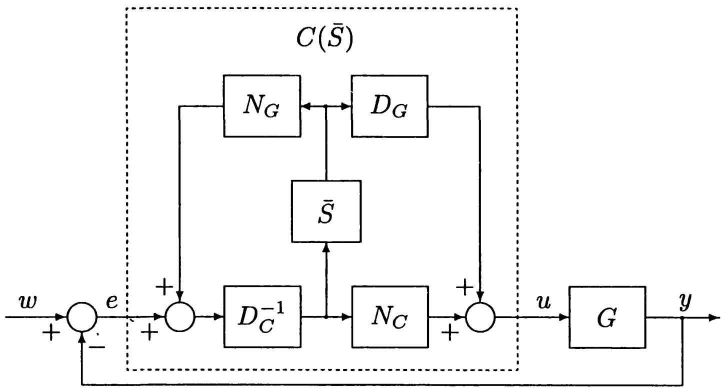

Fig. 2. Block diagram of the closed-loop system.

Theorem 3. Consider the minimisation of the cost function (11) with respect to the Youla–Kučera parameter $\bar{S}$ that is specified as a transfer function. Solve spectral factorisation equations (13), (14) for stable D_c and D_f and the bilateral Diophantine equation with unknown S_n, V^*

$$\psi D_f B^* X - \varphi D_f A^* F^* Y = S_n D_c^* + V^* D. \tag{22}$$

The optimal Youla–Kučera parameter is then given as

$$\bar{S} = FS, \quad S = \frac{S_n}{D_c D_f} \frac{M_1}{M_2} \in \mathcal{RH}_\infty. \tag{23}$$

Since D_c, D_f, M_2 are stable, it follows that S is a stable transfer function and fulfills the condition from the Youla–Kučera parametrisation.

Proof. To begin the proof, the two signals ($\tilde{u}$, e) used in the cost function (11) are derived using the equations (1), (3), (5), and (20) describing the closed-loop system (so that the desired signals are functions of only the external signal w)

$$\tilde{u} = \frac{Y_m + A_m F S}{M_1(AFX + BY)} AFw = \frac{Y_m + A_m F S}{M_1 D} AH \tag{24}$$

$$e = \frac{X_m - B_m S}{M_1(AFX + BY)} AFw = \frac{X_m - B_m S}{M_1 D} AH. \tag{25}$$

Minimising equation (11) with respect to all stable S represents minimising the following cost function in complex domain

$$J = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \left(\tilde{u}^*(s)\varphi\tilde{u}(s) + e^*(s)\psi e(s)\right) \mathrm{d}s = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \left(\varphi S_{\tilde{u}} + \psi S_e\right) \mathrm{d}s \tag{26}$$

where $S_{\tilde{u}}$ and S_e are spectral functions of the form

$$\begin{aligned} S_{\tilde{u}} &= \tilde{u}^*\tilde{u} = \left(\frac{Y_m + A_mFS}{M_1D}AH\right)^* \left(\frac{Y_m + A_mFS}{M_1D}AH\right) \\ &= S^*SA^*AH^*H\frac{A_m^*A_mF^*F}{M_1^*D^*M_1D} + S^*A^*AH^*H\frac{A_m^*F^*Y_m}{M_1^*D^*M_1D} \\ &+SA^*AH^*H\frac{A_mFY_m^*}{M_1^*D^*M_1D} + A^*AH^*H\frac{Y_m^*Y_m}{M_1^*D^*M_1D} \end{aligned} \quad (27)$$

$$\begin{aligned} S_e &= e^*e = \left(\frac{X_m - B_mS}{M_1D}AH\right)^* \left(\frac{X_m - B_mS}{M_1D}AH\right) \\ &= S^*SA^*AH^*H\frac{B_m^*B_m}{M_1^*D^*M_1D} - S^*A^*AH^*H\frac{B_m^*X_m}{M_1^*D^*M_1D} \\ &-SA^*AH^*H\frac{B_mX_m^*}{M_1^*D^*M_1D} + A^*AH^*H\frac{X_m^*X_m}{M_1^*D^*M_1D}. \end{aligned} \quad (28)$$

The direct minimisation of the cost function (26) with respect to a polynomial is a difficult task. Therefore, we complete the terms to squares.

$$\begin{aligned} &\varphi S_{\tilde{u}} + \psi S_e = \\ = \; & S^*S\frac{A^*AH^*H}{M_1^*D^*M_1D}\left(\varphi A_m^*A_mF^*F + \psi B_m^*B_m\right) \\ &+S^*\frac{A^*AH^*H}{M_1^*D^*M_1D}\left(\varphi A_m^*F^*Y_m - \psi B_m^*X_m\right) \\ &+S\frac{A^*AH^*H}{M_1^*D^*M_1D}\left(\varphi A_mFY_m^* - \psi B_mX_m^*\right) + \frac{A^*AH^*H}{M_1^*D^*M_1D}\left(\varphi Y_m^*Y_m + \psi X_m^*X_m\right) \\ = \; & S^*S\frac{A^*AH^*H}{M_1^*M_1^*M_1M_1}\left(\varphi A^*AF^*F + \psi B^*B\right) + S^*\frac{A^*AH^*H}{M_1^*M_1^*D}\left(\varphi A^*F^*Y - \psi B^*X\right) \\ &+S\frac{A^*AH^*H}{M_1M_1D^*}\left(\varphi AFY^* - \psi BX^*\right) + \frac{A^*AH^*H}{D^*D}\left(\varphi Y^*Y + \psi X^*X\right). \end{aligned} \quad (29)$$

Let us now consider the term (29) and its first part containing S^*S

$$\begin{aligned} S_1 &= \frac{A^*AH^*H}{M_1^*M_1^*M_1M_1}\left(\varphi A^*AF^*F + \psi B^*B\right) \\ &= \frac{D_f^*D_f}{M_1^*M_1^*M_1M_1}\left(\varphi A^*F^*AF + \psi B^*B\right) = \left(\frac{D_fD_c}{M_1M_1}\right)^*\left(\frac{D_fD_c}{M_1M_1}\right) \end{aligned}$$

where the stable polynomials D_c, D_f are defined from two spectral factorisation equations (13) and (14). The completing the squares approach gives thus

$$\begin{aligned} \varphi S_{\tilde{u}} + \psi S_e &= \left(\frac{D_fD_c}{M_1M_1}S + \frac{\varphi D_fA^*F^*Y}{DD_c^*} - \frac{\psi D_fB^*X}{DD_c^*}\right)^* \\ &\times \left(\frac{D_fD_c}{M_1M_1}S + \frac{\varphi D_fA^*F^*Y}{DD_c^*} - \frac{\psi D_fB^*X}{DD_c^*}\right) + y_d \end{aligned} \quad (30)$$

where y_d is the rest that is independent of S.

For reaching the minimum value of J, clearly the way is to put the first term equal to zero. From this equation, the optimal S can be determined. However, the simple putting the brace equal to zero would not do the job: the resulting S would not be stable. Therefore, we manipulate the second and third terms in brackets. These can be separated in

$$\frac{\psi D_f B^* X}{DD_c^*} - \frac{\varphi D_f A^* F^* Y}{DD_c^*} = \frac{S_n}{D} + \frac{V^*}{D_c^*}. \tag{31}$$

The first term (S_n/D) is stable and the second one (V^*/D_c^*) is unstable. Because the second term is unstable and S is required be stable it vanishes in the cost. The brace now reads

$$\left(\frac{D_f D_c}{M_1 M_1} S - \frac{S_n}{D}\right) \equiv \left(\frac{D_f D_c}{M_1 M_1} S - \frac{S_n}{M_1 M_2}\right). \tag{32}$$

Setting it to zero, S now reads

$$S = \frac{S_n}{D_c D_f} \cdot \frac{M_1}{M_2}. \tag{33}$$

Because the denominator is stable, so is S as well. □

Comparison of two approaches to LQ tracking problem is summarised by the following corollary.

Corollary 3. If the classical LQ controller (X_c, Y_c) is obtained from (15), (16) and the parametrised LQ controller (X_s, Y_s) is obtained from (20), (33) with stable polynomials D_c and D_f calculated from (13) and (14), then transfer functions of these controllers are identical.

Proof. It is not difficult to check that

$$\frac{Y_c}{FX_c} = \frac{Y_s}{FX_s}. \tag{34}$$

Transfer function of the classical controller (Y_c, X_c) can be obtained from equations (15) and (16)

$$\frac{Y_c}{FX_c} = \frac{\psi B^* D_f - AFV^*}{F(\varphi A^* F^* D_f + BV^*)}.$$

Using equations (31) and (33), we can rewrite the Youla–Kučera parameter S as follows

$$S = \frac{S_n}{D_c D_f} \cdot \frac{M_1}{M_2} = \frac{\psi D_f B^* X - \varphi D_f A^* F^* Y - DV^*}{D_c^* D_c D_f} \cdot \frac{M_1}{M_2}. \tag{35}$$

Putting (35) into (20) we have

$$\frac{Y_s}{FX_s} = \frac{YM_1 + AM_2 FS}{FXM_1 - BM_2 FS} = \frac{\psi B^* D_f - AFV^*}{F(\varphi A^* F^* D_f + BV^*)}$$ □

4. ILLUSTRATIVE EXAMPLE

In this section, an example is presented to to show all steps of the calculation in both cases of LQ design. Let us consider the controlled system described by the following transfer function

$$G(s) = \frac{B(s)}{A(s)} = \frac{3}{5s+1}.$$

The reference has been chosen as step change $w(t) = 1(t)$. The weighting coefficients φ and ψ in the cost function (11) have been selected as $\varphi = 0.7$, $\psi = 0.8$. Both stable polynomials $D_c(s)$ and $D_f(s)$ obtained from spectral factorisations (13), (14) are of the form

$$\begin{aligned} D_c(s) &= d_{c2}s^2 + d_{c1}s + d_{c0} \\ D_f(s) &= d_{f1}s + d_{f0} \end{aligned}$$

and their coefficients are given as

$$\begin{aligned} d_{c0} &= \sqrt{\psi b_0^2}\;; \quad d_{c2} = \sqrt{\varphi a_1^2} \\ d_{c1} &= \sqrt{\varphi + 2d_{c2}d_{c0}} \\ d_{f1} &= |a_1|\;; \quad d_{f0} = 1. \end{aligned}$$

The resulting degrees of both polynomials of the controller transfer function are $\deg Y_c(s) = \deg X_c(s) = 1$. Their coefficients have been calculated from the polynomial equation (17). The proper transfer function of the feedback classical LQ controller (with precompensator) is given as

$$C_c(s) = \frac{Y_c(s)}{F(s)X_c(s)} = \frac{4.472s + 0.894}{4.183s^2 + 4.811s}$$

For the Youla–Kučera parametrised LQ controller a nominal controller that stabilises the closed-loop is chosen as

$$C(s) = \frac{Y(s)}{F(s)X(s)} = \frac{0.8133s + 0.3333}{0.2s^2 + 0.56s}$$

and yields the closed-loop pole polynomial of the form

$$D(s) = M_1(s)M_2(s)$$

where $M_1(s) = (1+s)$ and $M_2(s) = (1+s)^2$.

The polynomial $S_n(s)$ is calculated from (22). This gives the optimal Youla–Kučera transfer function $S(s)$ as

$$S(s) = \frac{S_n(s)}{D_f(s)D_c(s)} \cdot \frac{M_1(s)}{M_2(s)} = \frac{-2.508s^2 - 2.624s - 1.103}{20.92s^3 + 28.24s^2 + 18.23s + 2.683} \cdot \frac{M_1(s)}{M_2(s)}.$$

Finally, calculation of the LQ controller $C_s(s)$ yields

$$C_s(s) = \frac{Y_s(s)}{F(s)X_s(s)} = \frac{Y(s)M_1(s) + A(s)M_2(s)F(s)S(s)}{F(s)X(s)M_1(s) - B(s)M_2(s)F(s)S(s)}$$

with the same controller polynomials as in the first case.

5. CONCLUSIONS

In this paper, we have presented a procedure to compute deterministic LQ controller from a stabilising controller using the Youla–Kučera parametrisation. The presented controller design procedure ensures stability of the controlled system and asymptotic tracking of the references most commonly used in practice. Two approaches have been compared. The same result has been obtained in both cases. The proposed approach can be applied in adaptive control framework.

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