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ON THE STABILIZABILITY OF SOME CLASSES OF BILINEAR SYSTEMS IN $\mathbb{R}^3$

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In this paper, we consider some classes of bilinear systems. We give sufficient condition for the asymptotic stabilization by using a positive and a negative feedbacks.

1. INTRODUCTION

Stabilizability of bilinear systems of the form

$$\dot{x} = Ax + uBx \tag{1}$$

(where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and A, B are constant real matrices $(n \times n)$) has widely studied in the past years by many authors (see e.g. [1 – 13]). In [4], the authors give a necessary and sufficient condition, algebraically computable, for the global stabilization of the planar bilinear systems

$$\begin{cases} \dot{z} = \tilde{A}z + v\tilde{B}z \\ z \in \mathbb{R}^2, \quad v \in \mathbb{R} \quad \text{and} \quad \tilde{A}, \quad \tilde{B} \in M(2, \mathbb{R}). \end{cases} \tag{2}$$

It turns out that the stabilizability by homogeneous feedback is equivalent to the asymptotic controllability to the origin which is equivalent to the stabilizability. Moreover, they show that there exists a large class of planar bilinear systems that are not C^1 stabilizable but stabilizable by means of homogeneous feedback of the form $v(z) = \frac{Q_1(z)}{Q_2(z)}$, where Q_1 is a quadratic form and Q_2 is a positive-definite quadratic form.

For the three dimensional case, in [3] the authors deal with a particular class of bilinear systems of form (1) with A diagonal and B skew symmetric. For these systems a necessary and sufficient condition for global asymptotic stabilization by constant feedback and a sufficient condition for stabilization by a family of linear feedbacks are given. Another interesting problem is considered in the literature. The question is: does the local asymptotic stabilizability of (1) imply the global asymptotic stabilizability? More precisely let us assume that there exists a feedback law (locally defined) $u : x \mapsto u(x)$ such that the closed system $\dot{x} = Ax + u(x)\,Bx$ (Σ)

is locally asymptotically stable about the origin. Does there exist a feedback law (globally defined) $\bar{u}(x)$ which makes the origin of (1) globally asymptotically stable?

To the closed-loop system (Σ) a positive-definite function V (locally defined) is associated, such that $\dot{V}(x)$ (the derivative of V along the trajectories of system (Σ)) is negative definite.

Hammouri and Marques [8], proved that local asymptotic stabilizability implies global asymptotic stabilizability under some assumption on the level surfaces of the Lyapunov function related to system (Σ). Andriano [1] assert, that the answer to the above question is yes without any assumption on the level surfaces of the Lyapunov function. In [5] the authors clarify the result of Andriano given in [1].

This work is a contribution to the study of stabilization of bilinear systems by homogeneous feedback. The results concern single-input bilinear systems of the form

$$\dot{x} = Ax + uBx \tag{3}$$

where $x \in \mathbb{R}^3$, $u \in \mathbb{R}$ and ${}^TA$, ${}^TB$ two matrices supposed have a same eigenvector (${}^TA$ denotes the transpose of matrix A). In a suitable basis matrices A and B can be written as

$$A = \begin{pmatrix} a_{(1.1)} & 0 & 0 \\ a_{(2.1)} & a_{(2.2)} & a_{(2.3)} \\ a_{(3.1)} & a_{(3.2)} & a_{(3.3)} \end{pmatrix}, \quad B = \begin{pmatrix} b_{(1.1)} & 0 & 0 \\ b_{(2.1)} & b_{(2.2)} & b_{(2.3)} \\ b_{(3.1)} & b_{(3.2)} & b_{(3.3)} \end{pmatrix}.$$

We define matrices $\tilde{A}$ and $\tilde{B}$ as

$$\tilde{A} = \begin{pmatrix} a_{(2.2)} & a_{(2.3)} \\ a_{(2.3)} & a_{(3.3)} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \tilde{B} = \begin{pmatrix} b_{(2.2)} & b_{(2.3)} \\ b_{(2.3)} & b_{(3.3)} \end{pmatrix}.$$

We suppose more that system (2) is not stabilizable by a constant feedback and $\tilde{B}$ is not diagonalizable.

In this paper we show how to compute the homogeneous feedback of the system (3) when the planar bilinear system (2) is stabilizable by a positive and negative feedback.

The paper is organized as follows. In Section 2, for the convenience of the reader we recall two results of constant use in the sequel.

Section 3: In the case where the eigenvalues of B associate to the common eigenvector of ${}^TA$ and ${}^TB$ is zero we give a necessary and sufficient condition for the stabilizability of system (3), the feedback is given explicitly. Next we prove that if the planar bilinear system is globally asymptotically stable (GAS) by a feedback $v(z)$ such that $b_{1.1}v(z) < 0$ then system (2) is GAS.

In Section 4, we suppose that the system (2) is not stabilizable by a constant feedback and $\tilde{B}$ is not diagonalizable. As an application of the last result of the Section 2, we give a necessary and sufficient condition, algebraically computable, for the global stabilization of the planar bilinear systems by a positive and negative feedback.

2. TWO RESULTS ON STABILIZATION

We recall the following theorem, because we need these results to prove that we can stabilize some bilinear system by a positive and negative feedback (see Theorems 5, 6, 7 and 8).

Theorem 1. Consider the two-dimensional system,

$$^T[\dot{z}_1, \dot{z}_2] = ^T[f_1(z_1, z_2), f_2(z_1, z_2)]$$

where $^T[f_1, f_2]$ is Lipschitz continuous and is homogeneous of degree p. Then the system is asymptotically stable if and only if one of the following is satisfied:

(i) The system does not have any one-dimensional invariant subspaces and

$$\begin{aligned} I &= \int_0^{2\pi} \frac{\cos\theta f_1(\cos\theta, \sin\theta) + \sin\theta f_2(\cos\theta, \sin\theta)}{\cos\theta f_2(\cos\theta, \sin\theta) - \sin\theta f_1(\cos\theta, \sin\theta)} \, \mathrm{d}\theta \\ &= \int_{-\infty}^{+\infty} \frac{f_1(1, s)}{f_2(1, s) - s f_1(1, s)} \, \mathrm{d}s < 0 \end{aligned}$$

or

(ii) The restriction of the system to each of its one-dimensional invariant subspaces is asymptotically stable.

For a proof see [2, 7].

In the sequel we use constantly a result of asymptotic stability using positive-semi-definite function. The theorem can be found in [9] or [10], we use the formulation of [10]. Consider the differential equation

$$(\Gamma) \begin{cases} \dot{x} = X(x) \\ X(0) = 0 \end{cases}$$

where X is a smooth vector field on $I\!R^n$. For a differentiable function V, we denote the action of X, considered as a differential operator, on V by XV, which is defined by

$$XV(x) = \frac{\mathrm{d}}{\mathrm{d}t} V(X_t(x))_{t=0}$$

$X_t(x_0)$ is the solution of (Γ) starting at x_0, i.e. $\frac{\mathrm{d}}{\mathrm{d}t} X_t(x_0) = X(X_t(x_0))$ and $X_0(x_0) = x_0$.

Theorem 2. We suppose that there exists a function $V \in C^1(\mathbb{R}^n, \mathbb{R})$ such that

(1) $V(x) \geq 0$ for all $x \in \mathbb{R}^n$ and $V(0) = 0$

(2) $\dot{V}(x) = XV(x) \leq 0$.

We denote by $\mathcal{L}$ the largest positively invariant set of X contained in $M = \{x \in \mathbb{R}^n : \dot{V}(x) = 0\}$

If the origin is asymptotically stable with respect to the system (Γ) restricted to $\mathcal{L}$, then the origin is asymptotically stable.

3. MAIN RESULT

Consider a single input bilinear system (3), we suppose that the ${}^TB$ and ${}^TA$ have a same eigenvector
we recall that, in a suitable basis of $\mathbb{R}^3$, the matrices A, B take the following forms

$$A = \begin{pmatrix} a_{(1.1)} & 0 & 0 \\ a_{(2.1)} & a_{(2.2)} & a_{(2.3)} \\ a_{(3.1)} & a_{(3.2)} & a_{(3.3)} \end{pmatrix}, \quad B = \begin{pmatrix} b_{(1.1)} & 0 & 0 \\ b_{(2.1)} & b_{(2.2)} & b_{(2.3)} \\ b_{(3.1)} & b_{(3.2)} & b_{(3.3)} \end{pmatrix}$$

and $\tilde{A}$, $\tilde{B}$ as follows

$$\tilde{A} = \begin{pmatrix} a_{(2.2)} & a_{(2.3)} \\ a_{(2.3)} & a_{(3.3)} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \tilde{B} = \begin{pmatrix} b_{(2.2)} & b_{(2.3)} \\ b_{(2.3)} & b_{(3.3)} \end{pmatrix}.$$

For the sake of clarity, we set $x = (x_1, x_2, x_3) = (x_1, z)$ where $z = (x_2, x_3)$. We denotes $a_{(1.1)} = \alpha$ and $\beta = b_{(1.1)}$

Theorem 3. In the case when $\beta = 0$ we can assume that:

The bilinear system (3) is GAS if and only if $\alpha < 0$ and the planar bilinear system $\dot{z} = \tilde{A}z + v\tilde{B}z$, (2) is GAS.

If system (2) is GAS by the feedback law $v(z) = \frac{Q_1(z)}{Q_2(z)}$ (where Q_1 is a quadratic form and Q_2 is a positive-definite quadratic form), then the feedback $u(x) = \frac{Q_1(z)}{Q_2(z)+x_1^2}$ stabilizes the system (3).

Proof. The system (3) takes the form

$$\begin{cases} \dot{x}_1 & = \alpha x_1 \\ \begin{pmatrix} \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} & = x_1 \begin{pmatrix} a_{(2.1)} \\ a_{(3.1)} \end{pmatrix} + u x_1 \begin{pmatrix} b_{(2.1)} \\ b_{(3.1)} \end{pmatrix} + \tilde{A} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} + u\tilde{B} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}. \end{cases} \tag{4}$$

It is clear that it is necessary for the stabilizability of (4) that $\alpha < 0$ and the system $\dot{z} = \tilde{A}z + v\tilde{B}z$ is stabilizable.

We suppose now that these two conditions are satisfied. Let $v(z) = \frac{Q_1(z)}{Q_2(z)}$ be a stabilizing homogeneous feedback for $\dot{z} = \tilde{A}z + v\tilde{B}z$, where Q_1 is a quadratic form and Q_2 is a positive-definite quadratic form. We define $u(x_1,x_2,x_3) = \frac{Q_1(z)}{Q_2(z)+x_1^2}$ on $I\!R^3$. This feedback is homogeneous and C^∞ in $I\!R^3 \setminus \{0\}$. We denote by $X(x) = A(x)+u(x)\,B(x)$ the vectors field of the closed-loop system. We shall prove, by using Theorem 2, that this system is asymptotically stable. It is clear that the vectors field X and Y where $Y(x) = (Q_2(z)+\|x_1\|^2)\,X(x)$ have the same orbits. We choose $V(x) = x_1^2$, V is clearly positive-semi-definite on $I\!R^3$. Since $\alpha < 0$, then

$$YV(x) = \dot{V}(x) = \alpha(Q_2(z) + \|x_1\|^2)\|x_1\|^2 \leq 0.$$

Let M be the set $M = \{x \in I\!R^3 : \dot{V}(x) = 0\}$. It is clear that M is $\{0\} \times I\!R^2$ in $I\!R^3$. Then the vectors field Y is reduced on M to $\dot{z} = Q_2(z)(\tilde{A}z + u(0,z)\tilde{B}z$.

Since $Q_2(z)$ is positive-definite and $\dot{z} = \tilde{A}z + v(z)\tilde{B}z$ is asymptotically stable, then Y and hence X, is asymptotically stable.

In the case when $\beta \neq 0$ and without loss of generality we can suppose that $a_{(1.1)} = \alpha < 0$. □

Theorem 4. If the planar bilinear system $\dot{z} = \tilde{A}z + v\tilde{B}z$ (2) is globally asymptotically stabilizable by a feedback law of the form $v(z) = \frac{Q_1(z)}{Q_2(z)}$ such that $\beta v(z) \leq 0$ $\forall\, z \in I\!R^2 - \{(0,0)\}$ then the feedback $u(x) = \frac{Q_1(z)}{Q_2(z)+x_1^2}$ stabilizes the system (3).

Proof. It is straightforward that the the closed-loop system (3) with the feedback $u(x) = \frac{Q_1(z)}{Q_2(z)+x_1^2}$

$$\begin{cases} \dot{x}_1 = \alpha x_1 + u\beta x_1 \\ \begin{pmatrix} \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = x_1 \begin{pmatrix} a_{(2.1)} \\ a_{(3.1)} \end{pmatrix} + ux_1 \begin{pmatrix} b_{(2.1)} \\ b_{(3.1)} \end{pmatrix} + \tilde{A}\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} + u\tilde{B}\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \end{cases} \qquad (5)$$

is GAS. □

The proof is organized as the proof of the preceding theorem. Since this system is in triangular forms (see [10]), and $\dot{z} = \tilde{A}z + u(0,z)\tilde{B}z$ is GAS, then the equation (5) is GAS.

4. STABILIZABILITY OF PLANAR BILINEAR SYSTEM BY A POSITIVE AND NEGATIVE FEEDBACK

In this section we consider the planar bilinear system $\dot{z} = \tilde{A}z + v\tilde{B}z$ (2) which it not be stabilizable by a constant feedback. We suppose that matrix $\tilde{B}$ is not diagonalizable.

As an application of Theorem 4, in this section we construct a positive and negative feedbacks who's stabilize the planar bilinear systems (2).

4.1. $\tilde{B}$ have no real eigenvalues

In a suitable basis of $I\!R^2$, the matrix $\tilde{B}$ takes the following form

$$\tilde{B} = \begin{pmatrix} \nu & \mu \\ -\mu & \nu \end{pmatrix}.$$

For the sake of clarity, we set $z = (z_1, z_2)$, $\tilde{z}_1 = e_1 z_1 + e_2 z_2$ and $\tilde{z}_2 = -e_2 z_1 + e_1 z_2$, where $e_1 = (a - d) - \sqrt{(b + c)^2 + (a - d)^2}$ and $e_2 = (b + c)$.

According to the assumption that system (2) is not stabilizable by a constant feedback, it is the classification of planar bilinear systems we are speaking of [4], we have

(i) $\mathrm{Tr}(\tilde{A}) \geq 0$, $\mathrm{Tr}(\tilde{B}) = 0$ and $(b + c)^2 - 4ad > 0$.

Theorem 5. If the condition (i) is satisfied then for $t_1 > 0$ and $t_2 > 0$ large enough and $\tilde{d}\sqrt{\frac{t_1}{t_2}} + \tilde{a} > 0$ the positive feedback law

$$v_1(z_1, z_2) = \frac{t_1 \tilde{z}_1^2 + (\tilde{d} - \tilde{a})\tilde{z}_1\tilde{z}_2 + t_2\tilde{z}_2^2}{\mu(\tilde{z}_1^2 + \tilde{z}_2^2)} + \frac{c - b}{2\mu} \quad \text{and the negative feedback law}$$

$$v_2(z_1, z_2) = -\frac{t_1 \tilde{z}_1^2 + (\tilde{a} - \tilde{d})\tilde{z}_1\tilde{z}_2 + t_2\tilde{z}_2^2}{\mu(\tilde{z}_1^2 + \tilde{z}_2^2)} + \frac{c - b}{2\mu} \quad \text{where}$$

$$\tilde{a} = (ae_1^2 + (c + b)e_1e_2 + de_2^2)/(e_1^2 + e_2^2) \quad \text{and}$$

$$\tilde{d} = (ae_2^2 - (c + b)e_1e_2 + de_1^2)/(e_1^2 + e_2^2)$$

stabilize the system (2).

Proof. We consider the closed-loop system (2) by the feedback $v_1(z_1, z_2)$

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} Z_1(z_1, z_2) \\ Z_2(z_1, z_2) \end{pmatrix} = (\tilde{z}_1^2 + \tilde{z}_2^2)\left[\tilde{A}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + v_1(z_1, z_2)\tilde{B}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right].$$

Since the function $(\tilde{z}_1^2 + \tilde{z}_2^2)$ is positive-definite then there is equivalence between asymptotic stability of the vector field $Z = (Z_1, Z_2)$ and the closed-loop bilinear system (2), defining the function F as follows

$$F(z_1, z_2) = z_1 Z_2(z_1, z_2) - z_2 Z_1(z_1, z_2)$$

a simple computation gives

$$F(z_1, z_2) = -(t_1\tilde{z}_1^2 + t_2\tilde{z}_2^2)(\tilde{z}_1^2 + \tilde{z}_2^2).$$

From Theorem 1 one can deduce that the vector field Z is GAS if and only if

$$I = \int_{-\infty}^{+\infty} \frac{Z_1(1, s)}{F(1, s)}\, \mathrm{d}s < 0.$$

It is easy to verify that

$$\tilde{a}\tilde{d} < 0 \qquad I = -\frac{\pi}{t_2}\frac{\tilde{d}t + \tilde{a}}{t(t+1)} \quad \text{where} \quad t = \sqrt{\frac{t_1}{t_2}}.$$

Since t_1 and t_2 have been chosen large enough such that $v_1(z_1, z_2) > 0$ and $\tilde{d}t + \tilde{a} > 0$, then $I < 0$.

From the fact that the vector field Z is GAS, we can assume that the differential equation

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{pmatrix} \tag{6}$$

$$= \left[\begin{pmatrix} a & -b \\ -c & d \end{pmatrix} - \left(\frac{t_1\bar{\xi}_1^2 + (\tilde{d} - \tilde{a})\bar{\xi}_1\bar{\xi}_2 + t_2\bar{\xi}_2^2}{\mu(\bar{\xi}_1^2 + \bar{\xi}_2^2)} + \frac{b-c}{2\mu} \right) \begin{pmatrix} 0 & -\mu \\ \mu & 0 \end{pmatrix} \right] \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

$\bar{\xi}_1 = e_1\xi_1 - e_2\xi_2$ and $\bar{\xi}_2 = e_2\xi_1 + e_1\xi_2$ is GAS. Under a linear change in the state space of the form $z_1 = \xi_1$ and $z_2 = -\xi_2$ the differential equation (6) becomes $\dot{z} = \tilde{A}z + v_2(z_1, z_2)\tilde{B}z$ which it GAS. □

4.2. Case where the eigenvalues of B are real without B being diagonalizable

In a suitable basis of $I\!R^2$, matrices $\tilde{A}$ and $\tilde{B}$ take the following forms

$$\tilde{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \tilde{B} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

According to the assumption that the system (2) is not stabilizable by a constant feedback, it is the classification of planar bilinear systems we are speaking of [4], we have

(i) $\mathrm{Tr}(\tilde{A}) > 0$, $\mathrm{Tr}(\tilde{B}) = 0$ and $\mathrm{Tr}(\tilde{A}\tilde{B}) = c \neq 0$;

(ii) $\mathrm{Tr}(\tilde{A}) = \mathrm{Tr}(\tilde{B}) = 0$ and $\mathrm{Tr}(\tilde{A}\tilde{B}) = c \neq 0$.

Without loss of generality, we can suppose that $\mathrm{Tr}(\tilde{A}\tilde{B}) = c > 0$.

4.2.1. Case when $\mathrm{Tr}(\tilde{A}) > 0$

We will treated separately the two subcases: $4bc + (a-d)^2 < 0$ and $4bc + (a-d)^2 \geq 0$.

The sub case when $4bc + (a-d)^2 \geq 0$.

Theorem 6. If the condition (i) is satisfied then the negative feedback

$$v(z) = -b - \frac{(a-d)^2}{4c} - \frac{(d+a)^2 P(z)}{4cQ(z)}$$

where

$$Q(z) = (cz_1 - (d+2a)z_2)^2 + \tfrac{29}{4}(d+a)^2 z_2^2 \quad \text{and}$$
$$P(z) = 26c^2 z_1^2 + (88ac + 140cd)z_1 z_2 + \left(\tfrac{849}{2}d^2 + 709ad + \tfrac{621}{2}a^2\right) z_2^2$$

stabilizes the system (2).

Proof. Suppose that, the condition (i) is satisfied. We consider the Lyapunov function,

$$\begin{aligned} V(z) &= \left(c^2 z_1^2 - \frac{dc+5ac}{2} z_1 z_2 - (\frac{17}{2}a^2 + \frac{39}{2}da + 10d^2) z_2^2\right)^2 \\ &\quad + \frac{133}{4}\left(\frac{d^2-a^2}{2} z_2^2 + c(d+a) z_1 z_2\right)^2 \end{aligned}$$

for the system (2), and the feedback law

$$v(z) = -b - \frac{(a-d)^2}{4c} - \frac{(d+a)^2 P(z)}{4cQ(z)}.$$

One can verify that, V is positive-definite and the feedback law is homogeneous of degree zero. A simple computation gives

$$\dot{V}(z) = \frac{-(a+d)D(z)R(z)}{Q(z)} < 0 \quad \forall z \neq 0$$

where $R(z) = \left(cz_1 + \frac{d-a}{2} z_2\right)^2 + \frac{19}{2}(a+d)^2 z_2^2$, and

$$\begin{aligned} D(z) &= \left(c^2 z_1^2 - (cd+3ac)z_1 z_2 - \left(\frac{33}{4}a^2 + \frac{39}{2}ad + \frac{41}{4}d^2\right) z_2^2\right)^2 \\ &\quad +33\left(\frac{d^2-a^2}{2} z_2^2 + c(d+a) z_1 z_2\right)^2 . \end{aligned}$$

This prove that the feedback $v(z)$ stabilizes the system (2).

Since $P(z) - Q(z) = 25\left(cz_1 + \frac{(71d+46a)}{25} z_2\right)^2 + \frac{21461}{100}(a+d)^2 z_2^2$, is positive-definite then

$$\frac{P(z)}{Q(z)} > 1, \ \forall z \neq 0.$$

Tacking into account the fact that $4bc + (a-d)^2 \geq 0$ and $\frac{P(z)}{Q(z)} > 1$ then

$$v(z) < -\frac{(d+a)^2}{4c} - b - \frac{(a-d)^2}{4c} < 0. \qquad \square$$

Proposition 1. The system (2) is not stabilizable by a positive feedback.

Proof. Consider the linear change of coordinates whose transformation matrix is given by

$$P = \begin{pmatrix} 1 & \frac{d-a}{2c} \\ 0 & 1 \end{pmatrix}.$$

The matrix $\tilde{B}$ keep its initial form and the matrix $\tilde{A}$ becomes

$$\tilde{A} = \begin{pmatrix} \frac{a+d}{2} & b + \frac{(a-d)^2}{4c} \\ c & \frac{a+d}{2} \end{pmatrix}.$$

In the new basis it is easy to verify that the set $H = \{(z_1, z_2) \in \mathbb{R}^2$ such that $z_1 \geq 0$ $z_2 = 2\}$ is invariant by the open loop system $\dot{z} = \tilde{A}z + v\tilde{B}z$ where v lie in $\mathbb{R}^+$. □

The sub case where $4bc + (a-d)^2 < 0$. Under a change in input state of the form $\tilde{v} = (\frac{2c}{\sqrt{-4bc-(a-d)^2}})v$ and if we consider the linear change of coordinates whose transformation matrix is given by

$$P = \begin{pmatrix} 1 & \frac{d-a}{2c} \\ 0 & \frac{\sqrt{-4bc-(a-d)^2}}{2c} \end{pmatrix}.$$

The matrix $\tilde{B}$ keep its initial form and the matrix $\tilde{A}$ becomes

$$\tilde{A} = \begin{pmatrix} \frac{a+d}{2} & -\tilde{c} \\ \tilde{c} & \frac{a+d}{2} \end{pmatrix}$$

where $\tilde{c} = \frac{\sqrt{-4bc-(a-d)^2}}{2}$.

In the new basis, we prove the following result.

Theorem 7. If the condition (i) is satisfied, then for $-t > 0$ large enough the negative feedback

$$\tilde{v}(z) = \left(\frac{a+d}{2}\right)\left(\frac{-z_1^2 + (-\frac{10\tilde{c}}{a+d} + p)z_1z_2 + (t+7)z_2^2}{\frac{2\tilde{c}}{a+d}z_1^2 - 5z_1z_2 + ((\frac{7a+7d}{\tilde{c}}))z_2^2}\right)$$

where

$$p = -15\frac{(a+d)}{c} - 8\frac{(a+d)^2 + c^2}{c(a+d)^2}$$

stabilizes the system (2).

Proof. Suppose that, the condition (i) is satisfied. We consider the closed-loop system (2) by the feedback $\tilde{v}(z)$

$$\begin{aligned} \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} &= \begin{pmatrix} Z_1(z_1, z_2) \\ Z_2(z_1, z_2) \end{pmatrix} \\ &= \left(\frac{2\tilde{c}}{a+d}z_1^2 - 5z_1z_2 + \left(\frac{7a+7d}{2\tilde{c}}\right)z_2^2\right)\left[\tilde{A}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + v(z_1,z_2)\tilde{B}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right]. \end{aligned}$$

Since the function $(\frac{2\tilde{c}}{a+d}z_1^2 - 5z_1z_2 + (\frac{7a+7d}{2\tilde{c}})z_2^2)$ is positive-definite then there is equivalence between asymptotic stability of the vector field $Z = (Z_1, Z_2)$ and the closed-loop bilinear system (2), defining the function F as follows

$$F(z_1, z_2) = z_1 Z_2(z_1, z_2) - z_2 Z_1(z_1, z_2)$$

a simple computation gives

$$F(z_1, z_2) = \left(\frac{2\tilde{c}^2}{a+d}\right) z_1^4 - 5\tilde{c}z_1^3 z_2 + \left(4a + 4d + \frac{2\tilde{c}^2}{a+d}\right) z_1^2 z_2^2 - \frac{pa+pd}{2} z_1 z_2^3 - \frac{ta+td}{2} z_2^4.$$

It is clear that for $-t > 0$ large enough F is a positive-definite function. From Theorem 1 one can deduce that the vector field Z is GAS if and only if $I = \int_{-\infty}^{+\infty} \frac{Z_1(1,y)}{F(1,y)}\, \mathrm{d}y < 0$. It is easy to verify that

$$\begin{aligned} I &= \int_{-\infty}^{+\infty} \frac{Z_1(1,y) - 4\frac{\partial F}{\partial z_2}(1,y)}{F(1,y)} \mathrm{d}y \\ &= \frac{a+d}{8} \int_{-\infty}^{+\infty} \frac{-(\frac{2\tilde{c}}{a+d}) - (8 + 2(\frac{2\tilde{c}}{a+d})^2)y + (\frac{28a+28d}{2\tilde{c}} + p)y^2}{F(1,y)} \mathrm{d}y. \end{aligned}$$

Since $p = -15\frac{(a+d)}{c} - 8\frac{(a+d)^2+c^2}{c(a+d)^2}$, then we can verify that $-(\frac{2\tilde{c}}{a+d}) - (8 + 2(\frac{2\tilde{c}}{a+d})^2)y + (\frac{28a+28d}{2\tilde{c}} + p)y^2 < 0 \ \forall y \in I\!R$. Consequently, the proof of theorem follows from Theorem 1. □

Theorem 8. If the condition (i) is satisfied, then for $t > 0$ large enough the positive feedback

$$\tilde{v}(z) = \frac{(2\tilde{c}(a+d) + \tilde{c}^3(a+d)/2 + 8\tilde{c}t)z_1^2 - 8(a+d)z_1z_2 - (16(a+d)/\tilde{c} + 8\tilde{c}t)z_2^2}{(\tilde{c}^2(a+d)/2)z_1^2 + 8tz_2^2}$$

stabilizes the system (2).

Proof. Suppose that, the condition (i) is satisfied. We consider the closed-loop system (2) by the feedback $\tilde{v}(z)$

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} Z_1(z_1, z_2) \\ Z_2(z_1, z_2) \end{pmatrix} = \left(\frac{\tilde{c}^2}{8(a+d)^2} z_1^2 + t z_2^2\right) \left[\tilde{A}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + v(z_1, z_2)\tilde{B}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right].$$

Since the function $\left(\frac{\tilde{c}^2}{8(a+d)^2} z_1^2 + t z_2^2\right)$ is positive-definite, then there is equivalence between asymptotic stability of the vector field $Z = (Z_1, Z_2)$ and the closed-loop bilinear system (2), defining the function F as follows

$$F(z_1, z_2) = z_1 Z_2(z_1, z_2) - z_2 Z_1(z_1, z_2)$$

a simple computation gives

$$F(z_1, z_2) = \left(\frac{a+d}{2}\right) \left(\frac{\tilde{c}}{a+d} z_1 + z_2\right)^2 \left(\frac{-\tilde{c}}{a+d} z_1^2 + 2z_1z_2 - 2\frac{a+d}{\tilde{c}} z_2^2\right).$$

It is easy to see that, if $(1,\xi)$ verify $\mathcal{F}(1,\xi) = \xi X_1(1,\xi) - X_2(1,\xi) = 0$ then there exists $\nu \in I\!R$ such that $(X_1(1,\xi), X_2(1,\xi)) = \nu(1,\xi)$.

In our case we have $\xi = \frac{-\tilde{c}}{a+d}$ and $\nu = \frac{\tilde{c}+(a+d)\xi/2}{\xi} = -(a+d)/2 < 0$. Consequently, the proof of theorem follows from Theorem 1. □

4.2.2. Case where $\mathrm{Tr}(\tilde{A}) = \mathrm{Tr}(\tilde{B}) = 0$

Under the assumptions that $\mathrm{Tr}(\tilde{A}) = \mathrm{Tr}(\tilde{B}) = 0$ and $\mathrm{Tr}(\tilde{A}\tilde{B}) = c > 0$, and in suitable basis of $I\!R^2$, the matrices $\tilde{A}$ and $\tilde{B}$ can be written as $\tilde{A} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, $\tilde{B} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ where $c > 0$ consider the linear change of coordinates whose transformation matrix is given by

$$P = \begin{pmatrix} 1 & \frac{-a}{c} \\ 0 & 1 \end{pmatrix}.$$

The matrix $\tilde{B}$ keep its initial form and the matrix $\tilde{A}$ becomes

$$\tilde{A} = \begin{pmatrix} 0 & b + \frac{a^2}{c} \\ c & 0 \end{pmatrix}.$$

In the new basis and in the case where $b + a^2/c < 0$ there is equivalence between the stabilizability of system $\dot{z} = (\tilde{A} + v\tilde{B})z$ by a positive feedback and the stabilizability of the system $\dot{z} = (\tilde{B} + v\tilde{A})z$ (7) where $v \in I\!R_+$. Moreover we can assume that there is equivalence between the stabilizability of system $\dot{z} = (\tilde{A} + v\tilde{B})z$ by a negative feedback and the stabilizability of the system $\dot{z} = (-\tilde{B} + v\tilde{A})z$ (8) where $v \in I\!R_+$. The stabilizability problem of systems (7) and (8) was treated in the subsection 4.1.

In the case when $b + a^2/c > 0$ we consider the change of feedback

$$\tilde{v}(z) = v(z) + \frac{a^2 + bc}{c} + c$$

the system (2) becomes

$$\dot{z} = (\bar{A} + \tilde{v}\tilde{B})z \quad \text{where} \quad \bar{A} = \begin{pmatrix} 0 & -c \\ c & 0 \end{pmatrix}.$$

The characteristic polynomial of matrix $\bar{A}$ is equal to $X^2 + c^2$, so matrix $\bar{A}$ admits a first integral, namely the positive-definite function

$$V(z_1, z_2) = \frac{1}{2}(z_1^2 + z_2^2).$$

Moreover, the rank of the family $\{\tilde{B}z, \mathrm{ad}\bar{A}\tilde{B}z, \ldots\}$ is equal to two on $I\!R^2 \setminus \{0\}$, hence for any positive constant δ the feedback law

$$\tilde{v}(z) = -\frac{\mathrm{L}_{\tilde{B}} V(z)}{\delta V(z_1, z_2)} = -\frac{z_1 z_2}{\delta(z_1^2 + z_2^2)}$$

stabilizes the system (2) (the proof is a modification of the result of [6] with the feedback rendered homogeneous; the proof is exactly the same). It follows for $\delta > 0$ large enough the negative feedback

$$v(z) = -\frac{z_1 z_2}{\delta(z_1^2 + z_2^2)} - \frac{a^2 + bc}{c} - c.$$

Proposition 2. The system (2) is not stabilizable by a positive feedback.

Proof. In the new basis it is easy to verify that the set $H = \{(z_1, z_2) \in \mathbb{R}^2$ such that $z_1 \geq 2\ z_2 \geq 2\}$ is invariant by the open loop system $\dot{z} = \tilde{A}z + v\tilde{B}z$ where v lie in $\mathbb{R}^+$. □

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