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VALIDATION SETS IN FUZZY LOGICS[1]

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The validation set of a formula in a fuzzy logic is the set of all truth values which this formula may achieve. We summarize characterizations of validation sets of S-fuzzy logics and extend them to the case of R-fuzzy logics.

1. BASIC NOTIONS

In order to express vagueness of information, we often enlarge the set $\{0,1\}$ of truth values to the unit interval $[0,1]$, obtaining fuzzy logic systems [1, 3, 8, 9, 20]. Fuzzy logics are naturally linked to the theory of fuzzy sets, where the membership of objects is described by "membership functions" the range of which is the interval $[0,1]$, see [10, 24]. In this paper we study two approaches to fuzzy logics: R-fuzzy logics studied mainly by Hájek [10], and S-fuzzy logics introduced by Butnariu, Klement and Zafrany [1]. We ask which are the sets of possible truth values of formulas in these logics.

Let us recall the basic notions used in the sequel.

Definition 1.1. A (propositional) fuzzy logic is an ordered pair $\mathcal{P} = (\mathcal{L}, \mathcal{Q})$ of a *language (syntax)* $\mathcal{L}$ and a *structure (semantics)* $\mathcal{Q}$ described as follows:

(i) The language of $\mathcal{P}$ is a pair $\mathcal{L} = (A, \mathcal{C})$, where A is a nonempty at most countable set of *atomic symbols* and $\mathcal{C}$ is a tuple of *connectives*.

(ii) The structure of $\mathcal{P}$ is a pair $\mathcal{Q} = ([0,1], \mathcal{M})$, where $[0,1]$ is the set of *truth values*, and the tuple $\mathcal{M}$ consists of the *interpretations (meanings)* of the connectives in $\mathcal{C}$.

For simplicity, we fix the set A of atomic symbols throughout this paper.

The tuple of connectives always will contain at least a conjunction which is interpreted by a *triangular norm* (*t-norm* for short), i. e., a commutative, associative, non-decreasing operation $T\colon [0,1]^2 \to [0,1]$ with neutral element 1 (see [13, 23]).

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Three basic t-norms are the minimum $T_{\mathbf{G}}$, the product $T_{\mathbf{P}}$ and the Łukasiewicz t-norm $T_{\mathbf{L}}$ given, respectively, by $T_{\mathbf{G}}(x,y) = \min(x,y)$, $T_{\mathbf{P}}(x,y) = xy$ and $T_{\mathbf{L}}(x,y) = \max(0, x+y-1)$.

A *triangular conorm* (*t-conorm* for short) is a commutative, associative, non-decreasing operation $S\colon [0,1]^2 \to [0,1]$ with neutral element 0.

There is an obvious duality between t-norms and t-conorms. Let $N_{\mathbf{S}}\colon [0,1] \to [0,1]$ be the *standard negation* defined by $N_{\mathbf{S}}(x) = 1 - x$. For each t-norm T, the function $S_T\colon [0,1]^2 \to [0,1]$ given by

$$S_T(x,y) = N_{\mathbf{S}}(T(N_{\mathbf{S}}(x), N_{\mathbf{S}}(y)))$$

is a t-conorm, called the *dual of* T. The duals of the three important t-norms are the maximum $S_{\mathbf{G}}$, the probabilistic sum $S_{\mathbf{P}}$ and the bounded sum $S_{\mathbf{L}}$ given, respectively, by $S_{\mathbf{G}}(x,y) = \max(x,y)$, $S_{\mathbf{P}}(x,y) = x + y - xy$ and $S_{\mathbf{L}}(x,y) = \min(1, x+y)$.

The class $\mathcal{F}_{\mathcal{P}}$ of *well-formed formulas* in a fuzzy logic $\mathcal{P}$ (*$\mathcal{P}$-formulas* for short) is defined in the standard way, starting from the atomic symbols and constructing new formulas using the connectives. For each function $t\colon A \to [0,1]$ which assigns a truth value to each atomic formula, there exists a unique *natural extension* of t to a *truth assignment* (*evaluation*) $\bar{t}\colon \mathcal{F}_{\mathcal{P}} \to [0,1]$.

All logics studied in this paper have their axiomatizations allowing to define provable formulas (theorems) and formulate and prove completeness theorems (see [1, 10] for more details). Here we concentrate on the properties of validations sets. The *$\mathcal{P}$-validation set* of a $\mathcal{P}$-formula φ is defined as

$$V_{\mathcal{P}}(\varphi) = \{\bar{t}(\varphi) \mid t \in [0,1]^A\}.$$

This paper deals with the question of which validation sets may occur in various fuzzy logics. The section dealing with S-fuzzy logics summarizes the results of [12] for comparison, while the section on R-fuzzy logics is new. Prior to this, let us clarify the situation in classical logic.

Proposition 1.2. Let $\mathcal{C}$ be classical logic. Each $\mathcal{C}$-validation set is of one of the following forms:

- $\{1\}$ iff the formula is a tautology,
- $\{0\}$ iff the negation of the formula is a tautology,
- $\{0,1\}$ otherwise.

As all fuzzy logics considered here extend classical logic (in the sense that all logical operations work on crisp values $\{0,1\}$ classically), each validation set necessarily contains 0 or 1.

2. S–FUZZY LOGICS

The following construction of propositional fuzzy logics was presented in [1]:

Definition 2.1. A *t-norm-based propositional fuzzy logic* (*S-fuzzy logic*) $\mathcal{S}_T$ is a fuzzy logic (in the sense of Definition 1.1) in which the basic connectives are unary $\neg$ (*negation*) and binary $\wedge$ (*conjunction*), interpreted respectively by the standard fuzzy negation N_{S} and a t-norm T.

All S-fuzzy logics $\mathcal{S}_T$ have the same syntax, they differ only by their semantics. The logics corresponding to the basic t-norms $T_{\mathrm{G}}, T_{\mathrm{L}}$ and T_{P} are *Gödel S-fuzzy logic* $\mathcal{S}_{\mathrm{G}}$, *Łukasiewicz S-fuzzy logic* $\mathcal{S}_{\mathrm{L}}$ and *product S-fuzzy logic* $\mathcal{S}_{\mathrm{P}}$.

Starting with the basic logical connectives $\neg$ and $\wedge$, we can define additional logical connectives in an S-fuzzy logic $\mathcal{S}_T$. The *disjunction* $\vee$ is defined by $\varphi \vee \psi = \neg(\neg\varphi \wedge \neg\psi)$; it is interpreted by the t-conorm S_T dual to T.

The *implication* $\rightarrow$ in $\mathcal{S}_T$ is defined as $\varphi \rightarrow \psi = \neg(\varphi \wedge \neg\psi)$; it is interpreted by the binary operation $I_T\colon [0,1]^2 \rightarrow [0,1]$ given by $I_T(x,y) = S_T(N_{\mathrm{S}}(x), y)$, which is often called the *S-implication* induced by the t-norm T.

In S-fuzzy logics different from Łukasiewicz S-fuzzy logic, the false statement cannot be obtained as a (nullary) derived connective, i. e., there is no formula evaluated by the constant function 0 (of course, it may be added to the definition).

Let us summarize results on $\mathcal{S}_T$-validation sets from [1] and [12]:

Theorem 2.2. The validation sets in Gödel S-fuzzy logic $\mathcal{S}_{\mathrm{G}}$ are of one of the following forms:

$$[0, \tfrac{1}{2}]\,, \qquad [\tfrac{1}{2}, 1]\,, \qquad [0,1]\,.$$

The validation sets in product S-fuzzy logic $\mathcal{S}_{\mathrm{P}}$ are of one of the following forms:

$$[0,a]\,, \qquad [b,1]\,, \qquad [0,1]\,,$$

where $a, b \in\,]0,1[$. The validation sets in Łukasiewicz S-fuzzy logic $\mathcal{S}_{\mathrm{L}}$ are of one of the following forms:

$$\{0\}\,, \qquad \{1\}\,, \qquad [0,a]\,, \qquad [b,1]\,, \qquad [0,1]\,,$$

where $a, b \in\,]0,1[$. The possible values of the bounds a, b form a countable dense subset of $[0,1]$.

3. R-FUZZY LOGICS

A reasonable way of constructing connectives in fuzzy logics is to start with a continuous t-norm T and to use the *residuum* (*R-implication*, see [4, 22]) defined by

$$R_T(x,y) = \sup\{z \in [0,1] \mid T(x,z) \le y\}\,. \tag{1}$$

as the interpretation of the implication. It is immediate that we have

$$R_T(x,y) = 1 \qquad \text{if and only if} \qquad x \le y\,.$$

The following approach to fuzzy logics with residual implications is described in detail in [10].

Definition 3.1. A *residuum-based propositional fuzzy logic* (*R-fuzzy logic*) $\mathcal{R}_T$ is a fuzzy logic (in the sense of Definition 1.1) in which the basic connectives are the nullary connective $\mathbf{0}$ (*false statement*) and the binary connectives $\wedge$ (*conjunction*) and $\rightarrow$ (*implication*) with respective interpretations 0, T, R_T, where T is a t-norm and R_T is the corresponding residuum.

Well-formed formulas in an R-fuzzy logic will be called *$\mathcal{R}$-formulas*. Since their definition is independent of T, we omit this index.

The R-fuzzy logics corresponding to the basic t-norms $T_{\mathbf{G}}, T_{\mathbf{L}}$, and $T_{\mathbf{P}}$ are *Gödel R-fuzzy logic* $\mathcal{R}_{\mathbf{G}}$, *Łukasiewicz R-fuzzy logic* $\mathcal{R}_{\mathbf{L}}$, and *product R-fuzzy logic* $\mathcal{R}_{\mathbf{P}}$.

Using the basic logical connectives $\wedge, \rightarrow$ and $\mathbf{0}$, we can define derived logical connectives in an R-fuzzy logic $\mathcal{R}_T$.

The negation $\neg$ in $\mathcal{R}_T$ is defined as an implication with consequence $\mathbf{0}$, i.e., $\neg\varphi = \varphi \rightarrow \mathbf{0}$. Its interpretation is the negation N_T given by $N_T(x) = R_T(x, 0)$. For $T = T_{\mathbf{L}}$, i.e., in Łukasiewicz R-fuzzy logic $\mathcal{R}_{\mathbf{L}}$, we obtain the standard negation $N_{\mathbf{S}}$. For $T_{\mathbf{G}}$ and for all strict t-norms T, we obtain the *Gödel negation*,

$$N_{\mathbf{G}}(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x > 0. \end{cases} \tag{2}$$

In each R-fuzzy logic $\mathcal{R}_T$, the derived binary connective $\vee_{\mathbf{M}}$ defined by

$$\varphi \vee_{\mathbf{M}} \psi = [(\varphi \rightarrow \psi) \rightarrow \psi] \wedge [[(\varphi \rightarrow \psi) \rightarrow \psi] \rightarrow [(\psi \rightarrow \varphi) \rightarrow \varphi]] \tag{3}$$

is evaluated by the maximum, i.e. by $S_{\mathbf{G}}$ (see [10]),

$$\bar{t}(\varphi \vee_{\mathbf{M}} \psi) = \max(\bar{t}(\varphi), \bar{t}(\psi)).$$

Observe that the S-implication $I_{T_{\mathbf{L}}}$ coincides with the R-implication $R_{T_{\mathbf{L}}}$. So the interpretation of logical connectives in Łukasiewicz S-fuzzy logic $\mathcal{S}_{\mathbf{L}}$ and Łukasiewicz R-fuzzy logic $\mathcal{R}_{\mathbf{L}}$ is identical (although not the same connectives are considered as the basic ones). One difference between Łukasiewicz fuzzy logics $\mathcal{R}_{\mathbf{L}}$ and $\mathcal{S}_{\mathbf{L}}$ is that the nullary connective $\mathbf{0}$ is not considered an $\mathcal{S}$-formula. Nevertheless, it can be introduced as a derived logical connective putting, e.g., $\mathbf{0} = \neg\varphi \wedge \varphi$ for a fixed $\mathcal{S}$-formula φ.

In Gödel R-fuzzy logic $\mathcal{R}_{\mathbf{G}}$, the interpretation $R_{\mathbf{G}}$ of the implication is defined by

$$R_{\mathbf{G}}(x, y) = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{otherwise.} \end{cases}$$

The R-implication $R_{\mathbf{G}}$ (called the *Gödel implication*) is not continuous in the points (x, x) with $x \in [0, 1[$.

In product R-fuzzy logic $\mathcal{R}_{\mathbf{P}}$, we obtain the interpretation $R_{\mathbf{P}}$ of the implication defined by

$$R_{\mathbf{P}}(x, y) = \begin{cases} 1 & \text{if } x \leq y, \\ \dfrac{y}{x} & \text{otherwise.} \end{cases}$$

The R-implication $R_{\mathbf{P}}$ (also called the *Goguen implication*) is not continuous in the point $(0, 0)$.

The notion of $\mathcal{R}_T$-validation set depends on the choice of T. In contrast to the situation of S-fuzzy logics (see Section 2), the validation set $V_{\mathcal{R}_T}(\varphi)$ of an $\mathcal{R}$-formula φ in $\mathcal{R}_T$ is not necessarily an interval.

In view of the equivalence of the semantics of Łukasiewicz S- and R-fuzzy logics, we have:

Theorem 3.2. The validation sets in Łukasiewicz R-fuzzy logic $\mathcal{R}_{\mathbf{L}}$ are of one of the following forms:

$$\{0\}\,, \quad \{1\}\,, \quad [0,a]\,, \quad [b,1]\,, \quad [0,1]\,,$$

where $a, b \in \,]0,1[$. The possible values of the bounds a, b form a countable dense subset of $[0,1]$.

In Gödel R-fuzzy logic, the situation becomes different because of the lack of an operation interpreted by the standard fuzzy negation.

Theorem 3.3. The validation sets in Gödel R-fuzzy logic $\mathcal{R}_{\mathbf{G}}$ are of one of the following forms:

$$\{0\}\,, \quad \{1\}\,, \quad \{0,1\}\,, \quad]0,1]\,, \quad [0,1]\,.$$

Proof. First, we prove that all the above-mentioned cases occur. Let p be an atomic symbol. Then

$$\begin{aligned} V_{\mathcal{R}_{\mathbf{G}}}(\mathbf{0}) &= \{0\}\,,\\ V_{\mathcal{R}_{\mathbf{G}}}(\mathbf{0}\to\mathbf{0}) &= \{1\}\,,\\ V_{\mathcal{R}_{\mathbf{G}}}(p\to\mathbf{0}) &= \{0,1\}\,,\\ V_{\mathcal{R}_{\mathbf{G}}}(((p\to\mathbf{0})\to\mathbf{0})\to p) &= \,]0,1]\,,\\ V_{\mathcal{R}_{\mathbf{G}}}(p) &= [0,1]\,. \end{aligned}$$

Second, we have to prove that all $\mathcal{R}_{\mathbf{G}}$-validation sets are of one of the above forms. For this, it is sufficient to prove the following implication:
If φ is an $\mathcal{R}$-formula and t an $\mathcal{R}_{\mathbf{G}}$-evaluation such that $\bar{t}(\varphi) \in \,]0,1[$, then for each $b \in \,]0,1]$ there is an $\mathcal{R}_{\mathbf{G}}$-evaluation t_b such that $\bar{t}_b(\varphi) = b$.

The proof will be done separately for $b \in \,]0,1[$ and for $b = 1$.

First, assume that $b \in \,]0,1[$ and $\bar{t}(\varphi) = a \in \,]0,1[$. We may find an order automorphism (i.e., an increasing bijection) $h\colon [0,1] \to [0,1]$ such that $h(a) = b$. A routine verification shows that h commutes with the interpretations of all basic connectives, i.e.,

$$\begin{aligned} h(0) &= 0\,,\\ h(T_{\mathbf{G}}(a,b)) &= T_{\mathbf{G}}(h(a),h(b))\,,\\ h(R_{\mathbf{G}}(a,b)) &= R_{\mathbf{G}}(h(a),h(b))\,. \end{aligned}$$

We define the evaluation t_b of atomic formulas

$$t_b(p) = h(t(p))\,. \tag{4}$$

The formula

$$\bar{t}_b(\rho) = h(\bar{t}(\rho)) \tag{5}$$

holds for all atomic formulas and also for **0**, because

$$\bar{t}_b(\mathbf{0}) = 0 = h(0) = h(\bar{t}(\mathbf{0})) \; .$$

Suppose that ρ, ψ are formulas for which (5) holds. Then

$$\begin{aligned}\bar{t}_b(\rho \wedge \psi) &= T_{\mathbf{G}}(\bar{t}_b(\rho), \bar{t}_b(\psi)) = T_{\mathbf{G}}(h(\bar{t}(\rho)), h(\bar{t}(\psi))) \\ &= h(T_{\mathbf{G}}(\bar{t}(\rho), \bar{t}(\psi))) = h(\bar{t}(\rho \wedge \psi)) \; , \\ \bar{t}_b(\rho \to \psi) &= R_{\mathbf{G}}(\bar{t}_b(\rho), \bar{t}_b(\psi)) = R_{\mathbf{G}}(h(\bar{t}(\rho)), h(\bar{t}(\psi))) \\ &= h(R_{\mathbf{G}}(\bar{t}(\rho), \bar{t}(\psi))) = h(\bar{t}(\rho \to \psi)) \; ,\end{aligned}$$

thus also $\rho \wedge \psi$ and $\rho \to \psi$ satisfy (5). The latter two equalities are inductive steps which allow us to prove (by induction over the complexity of formulas) that (5) holds for all $\mathcal{R}$-formulas. In particular,

$$\bar{t}_b(\varphi) = h(\bar{t}(\varphi)) = h(a) = b \; .$$

Second, assume that $b = 1$ and $\bar{t}(\varphi) = a \in \,]0,1[$. We proceed analogously to the previous case. We define an order preserving mapping (now not a bijection)

$$h(a) = \begin{cases} 0 & \text{if} \quad a = 0 \; , \\ 1 & \text{if} \quad a \in \,]0,1] \; . \end{cases}$$

Again, h commutes with the interpretations of all basic connectives.

We define an evaluation t_b by (4) and by induction over the complexity of formulas we obtain (5) for all $\mathcal{R}$-formulas. In particular,

$$\bar{t}_b(\varphi) = h(\bar{t}(\varphi)) = h(a) = b = 1 \; .$$

We have proved that whenever an $\mathcal{R}_{\mathbf{G}}$-validation set contains a number from $]0,1[$, it contains the whole $]0,1]$, thus it can be only $]0,1]$ or $[0,1]$. This finishes the proof of the theorem. □

Theorem 3.4. The validation sets in product R-fuzzy logic $\mathcal{R}_{\mathbf{P}}$ are of one of the following forms: $\{0\}\,,\quad \{1\}\,,\quad \{0,1\}\,,\quad]0,1]\,,\quad [0,1]\,.$

P r o o f. The proof follows the method from Theorem 3.3; the only difference is that not all order automorphisms commute with the product t-norm $T_{\mathbf{P}}$. Nevertheless, there are such automorphisms, namely

$$h(a) = a^r \; ,$$

where $r \in \,]0,\infty[$. Then

$$h(T_{\mathbf{P}}(a,b)) = (a \cdot b)^r = a^r \cdot b^r = T_{\mathbf{P}}(h(a), h(b)) \; .$$

(These are the only automorphisms with this property, see [6].) Moreover, these automorphisms commute also with the Goguen implication $R_\mathbf{P}$. Indeed, $R_\mathbf{P}(a,b) = 1$ iff $a \le b$. This condition is equivalent to $h(a) \le h(b)$ and in this case we obtain

$$h(R_\mathbf{P}(a,b)) = h(1) = 1 = R_\mathbf{P}(h(a),h(b)) .$$

In the remaining case, $a > b$, we have $h(a) > h(b)$ and

$$h(R_\mathbf{P}(a,b)) = h\left(\frac{b}{a}\right) = \frac{b^r}{a^r} = \frac{h(b)}{h(a)} = R_\mathbf{P}(h(a),h(b)) .$$

Thus it suffices to take $r = \frac{\log b}{\log a}$ for $b \in]0,1[$; the case of $b = 1$ remains unchanged. Arguments analogous to those of Theorem 3.3 show that the characterization of $\mathcal{R}_\mathbf{P}$-validation sets is the same as that of $\mathcal{R}_\mathbf{G}$-validation sets. □

4. CONCLUDING REMARKS

We gave a characterization of validation sets for the most frequently studied fuzzy logics. Still there are open questions for further study. There are many other fuzzy logics for which characterizations of validation sets are yet unknown. In particular, one might consider logics in which conjunction is interpreted by a t-norm different from the three basic ones used in this paper. We already know that the characterizations of validations sets in logics using a strict t-norm instead of the product remain basically the same. (This is trivial in case of R-fuzzy logics because they are isomorphic to product logic. In S-fuzzy logics, the situation is different as the isomorphism need not preserve the standard negation; still the same results concerning validation sets are obtained.) Recently R-fuzzy logics were studied in which conjunction is interpreted by a continuous t-norm which is an ordinal sum of the basic t-norms (Łukasiewicz and product). Also the case of discontinuous t-norms might be of interest.

Following [18], vector-valued evaluations of series of formulas may be introduced, leading to validation sets that are subsets of vector spaces. This might lead to a substantial generalization related to other questions of satisfiability, compactness, etc.

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REFERENCES

[1] D. Butnariu, E. P. Klement, and S. Zafrany: On triangular norm-based propositional fuzzy logics. Fuzzy Sets and Systems *69* (1995), 241–255.

[2] C. C. Chang: Algebraic analysis of many valued logics. Trans. Amer. Math. Soc. *88* (1958), 467–490.

[3] R. Cignoli, I. M. L. D'Ottaviano, and D. Mundici: Algebraic Foundations of Many-valued Reasoning. (Trends in Logic 7.) Kluwer, Dordrecht 1999.

[4] D. Dubois and H. Prade: A review of fuzzy set aggregation connectives. Inform. Sci. *36* (1985), 85–121.
[5] F. Esteva, L. Godo, P. Hájek, and M. Navara: Residuated fuzzy logics with an involutive negation. Arch. Math. Logic *39* (2000), 103–124.
[6] M. Gehrke, C. Walker, and E. A. Walker: DeMorgan systems on the unit interval Internat. J. Intelligent Syst. *11* (1996), 733–750.
[7] K. Gödel: Zum intuitionistischen Aussagenkalkül. Anz. Österreich. Akad. Wiss. Math.-Natur. Kl. *69* (1932), 65-66.
[8] S. Gottwald: Mehrwertige Logik. Akademie–Verlag, Berlin 1989.
[9] S. Gottwald: Fuzzy Sets and Fuzzy Logic. Foundations of Application – from a Mathematical Point of View. Vieweg, Braunschweig – Wiesbaden 1993.
[10] P. Hájek: Metamathematics of Fuzzy Logic. (Trends in Logic 4.) Kluwer, Dordrecht 1998.
[11] P. Hájek, L. Godo, and F. Esteva: A complete many-valued logic with product-conjunction. Arch. Math. Logic *35* (1996), 191–208.
[12] J. Hekrdla, E. P. Klement, and M. Navara: Two approaches to fuzzy propositional logics. Multiple–valued Logic, accepted.
[13] E. P. Klement, R. Mesiar, and E. Pap: Triangular Norms. (Trends in Logic 8.) Kluwer, Dordrecht 2000.
[14] E. P. Klement and M. Navara: Propositional fuzzy logics based on Frank t-norms: A comparison. In: Fuzzy Sets, Logics and Reasoning about Logics (D. Dubois, E. P. Klement, and H. Prade, eds., Applied Logic Series 15), Kluwer, Dordrecht 1999, pp. 17–38.
[15] E. P. Klement and M. Navara: A survey on different triangular norm-based fuzzy logics. Fuzzy Sets and Systems *101* (1999), 241–251.
[16] C. M. Ling: Representation of associative functions. Publ. Math. Debrecen *12* (1965), 189–212.
[17] J. Łukasiewicz: Bemerkungen zu mehrwertigen Systemen des Aussagenkalküls. Comptes Rendus Séances Société des Sciences et Lettres Varsovie cl. III *23* (1930), 51–77.
[18] M. Navara: Satisfiability in fuzzy logics. Neural Network World *10* (2000), 845–858.
[19] M. Navara: Product Logic is Not Compact. Research Report No. CTU–CMP–2001–09, Center for Machine Perception, Czech Technical University, Prague 2001.
[20] H. T. Nguyen and E. Walker: A First Course in Fuzzy Logic. CRC Press, Boca Raton 1997.
[21] V. Novák: On the syntactico-semantical completeness of first-order fuzzy logic. Part I – Syntactical aspects; Part II – Main results. Kybernetika *26* (1990), 47–66, 134–154.
[22] W. Pedrycz: Fuzzy relational equations with generalized connectives and their applications. Fuzzy Sets and Systems *10* (1983), 185–201.
[23] B. Schweizer and A. Sklar: Probabilistic Metric Spaces. North–Holland, Amsterdam 1983.
[24] L. A. Zadeh: Fuzzy sets. Inform. and Control *8* (1965), 338–353.

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