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# OBSERVATILITY AND OBSERVERS FOR NONLINEAR SYSTEMS WITH TIME DELAYS[1]

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Basic properties on linearization by output injection are investigated in this paper. A special structure is sought which is linear up to a suitable output injection and under a suitable change of coordinates. It is shown how an observer may be designed using theory available for linear time delay systems.

## 1. INTRODUCTION

In modern control theory, it has been a common practice to use ordinary differential equations to model dynamic systems. However, most of real systems have delays associated –normally introduced by natural response time (of sensors, actuators, etc,...), and transport (of information, mass, etc,...) phenomena– and a more accurate model would require the use of functional-differential equations. A lot of examples have been reported in the literature in a wide variety of applications like obtaining models in biological sciences (ecology, epidemiology, immunology, physiology, ... [see, e.g. [3] and references therein]), engineering (cold rolling mills, artificial neural networks, optimal control of flow [traffic, water resources systems], shunted transmission lines, information transmission over the internet, ... [see, e.g. [9, 11, 17] and references therein]) chemistry [2, 10, 16], just to name some.

These systems – which are referred to as *time-delay systems* – have been attracting an increasing number of researchers in recent years. In fact, some general results are already available specially in the linear case. Some results are also available when dealing with nonlinear time-delay systems in particular for control problems like feedback linearization, disturbance decoupling, and noninteracting control – see contributions in [6, 18, 21]. However, the proposed solutions assume that the full state variables are available for measurement and can be used in the control design. Such a request is seldom fulfilled in practice and one issue then consists in the design of a nonlinear observer. For linear time-delay systems, the observability has been widely investigated [5, 15, 22]. However, in the nonlinear case, and despite a couple of contributions [1, 7], this important problem remains open.

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[1]This work was performed while the first author was at the IRCCyN, in Nantes, France.

In the present work, structural properties of nonlinear time-delay systems are investigated, and preliminary steps on observability and observer design are given. As it is the case in the situation of nonlinear systems without delays, one important issue goes through the search of so-called linearizing coordinates, that is, a state space representation in which the given system is described by linear dynamics – up to additive nonlinear output injections. Such a structure, whenever it exists, displays the interesting feature to switch to linear theoretic arguments for the effective design of a stable observer (with linear dynamics with time delays).

A major concern in this paper is about the theoretical framework which shows to be adequate for the general analysis of nonlinear time-delay systems and more specifically for the analysis of observability or the observer design. For instance, in the current literature, several non equivalent definitions of coordinate transformations exist; the transformation which is defined in Section 2.2 is invertible and general enough to involve the time-shift operator.

In Section 2 we introduce both the mathematical setting for the analysis of nonlinear time-delay systems and basic transformations as state transformations. Section 3 is devoted to observability and observer design. An illustrative academic example is displayed in Section 4. Final remarks are presented in Conclusions.

## 2. PRELIMINARIES

### 2.1. Class of systems

A time-delay system is a dynamic system whose evolution in time depends not only on its actual state but also on the past. Mathematically, a time-delay system is described by means of a set of delay-differential equations [8].

Consider the class of systems given by

$$\Sigma : \left\{ \begin{array}{rcl} \dot{x}(t) & = & f(x(t), x(t-1), \dots, x(t-\tau)) \\ & & + \sum\limits_{i=0}^{\tau} g_i(x(t), x(t-1), \dots, x(t-\tau)) u(t-i) \\ y(t) & = & h(x(t), x(t-1), \dots, x(t-\tau)) \\ x(t) & = & \varphi, \quad u(t) = u_0, \quad \forall\, t \in [t_0 - \tau, t_0] \end{array} \right.$$

where only a finite number $\tau \in I\!N$ of *constant time delays* occur. The state $x \in I\!R^n$, the control input $u$ and the output $y \in I\!R$. The entries of $f$ and $g_i$ are meromorphic functions of their arguments. $\varphi$ is a continuous function of initial conditions.

For simplicity, the following notation will be used.

$$\left. \begin{array}{rcl} x_{i,j} & = & x_i(t-j), \\ u_j & = & u(t-j), \\ y_j & = & y(t-j), \\ \mathbf{x}_j & = & \{x_{1,j}, \dots, x_{n,j}\} \\ \mathbf{z}_j & = & \{\mathbf{x}_j, u_j, \dot{u}_j, \dots, u_j^{(k)}\} \end{array} \right\} \quad \begin{array}{l} i = 1 \dots n, \\ j = 0, 1, \dots \\ k \in I\!N \end{array}$$

We also define

$$\mathbf{x} = \{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_\tau\}.$$

$\mathbf{z}, \mathbf{u}$ and $\mathbf{y}$ are defined in a similar way.

Let $\mathcal{K}$ denote the field of meromorphic functions depending on a *finite* number in $\{\mathbf{z}\}$. Any element $a$ of $\mathcal{K}$ can be denoted by

$$a(\mathbf{z}_0, \dots, \mathbf{z}_\tau).$$

Denote by $\mathcal{E}$ the vector space spanned over $\mathcal{K}$ :

$$\mathcal{E} = \mathrm{span}_{\mathcal{K}} \{\mathrm{d}\xi,\ \xi \in \mathcal{K}\}$$

where d is the standard differential operator.

Define the *shift* operators $\delta$ and $\nabla$ as:

$$\begin{aligned} \delta a(\mathbf{z}_0, \dots \mathbf{z}_s) &:= a(\mathbf{z}_1, \dots, \mathbf{z}_{s+1}) \\ \nabla a(\mathbf{z}_0, \dots, \mathbf{z}_s) &:= a(\mathbf{z}_1, \dots, \mathbf{z}_{s+1})\nabla \\ \nabla \mathrm{d} &:= \mathrm{d}\delta. \end{aligned} \tag{1}$$

Let $\mathcal{K}[\nabla]$ denote the ring of polynomials of the operator $\nabla$ with coefficients over the field $\mathcal{K}$.

The differential of any function $\psi(\mathbf{z})$ may then be written as

$$\mathrm{d}\psi(\mathbf{z}) = \left[\sum_i \frac{\partial \psi(\mathbf{z})}{\partial \mathbf{z}_i} \nabla^i\right] \mathrm{d}\mathbf{z}_0.$$

$\mathcal{M}$ is defined as the left module over the ring $\mathcal{K}[\nabla]$:

$$\mathcal{M} = \mathrm{span}_{\mathcal{K}[\nabla]}\{\mathrm{d}\xi \mid \xi \in \mathcal{K}\}.$$

By taking the differential of $\dot{x}(t)$ one gets the so-called linearized tangent system:

$$\Sigma : \begin{cases} \mathrm{d}\dot{\mathbf{x}}_0 = F(\nabla)\mathrm{d}\mathbf{x}_0 + g(\nabla)\mathrm{d}\mathbf{u}_0 \\ \mathrm{d}y_0 = h(\nabla)\mathrm{d}\mathbf{x}_0. \end{cases}$$

### 2.2. Change of coordinates

**Definition 1.** (Change of coordinates) Consider system $\Sigma$ with state coordinates $\mathbf{x}_0$. $\xi_0 = \psi(\mathbf{x}), \psi \in \mathcal{K}^n$ is a *causal* change of coordinates for system $\Sigma$ if there exist a function $\psi^{-1}(\cdot) \in \mathcal{K}^n$ and a delay $\tau \in I\!N$ such that

$$\mathbf{x}_\tau = \psi^{-1}(\xi_0).$$

It is a *bicausal* change of coordinates if $\tau = 0$.

**Definition 2.** (Unimodular matrix) A matrix $A \in \mathcal{K}^{n\times n}[\nabla]$ is said to be unimodular if it has an inverse $A^{-1} \in \mathcal{K}^{n\times n}[\nabla]$.

Whether a matrix is unimodular may be tested with the algorithm provided in Appendix B.

**Remark 1.**

- If the linear map $T[\nabla] \in \mathbb{R}^{n\times n}[\nabla]$ is causal then

$$Smith\{T[\nabla]\} = diag\{\nabla^{\mu_i}\}, \text{ and} \tag{2}$$

- If $\boldsymbol{\xi}_0 = \varphi(\mathbf{x})$ is a bicausal change of coordinates, then the matrix $S[\nabla]$ defined by

$$\mathrm{d}\mathbf{x}_0 = S[\nabla]\mathrm{d}\boldsymbol{\xi}_0,$$

  is unimodular.

### 2.3. Closure

We end this section by recalling the notion of closure, introduced in control theory in [4] for the study of linear time-delay systems.

Let $M$ be a module defined over a ring $R$. The $R$-closure or closure over $R$ of a submodule $A \subset M$, noted $cls_R\{A\}$, is defined as

$$cls_R\{A\} := \{x \in M \mid \exists P \in R,\ Px \in A\}$$

When a submodule $A$ is equal to its $R$-closure, one says that it is closed over $R$.

## 3. OBSERVABILITY FOR NONLINEAR TIME–DELAY SYSTEMS

System $\Sigma$ is said to be observable when the state $x(t)$ can be expressed as a function of the derivatives of the output and the input, and their forward shifts:

$$x(t) = \psi(y^{(k)}(t+\tau), u^{(\ell)}(t+\sigma), k, \ell, \tau, \sigma \in \mathbb{N}).$$

**Definition 3.** Assume that system $\Sigma$ is observable; then it is said to be linearizable by additive output injections if the output may be written under the form [23]

$$y_0^{(n)} = \sum_{j=1}^{n} \phi_j^{(j-1)}(\mathbf{y}, \mathbf{u}). \tag{3}$$

For nonlinear time–delay systems, this may be checked with a straightforward adaptation of the algorithm given in [24], presented in [19]. For the sake of completeness, this algorithm is recalled in Appendix A.

### 3.1. Observation scheme

Assume that system $\Sigma$ is observable. In this section we will consider a transformation that will be helpful to construct an observer for the original nonlinear time delay system $\Sigma$. Assume that the system $\Sigma$ is linearizable by additive output injections (cf. Appendix A). Then at each step $k$ of the linearization algorithm we define

$$\mathrm{d}h_k(\mathbf{x}) = \mathrm{d}y_0^{(k)} - \sum_{i=1}^{k} \mathrm{d}\phi_{\mathrm{i}}^{(\mathrm{k-i})}(\mathbf{y}, \mathbf{u}). \tag{4}$$

Let $\{\mathrm{d}\xi_{1,0}, \ldots, \mathrm{d}\xi_{n,0}\}$ be a basis for

$$cls_{R[\nabla]}\{\mathrm{d}h_1(\mathbf{x}), \ldots, \mathrm{d}h_n(\mathbf{x})\},$$

so we have

$$\begin{bmatrix} \mathrm{d}h_1 \\ \vdots \\ \mathrm{d}h_n \end{bmatrix} = T[\nabla] \begin{bmatrix} \mathrm{d}\xi_{1,0} \\ \vdots \\ \mathrm{d}\xi_{n,0} \end{bmatrix}, \qquad T[\nabla] \in I\!R^{n\times n}[\nabla]. \tag{5}$$

From this point we will assume that $\{\mathrm{d}\boldsymbol{\xi}_0\}$ is a causal change of coordinates for $\mathrm{d}\mathbf{x}_0$. Taking the time derivative of functions $\xi_{k,0}$ we have, from equation (4),

$$\dot{\xi}_{k,0} = \alpha_k(\nabla)\boldsymbol{\xi}_0 + \phi_k(\mathbf{y}, \mathbf{u})$$

with $\alpha_k(\nabla) \in \mathcal{K}^{n\times 1}[\nabla]$, producing the new representation,

$$\dot{\boldsymbol{\xi}}_0 = A(\nabla)\boldsymbol{\xi}_0 + \phi(\mathbf{y}, \mathbf{u}) \tag{6}$$

where $A(\nabla) := [\alpha_1^T(\nabla) \ \cdots \alpha_n^T(\nabla)]^T$ and $\phi(\mathbf{y}, \mathbf{u})$ is the vector of functions given by

$$\phi(\mathbf{y}, \mathbf{u}) = \begin{bmatrix} \phi_1(\mathbf{y}, \mathbf{u}) \\ \vdots \\ \phi_n(\mathbf{y}, \mathbf{u}) \end{bmatrix}.$$

### 3.2. Observer design

From the previous section it is possible to define now the notion of an observer.

**Definition 4.** The dynamic system

$$\begin{aligned} \dot{\hat{\boldsymbol{\xi}}}_0 &= \gamma(\hat{\boldsymbol{\xi}}, \mathbf{y}, \mathbf{u}) \\ \delta^\tau \hat{\mathbf{x}}_0 &= \varphi(\hat{\boldsymbol{\xi}}, \mathbf{y}, \mathbf{u}), \end{aligned}$$

is said to be an observer for system $\Sigma$ if $e_\tau(t) \to 0$ as $t \to \infty$, where $e_\tau(t) := \delta^\tau \hat{\mathbf{x}}_0 - \delta^\tau \mathbf{x}_0$, for some $\tau \in I\!N$.

A characteristic of this general observer is that, as in the linear time-delay systems, we may obtain the estimation of the state at time $t - \tau$. This characteristic is related in the linear case with the well-known notion of weak observability [15].

From system (6) it is possible to propose a compensator that gives the estimation $\hat{\boldsymbol{\xi}}_0$ of the transformed state $\boldsymbol{\xi}_0$. This can be done as follows.

**Theorem 1.** Assume that system $\Sigma$ is linearizable by $\phi_i(\mathbf{y},\mathbf{u})$-injections. If

(i) $\boldsymbol{\xi}$ is a bicausal change of coordinates for $\mathbf{x}$ :

$$\boldsymbol{\xi}_0 = \varphi(\mathbf{x}), \quad \mathbf{x}_0 = \varphi^{-1}(\boldsymbol{\xi}), \quad \text{and} \tag{7}$$

(ii) $Smith\{T[\nabla]\} = diag\{\nabla^{\mu_i}\}$, for some $\mu_i$'s, $T[\nabla]$ given by (5),

then

$$\begin{aligned} \dot{\hat{\boldsymbol{\xi}}}_0 &= A(\nabla)\hat{\boldsymbol{\xi}}_0 + \phi(\mathbf{y},\mathbf{u}) - k(\nabla)(y_0 - C(\nabla)\hat{\boldsymbol{\xi}}_0) \\ \hat{\mathbf{x}}_0 &= \varphi^{-1}(\hat{\boldsymbol{\xi}}) \end{aligned} \tag{8}$$

with $A(\nabla) + k(\nabla)C(\nabla)$ an stable matrix, is an observer for system $\Sigma$.

The proof of Theorem 1 follows from the above and is left to reader. Stability of a matrix $\tilde{A}(\nabla)$ is meant in the sense that any solution of

$$\dot{x} = \tilde{A}(\nabla)x,$$

is stable for any initial condition.

Note that (i) does not depend on the choice of basis for $cls_{\mathbb{R}[\nabla]}$ because any two basis of the same submodule are related by an unimodular transformation matrix.

## 4. ILLUSTRATIVE EXAMPLES

As a first example, consider the system

$$\begin{aligned} \dot{x}_{1,0} &= u_0 \\ y_0 &= x_{1,0} + x_{1,1} \end{aligned}$$

for which $h_1 = x_{1,0} + x_{1,1}$ and $\phi_1 = u_0 + u_1$. The state of this system may not be recovered from the measurement of its output, unless the initial conditions are known. This is displayed by choosing $\xi_{1,0} = x_{1,0}$ and writing

$$\mathrm{d}h_{1,0} = [1+\nabla]\mathrm{d}\xi_{1,0}.$$

Because condition (2) is not satisfied, we cannot go any further.

Now consider the system

$$\begin{aligned} \dot{x}_{1,0} &= 0.2x_{1,1} + 0.1x_{2,1} + 0.5x_{2,1}x_{2,2} + 0.2x_{2,2}^2 + u_1 \\ \dot{x}_{2,0} &= -0.25x_{2,1} \\ y_0 &= x_{1,1} - x_{2,2}^2. \end{aligned} \tag{9}$$

In this case, we have

$$\begin{aligned} y_0 &= x_{1,1} - x_{2,2}^2 \\ \dot{y}_0 &= (0.2y_2 + u_2) + 0.1x_{2,2} \\ \ddot{y}_0 &= (0.2\dot{y}_2 + \dot{u}_2) - 0.025x_{2,3} \\ &= (0.2\dot{y}_2 + \dot{u}_2) - 0.25\delta(\dot{y}_0 - (0.2y_2 + u_2)) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}(-0.25y_1 + 0.2y_2 + u_2) + 0.25(0.2y_3 + u_3), \end{aligned}$$

so

$$\begin{aligned} h_1 &= x_{1,1} - x_{2,2}^2 \\ h_2 &= \dot{y}_0 - (-0.25y_1 + 0.2y_2 + u_2) \\ &= 0.25(x_{1,1} - x_{2,2}^2) + 0.1x_{2,2}. \end{aligned}$$

We choose $\{\xi_{1,0} = x_{1,0} - x_{2,1}^2, \xi_{2,0} = x_{2,0}\}$ as a basis for $cls_{\mathcal{R}[\nabla]}\{h_1, h_2\}$. Then

$$\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} \nabla & 0 \\ 0.25\nabla & 0.1\nabla^2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}.$$

By computing the Smith's normal form for the transformation matrix, we find that the invariant polynomials are $\{\nabla, \nabla^3\}$, so condition (2) is fulfilled. Also, condition (7) is satisfied:

$$\begin{aligned} x_{1,0} &= \xi_{1,0} + \xi_{2,1}^2 \\ x_{2,0} &= \xi_{2,0}. \end{aligned}$$

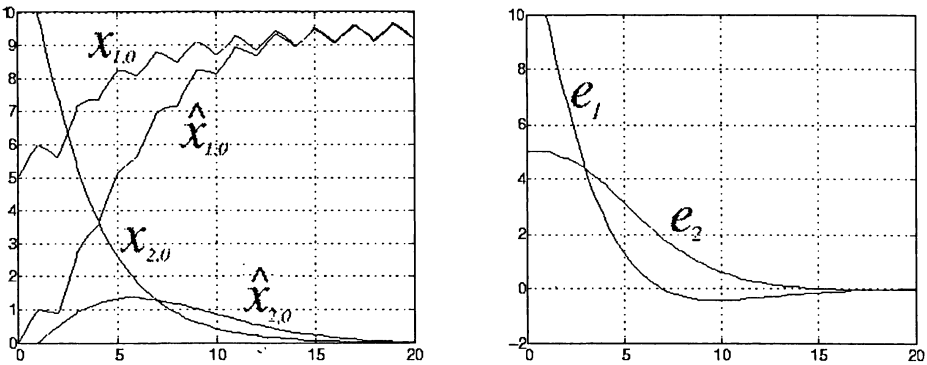
Under the new coordinates, the system reads

$$\begin{aligned} \dot{\boldsymbol{\xi}}_0 &= \begin{bmatrix} 0 & 0.1\nabla \\ 0 & -0.25\nabla \end{bmatrix} \boldsymbol{\xi}_0 + \begin{bmatrix} -0.25y_0 + 0.2y_1 + u_1 \\ 0 \end{bmatrix} \\ y_0 &= [\nabla \quad 0]\boldsymbol{\xi}_0 \end{aligned}$$

and

$$\begin{aligned} \dot{\hat{\boldsymbol{\xi}}}_0 &= \begin{bmatrix} 0 & 0.1\nabla \\ 0 & -0.25\nabla \end{bmatrix} \hat{\boldsymbol{\xi}}_0 + \begin{bmatrix} -0.25y_0 + 0.2y_1 + u_1 \\ 0 \end{bmatrix} \\ &\quad - \begin{bmatrix} k_1(\nabla) \\ k_2(\nabla) \end{bmatrix} (y_0 - \hat{\xi}_{1,0}) \\ \hat{x}_{1,0} &= \xi_{1,0} + \xi_{2,1}^2 \\ \hat{x}_{2,0} &= \xi_{2,0} \end{aligned}$$

is an observer for system (9) if the matrix $\begin{bmatrix} k_1(\nabla) & 0.1\nabla \\ k_2(\nabla) & -0.25\nabla \end{bmatrix}$ is stable. From [12], we find $k_1(\nabla) = -0.25\nabla$ and $k_2(\nabla) = -0.1\nabla$. A numerical simulation was carried out using Matlab–Simulink software, considering the initial conditions $x(t) = [5 \quad 10]$, and $\hat{x}(t) = [0 \quad 0]$ for $t \in [-2, 0]$. A square signal (peak-to-peak amplitude $=1$, frequency $= 0.5$ Hz, mean $= 0.5$ duty cycle $= 50\,\%$) was set as input signal. The state evolution (for the system $(x_{1,0}, x_{2,0})$ and the observer $(\hat{x}_{1,0}, \hat{x}_{2,0})$) and converging errors ($e_1 := x_{1,0} - \hat{x}_{1,0}$ and $e_2 := x_{2,0} - \hat{x}_{2,0}$) are depicted in Figure 1.



(a) (b)

**Fig. 1.** (a) Actual and observed state; (b) Convergence error.

## 5. CONCLUSIONS

A class of time delay systems has been considered, which includes time delays on the input and the state variables. This frame is general enough to model complex systems. It is closed under state feedback transformations. Note that the framework is also adequate and necessary for systems which display time delays in the input only since a delay free state feedback will naturally introduce delays on the state variables of the closed loop system. The proposed approach has been used to study the observability of nonlinear time-delay systems. A definition of observability and a methodology to construct an observer for systems satisfying a checkable condition are presented. It goes through a generalized notion of linearization via output injections, which has been a key issue for almost 20 years in the design of observers for nonlinear systems without delays [13, 14].

## APPENDIX A. ALGORITHM FOR LINEARIZATION BY ADDITIVE OUTPUT INJECTIONS

Define

$$\begin{aligned} E^0 &= 0 \\ E^k &= \mathrm{span}_{\mathcal{K}[\nabla]}\{\mathrm{d}y(t), \dots, \mathrm{d}y^{(k-1)}(t), \mathrm{d}u(t), \dots \mathrm{d}u^{(k-1)}(t)\}. \end{aligned}$$

Then, if (3) holds, it follows that

$$\mathrm{d}y^{(\bar{n})} = \mathrm{d}\left[\Phi_1(y,u)\right]^{(\bar{n}-1)} + \cdots + \mathrm{d}\Phi_{\bar{n}}(y,u). \tag{10}$$

Assume also

$$\dim_{\mathcal{K}[\nabla]} E^{\bar{n}} = 2\bar{n}. \tag{11}$$

Then, the following algorithm gives a way to check (10).

**Algorithm 1.** (Linearization by additive output injections)

Initial check: $\mathrm{d}y^{(\bar{n})} \in E^{\bar{n}}$. If no, stop! Otherwise, denote $\omega_1 = \mathrm{d}y^{(\bar{n})}$.
*Step 1:*
Pick functions $\xi_1, \theta_1 \in \mathcal{K}[\nabla]$ such that

$$\omega_1 - \xi_1 \mathrm{d}y^{(\bar{n}-1)} - \theta_1 \mathrm{d}u^{(\bar{n}-1)} \in E^{\bar{n}-1}. \tag{12}$$

Define the differential one-form $\bar{\omega}_1$ as

$$\bar{\omega}_1 = \xi_1 \mathrm{d}y + \theta_1 \mathrm{d}u.$$

Check: $\mathrm{d}\bar{\omega}_1 = 0$. If no, stop!

*Step $\ell$ ($\ell = 2, \ldots, \bar{n}$):*

Let $\Phi_{\ell-1}(y, u)$ be such that $\mathrm{d}\Phi_{\ell-1} = \bar{\omega}_{\ell-1}$.
Denote $\omega_\ell$ as

$$\omega_\ell = \omega_{\ell-1} - \mathrm{d}\Phi_{\ell-1}^{(\bar{n}-\ell+1)}.$$

Choose $\xi_\ell, \theta_\ell \in \mathcal{K}[\nabla]$ such that

$$\omega_\ell - \xi_\ell \mathrm{d}y^{(\bar{n}-\ell)} - \theta_\ell \mathrm{d}u^{(\bar{n}-\ell)} \in E^{\bar{n}-\ell}.$$

Define the differential one-form $\bar{\omega}_\ell$ as

$$\bar{\omega}_\ell = \xi_\ell \mathrm{d}y - \theta_\ell \mathrm{d}u.$$

Check: $\mathrm{d}\bar{\omega}_\ell = 0$. If no, stop!

This algorithm allows to check whether a system can be written as (10). The conditions are stated in terms of integrability conditions of some differential one-forms.

**Lemma 1.** (cf. [19]) Under assumption (11), $\mathrm{d}y^{(\bar{n})} \in \mathcal{E}$ may be written under the form (10) if and only if $\mathrm{d}y^{(\bar{n})} \in E^{\bar{n}}$ and

$$\mathrm{d}\bar{\omega}_i = 0, \; i = 1, \ldots, \bar{n}. \tag{13}$$

## APPENDIX B. INVERSION OF MATRICES WITH ENTRIES IN $\mathcal{K}[\nabla]$

**Algorithm 2.** Matrix inversion

Set $\mathcal{A} = [A \mid I]$
**for** $i = 1$ to $n$ **do**
  **if** $[a_{ii} \cdots a_{ni}] = \mathbf{0}$ **then**
    Problem has no solution. Algorithm ends.
  **else**
    Swap rows of $\mathcal{A}$ to have $pol.d^\circ(a_{ii}) \leq \min\{pol.d^\circ(a_{ji}),\ j > i, a_{ji} \neq 0\}$
    **while** $\exists\ j > i$ such that $a_{ji} \neq 0$ **do**
      Swap rows $i+1$ and $j$ of $\mathcal{A}$
      Find $P, Q, R, S \in \mathcal{K}[\nabla] \backslash \{0\}$ such that $\begin{bmatrix} P & Q \\ R & S \end{bmatrix} \begin{bmatrix} a_{ii} \\ a_{ji} \end{bmatrix} = \begin{bmatrix} d[\nabla] \\ 0 \end{bmatrix}$,
      where $d[\nabla]$ is the left-g.c.d. of $a_{ii}$ and $a_{ji}$.
      Set $\mathcal{A} \leftarrow \begin{bmatrix} I\!I_{i-1} & & & (0) \\ & P & Q & \\ & R & S & \\ (0) & & & I\!I_{n-i-1} \end{bmatrix} \mathcal{A}$
    **end while**
    **if** $a_{ii} \notin \mathcal{K}$ **then**
      Problem has no solution. Algorithm ends.
    **else**
      Divide (from the left) row $i$ of $\mathcal{A}$ by $a_{ii}$.
      Set $\mathcal{A} \leftarrow \left[\begin{array}{cc|c} & -a_{1i} & \\ Id_{i-1} & \vdots & (0) \\ & -a_{i-1,i} & \\ \hline & 1 & (0) \\ (0) & & I\!I_{n-i-1} \end{array}\right] \mathcal{A}$
    **end if**
  **end if**
**end for**

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