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ON THE DESCRIPTION AND ANALYSIS OF MEASUREMENTS OF CONTINUOUS QUANTITIES

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The measurement of continuous quantities is the basis for all mathematical and statistical analysis of phenomena in engineering and science.Therefore a suitable mathematical description of measurement results is basic for realistic analysis methods for such data. Since the result of a measurement of a continuous quantity is not a precise real number but more or less non-precise, it is necessary to use an appropriate mathematical concept to describe measurements. This is possible by the description of a measurement result by a so-called non-precise number. A non-precise number is a generalization of a real number and is defined by a so-called characterizing function. In case of vector valued quantities the concept of so-called non-precise vectors can be used. Based on these concepts more realistic data analysis methods for measurement data are possible.

1. INTRODUCTION

The result of the measurement of a physical quantity is usually thought to be a real number $x \in \mathbb{R}$ times a measurement unit. Of course engineers know that these numbers are uncertain in applications. Therefore stochastic models are used to describe data uncertainty by assuming that the results of repeated measurements of the same quantity are realisations of a stochastic quantity X. From a sample $x_1, \cdots, x_n$ of measurements different probability statements are made about the quantity of interest.

But there is a basic problem with this because the result of one measurement of a one-dimensional continuous quantity is not a precise real number but more or less non-precise. This imprecision is unavoidable also on the macroscopic level. Therefore it has to be taken into account before analysing measurement data. It should be noted that this kind of uncertainty is different from stochastic uncertainty and errors.

Historically a concept to describe real measurement data is the description of data in form of intervals and the related mathematical concept is the concept of interval mathematics. But in many situations the boundaries of such intervals are not precise and therefore a more general concept is necessary. Since intervals are subsets of the set $\mathbb{R}$ of real numbers a generalization of subsets of $\mathbb{R}$ is useful.

In the year 1951 K. Menger published the idea of so-called ensembles flous which

are generalizations of classical subsets in the following way. A classical subset A of a given set M is characterized by its *indicator function* $I_A(\cdot)$, where the possible values of an indicator function is the binary set $\{0,1\}$:

$$I_A(x) = \left\{ \begin{array}{lll} 1 & \text{for} & x \in A \\ 0 & \text{for} & x \notin A \end{array} \right\} \qquad \text{for all } x \in M.$$

K. Menger allowed a generalization of this to take care of uncertainty and non-precise boundaries of A. Therefore he defined a so-called *ensemble flou* by a real valued function defined on the set M which is allowed to assume any value from the interval $[0,1]$. Such functions where later called *membership functions* $\mu(\cdot)$ by L. Zadeh who made the topic popular in the 1960's,

$$\mu : M \to [0,1].$$

The generalized subset $A^\star$ defined by $\mu(\cdot)$ is called *fuzzy set* in English.

Now the idea of K. Menger and L. Zadeh can be used to describe non-precise measurements by non-precise numbers.

2. NON–PRECISE NUMBERS

A non-precise number $x^\star$ is defined by its so-called *characterizing function* $\xi(\cdot)$ which is a real function of one real variable x fulfilling the following:

(1) $0 \leq \xi(x) \leq 1$ for all $x \in I\!R$.

(2) There exists at least one $x \in I\!R$ with $\xi(x) = 1$.

(3) For all $\delta \in (0,1]$ the so-called δ-cut $C_\delta\Big(\xi(\cdot)\Big)$ defined by

$$C_\delta\Big(\xi(\cdot)\Big) := \Big\{x \in I\!R : \ \xi(x) \geq \delta\Big\}$$

is a closed finite interval $[a_\delta, b_\delta]$.

Precise real numbers as well as intervals are special cases of non-precise numbers. The characterizing function of a precise real number x_0 is the one point indicator function $I_{\{x_0\}}(\cdot)$ and the characterizing function of an interval $[a,b]$ is the indicator function $I_{[a,b]}(\cdot)$. Therefore the concept of non-precise numbers is a suitable concept for real measurement data. The area under the characterizing function of a non-precise number $x^\star$ is the amount of measurement uncertainty of $x^\star$ concerning imprecision.

3. CONSTRUCTION OF CHARACTERIZING FUNCTIONS

A crucial point is how to obtain the characterizing function of a measurement result. This depends on the application area but some general remarks can be given.

Classical measurement instruments with pointers – also precision measurement instruments – don't produce precise numbers as measurement result. Important

examples are strength measurements of materials. Looking realistically at the measurement process a non-precise number is obtained as individual measurement result.

To obtain the characterizing function of a single measurement a precise look at the pointer shows a vibration of it and from recording this the characterizing function can be obtained.

For digital measurement equipments the resulting data $x_i^\star$ are "numbers" with finitely many digits, i. e. they are data in form of intervals

$$x_i^\star = \left[\underline{x}_i, \overline{x}_i\right],$$

where $\underline{x}_i$ is the reading on the instrument completed by zeros for all digits after the last reported decimal, and $\overline{x}_i$ is the reading of the instrument completed by 9's for all digits after the last reported decimal. The characterizing function is the indicator function $I_{[\underline{x}_i,\overline{x}_i]}(\cdot)$.

In case of analog equipments with a screen the result can be a light point on an oscillograph. The light intensity of this "point" can be used to obtain the characterizing function $\xi(\cdot)$ from the light intensity function $\varphi(\cdot)$. In applications the light intensity is bounded for all $x \in I\!R$ and the values $\xi(x)$ of the characterizing function are obtained by

$$\xi(x) = \frac{\varphi(x)}{\max_{x \in I\!R} \varphi(x)} \qquad \text{for all} \quad x \in I\!R\ .$$

If the light intensity is increasing up to a certain value and decreasing afterwards, then the resulting function $\xi(\cdot)$ is a characterizing function as defined above.

More generally often color intensity pictures are obtained as results of measurement processes. For example hardness measurements of materials or results of remote sensing. Here the color intensity can be used to obtain the characterizing function.

4. NON–PRECISE VECTORS

Measurements of observations of vector valued continuous quantities $\boldsymbol{x} = (x_1, \cdots, x_k)$ are also not precise. Therefore a generalization of the concept of a vector, whose components are thought to be real numbers, is necessary.

For continuous vector quantities with dimension k *non-precise vectors* $\boldsymbol{x}^\star$ are defined by so-called *vector-characterizing functions.*

The vector-characterizing function $\zeta(\cdot, \cdots, \cdot)$ of a non-precise vector is a real valued function of k real variables $x_1, \cdots, x_k$ obeying the following:

(1) $0 \le \zeta(x_1, \cdots, x_k) \le 1 \qquad$ for all $\quad (x_1, \cdots, x_k) \in I\!R^k$.

(2) There exists at least one k-tuple $(x_1, \cdots, x_k) \in I\!R^k$ with $\zeta(x_1, \cdots, x_k) = 1$.

(3) For all $\delta \in (0, 1]$ the so-called δ-cut $C_\delta\Big(\zeta(\cdot, \cdots, \cdot)\Big)$, defined by $C_\delta\Big(\zeta(\cdot, \cdots, \cdot)\Big) := \{(x_1, \cdots, x_k) \in I\!R^k : \zeta(x_1, \cdots, x_k) \ge \delta\}$ is a closed compact and convex subset of $I\!R^k$.

For a classical precise k-dimensional vector $\overset{\circ}{x} = (\overset{\circ}{x}_1, \cdots, \overset{\circ}{x}_k)$ the vector-characterizing function is the one-point indicator function $I_{\{\overset{\circ}{x}_1, \cdots, \overset{\circ}{x}_k\}}(\cdot, \cdots, \cdot)$ and for a *k-dimensional interval*

$$[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_k, b_k] = \times_{i=1}^{k} [a_i, b_i]$$

the vector-characterizing function is the indicator function

$$I_{\times_{i=1}^{k}[a_i, b_i]}(\cdot, \cdots, \cdot) .$$

General vector-characterizing functions are obtained for data given as color intensity pictures. In case of a 2-dimensional point on a radar screen the position of an equipment is represented by a light "point" on the radar screen. The light intensity determines the vector-characterizing function of the non-precise 2-dimensional vector $x^\star$. Let $\phi(x_1, x_2)$ denote the light intensity in the plane $\mathbb{R}^2$. Then the vector-characterizing function $\zeta(\cdot, \cdot)$ is given by its values

$$\zeta(x_1, x_2) = \frac{\phi(x_1, x_2)}{\max\limits_{(x_1, x_2) \in \mathbb{R}^2} \phi(x_1, x_2)} \quad \text{for all} \quad (x_1, x_2) \in \mathbb{R}^2 .$$

5. FUNCTIONS OF NON-PRECISE VALUES

In analysing measurements, functions of the obtained measurements are essential. Based on classical assumptions functions $g(x_1, \cdots, x_n)$ of n real numbers $x_1, \cdots, x_n$ are considered.

For the realistic situation of non-precise measurement results $x_1^\star, \cdots, x_n^\star$ of one-dimensional quantities, functions $g(x_1^\star, \cdots, x_n^\star)$ have to be considered where $g(\cdot, \cdots, \cdot)$ is a real valued function. For non-precise argument values $x_1^\star, \cdots, x_n^\star$ the resulting value $g(x_1^\star, \cdots, x_n^\star)$ naturally is non-precise, i. e.

$$y^\star = g(x_1^\star, \cdots, x_n^\star) ,$$

where $y^\star$ is a non-precise number under certain conditions for $g(\cdot, \cdots, \cdot)$.

In order to obtain the characterizing function $\eta(\cdot)$ of $y^\star$ the so-called *extension principle* developed by L. Zadeh in the 1970's can be applied, compare [1] and [10]. Before applying the extension principle the non-precise values $x_1^\star, \cdots, x_n^\star$ with corresponding characterizing functions $\xi_1(\cdot), \cdots, \xi_n(\cdot)$ have to be combined into a non-precise vector $x^\star$ in $\mathbb{R}^n$. This can be done in the following way:

The vector-characterizing function $\zeta(\cdot, \cdots, \cdot)$ of $x^\star$ is obtained from the characterizing functions $\xi_1(\cdot), \cdots, \xi_n(\cdot)$ by its values

$$\zeta(x_1, \cdots, x_n) = \min [\xi_1(x_1), \cdots, \xi_n(x_n)] \quad \text{for all} \quad (x_1, \cdots, x_n) \in \mathbb{R}^n .$$

Remark 5.1. By this definition $\zeta(\cdot, \cdots, \cdot)$ is a vector-characterizing function. The δ-cuts $C_\delta(x^\star)$ of the corresponding non-precise vector $x^\star$ are related to the δ-cuts $C_\delta(x_i^\star)$ in the following way:

$$C_\delta(x^\star) = \times_{i=1}^{n} C_\delta(x_i^\star) \qquad \text{for all} \quad \delta \in (0, 1] ,$$

i. e. they are the Cartesian products of the δ-cuts of the n non-precise numbers $x_1^\star, \cdots, x_n^\star$.

Based on the so-called *combined non-precise vector* $\boldsymbol{x}^\star$ the characterizing function of the non-precise value of a function can be calculated.

Let $g(x_1, \cdots, x_n)$ be a real valued continuous function. Then for non-precise argument values $x_1^\star, \cdots, x_n^\star$ with combined non-precise vector $\boldsymbol{x}^\star$ the values $\eta(y)$ of the characterizing function $\eta(\cdot)$ of the non-precise value $y^\star = g(x_1^\star, \cdots, x_n^\star)$ are given by the extension principle, and using the notation $\boldsymbol{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$ by

$$\eta(y) = \left\{ \begin{array}{ll} \sup\{\zeta(\boldsymbol{x}) \colon g(\boldsymbol{x}) = y\} & \text{if} \quad g^{-1}(\{y\}) \neq \emptyset \\ 0 & \text{if} \quad g^{-1}(\{y\}) = \emptyset \end{array} \right\} \quad \text{for all} \quad y \in \mathbb{R} \, .$$

Proposition 5.1. Under the conditions above the function $\eta(\cdot)$ of $y^\star$ is a characterizing function as defined in Section 2 whose δ-cuts are

$$C_\delta(y^\star) = \left[\min_{\boldsymbol{x} \in C_\delta(\boldsymbol{x}^\star)} g(\boldsymbol{x}) \, , \max_{\boldsymbol{x} \in C_\delta(\boldsymbol{x}^\star)} g(\boldsymbol{x}) \right] \quad \text{for all} \quad \delta \in (0, 1].$$

The proof is given in [5].

Remark 5.2. The resulting characterizing function $\eta(\cdot)$ need not be continuous. An example for one-dimensional non-precise argument $x^\star$ is given in Figure 1.

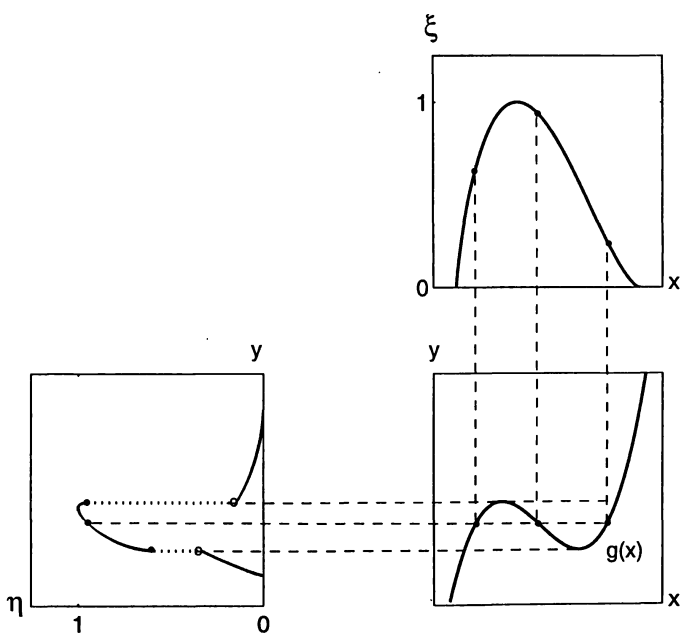

Fig. 1. Characterizing function of a derived non-precise value $y^\star = g(x^\star)$.

6. ANALYSING MEASUREMENT RESULTS

For repeated measurements of a quantity, for example a geodetic length or angle, the results are usually used to calculate the arithmetic mean or some weighted mean value. For n measurements $x_1^\star, \cdots, x_n^\star$ these are n non-precise numbers. The arithmetic mean of n precise numbers $x_1, \cdots, x_n$,

$$\overline{x}_n := \frac{1}{n} \sum_{i=1}^{n} x_i$$

has to be adapted to the situation of non-precise numbers. This is possible using the concept from Section 5. Here the function $g(\cdot, \cdots, \cdot)$ is

$$g(x_1, \cdots, x_n) = \frac{1}{n} \sum_{i=1}^{n} x_i \ .$$

Let the n non-precise measurements $x_i^\star$ have characterizing functions $\xi_i(\cdot)$, these have to be combined to a non-precise vector $\boldsymbol{x}^\star$ with vector-characterizing function $\zeta(\cdot, \cdots, \cdot)$.

Based on $\zeta(\cdot, \cdots, \cdot)$ the value $g(x_1^\star, \cdots, x_n^\star) = \frac{1}{n} \sum_{i=1}^{n} x_i^\star$ is a non-precise number whose characterizing function $\eta(\cdot)$ is obtained via the extension principle from Section 5.

An example of non-precise measurements is given in Figure 2.

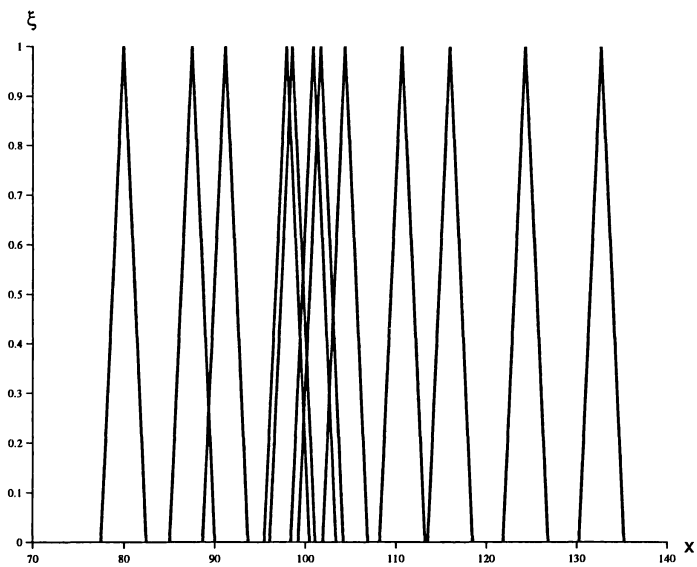

Fig. 2. Characterizing functions of 12 non-precise numbers.

In Figure 3 the characterizing function of the non-precise arithmetic mean $\overline{x}_{12}^\star$ of the 12 non-precise measurements from Figure 2 is depicted.

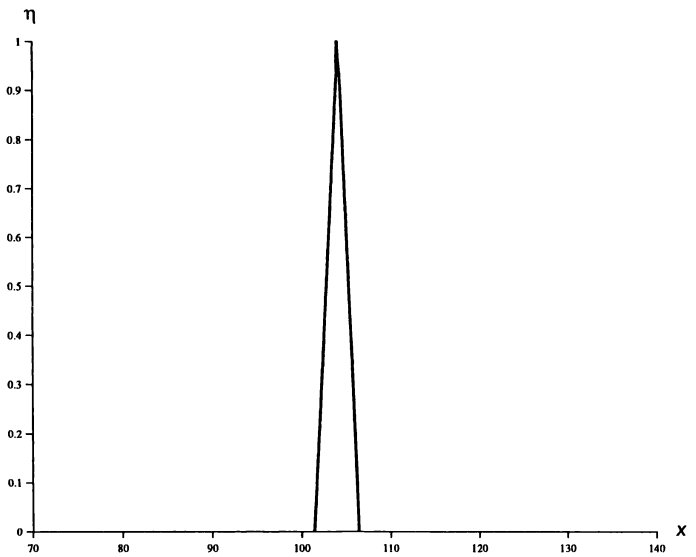

Fig. 3. Non-precise sample mean $\overline{x}_{12}^{\star}$.

The resulting characterizing function is the optimal realistic information which can be obtained from the non-precise measurements.

7. STATISTICAL TESTING OF HYPOTHESES

One of the main points in model building is the formulation of hypotheses. In order to decide if a model (hypothesis) is acceptable, so-called statistical tests are performed.

Statistical tests are decision rules which are usually depending on a test statistic $g(x_1, \cdots, x_n)$ which is a function of the observations $x_1, \cdots, x_n$.

In standard statistics the data are assumed to be generated by a random sample $X_1, \cdots, X_n$ of the considered model $X \sim P_\theta$; $\theta \in \Theta$. The decision is based on the value t which a function of the sample, i. e.

$$T = g(X_1, \cdots, X_n)$$

assumes.

Usually the space of possible values of the test statistic T is decomposed into an acceptance region A and its complement, the rejection region A^c. For precise data $x_1, \cdots, x_n$ the value $t = g(x_1, \cdots, x_n)$ of the test statistic is precise also. Therefore it is possible to decide whether the value t is in the acceptance region or not.

In case of non-precise data $x_1^{\star}, \cdots, x_n^{\star}$ the value $t^{\star} = g(x_1^{\star}, \cdots, x_n^{\star})$ of a test statistic, obtained by the method from Section 5, becomes non-precise. This makes a major problem in the usual setting of testing models, because it is not always

possible to decide whether the non-precise value $t^\star$ with characterizing function $\xi(\cdot)$ belongs to the acceptance region or not. An example is given in Figure 4.

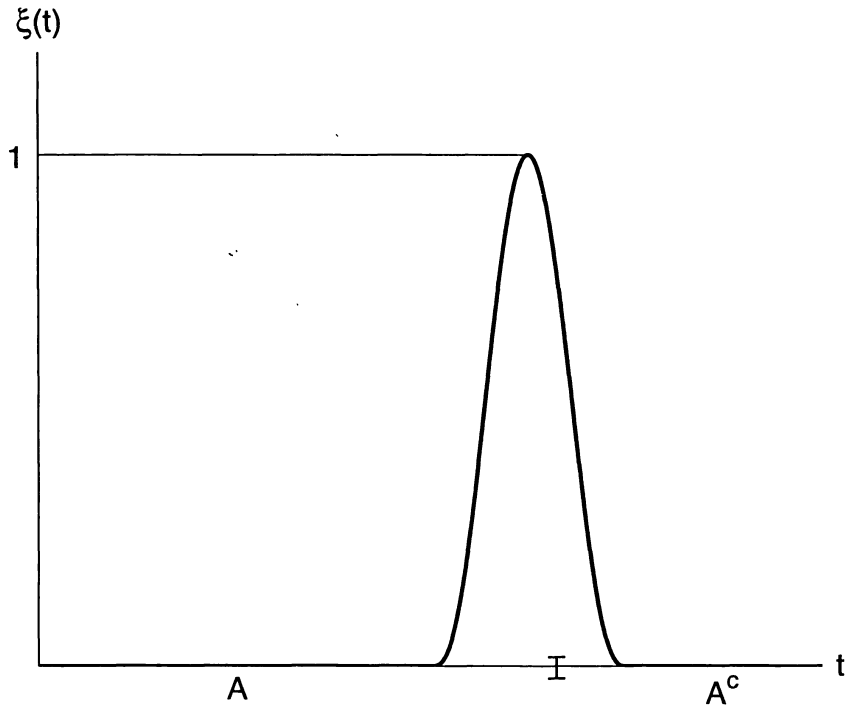

Fig. 4. Non-precise value of a test statistic.

In order to find a solution for this problem the concept of p-values is useful.

Based on a non-precise value $t^\star$ of a test statistic in form of a fuzzy number it is still possible to find a precise p-value. Then the decision can be found in the same way as for precise data where a precise value t of the test statistic is obtained.

The p-value for a non-precise value $t^\star$ with characterizing function $\xi(\cdot)$ is the smallest significance level α at which the hypothesis would be rejected. This significance level can be obtained from the characterizing function $\xi(\cdot)$ which is assumed to have finite support $\text{supp}(\xi(\cdot))$. For non-precise data $x_1^\star, \cdots, x_n^\star$ the p-value is the smallest significance level $\alpha(x_1^\star, \cdots, x_n^\star)$ for which $\text{supp}(\xi(\cdot))$ is included in the rejection region A^c. In Figure 5 this is explained by an example of a one-sided test problem, where the precise number t_0 determines the p-value.

By this construction the test decision can be made in the same way as in case of precise data.

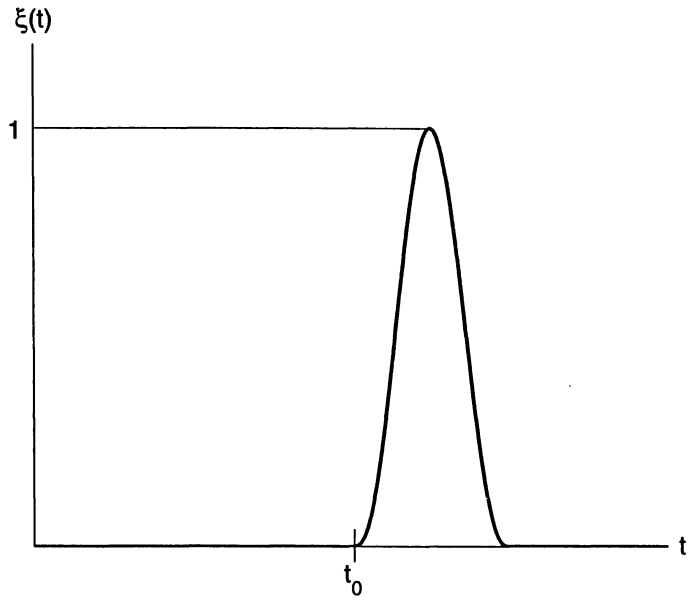

Fig. 5. p-value for a non-precise value of a test statistic.

8. CONCLUSION

In order to obtain realistic results from the analysis of measurements of continuous quantities, the single observations, which are always more or less non-precise, have to be described quantitatively with a suitable mathematical model. This is possible using so-called non-precise numbers and non-precise vectors. The results of such analyses are non-precise numbers which describe adequately the information from measurements. Moreover using the p-value approach to statistical tests allows to test hypotheses concerning mathematical models also for non-precise measurements.

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