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INVARIANT COPULAS

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Copulas which are invariant with respect to the construction of the corresponding survival copula and other related dualities are studied. A full characterization of invariant associative copulas is given.

1. INTRODUCTION

The investigation of copulas and their applications is a rather recent subject of mathematics. From one point of view, copulas are functions that join or 'couple' one-dimensional distribution functions and the corresponding joint distribution function. Alternatively, copulas are two-dimensional distribution functions with uniform marginals. Copulas are interesting not only in probability theory and statistics, but also in many other fields requiring the aggregation of incoming data, such as multi-criteria decision making [4], probabilistic metric spaces [6, 17] and the theory of generalized measures and integrals [8]. Associative copulas are special continuous triangular norms (t-norms for short, see [7]), and hence they are applied in several domains where t-norms (t-conorms) play a role. For an extensive overview over copulas we recommend [16].

The invariance of copulas with respect to a given operator plays an important role in several applications, e.g., in fuzzy similarity measurement [3] and in the aggregation based on the Choquet integral [8].

The aim of this contribution is the characterization of copulas which are invariant with respect to the construction of the corresponding survival copula [16] and other related dualities.

The paper is organized as follows. First we recall the notion of copulas and some operators leading to new copulas. Next we discuss the invariance of general copulas with respect to these operators, and finally a full characterization of invariant associative copulas (i.e., of copulas which are also t-norms) is given.

2. COPULAS

The notion of n-copulas was introduced in [18, 19] in the course of the investigation of the relationship between multivariate distribution functions and their marginal distributions [16]. Since we shall restrict ourselves to the two-dimensional case, we shall briefly speak about copulas instead of 2-copulas.

Definition 2.1. A binary operation $C : [0,1]^2 \longrightarrow [0,1]$ is called a *copula* if 1 is its neutral element, 0 is its zero element, and if C is 2-increasing, i. e., if for all $x_1, x_2, y_1, y_2 \in [0,1]$ with $x_1 \leq x_2$ and $y_1 \leq y_2$ we have

$$C(x_1, y_1) + C(x_2, y_2) - C(x_1, y_2) - C(x_2, y_1) \geq 0. \tag{1}$$

The importance of copulas in probability and statistics comes from *Sklar's Theorem* (appearing first in [18], see also [16, Theorem 2.4.1]) which shows that the joint distribution of a random vector and the corresponding marginal distributions are necessarily linked by a copula.

It is interesting that each copula itself can be regarded as a joint distribution (if we extend its domain in a natural way to $[-\infty, \infty]^2$) whose marginal distribution functions coincide with the uniform distribution on $[0,1]$.

This connection between copulas, joint and marginal distributions provides a natural interpretation of property (1): it simply makes sure that the probability of the rectangle $[x_1, x_2] \times [y_1, y_2]$

$$P_C([x_1, x_2] \times [y_1, y_2]) = C(x_1, y_1) + C(x_2, y_2) - C(x_1, y_2) - C(x_2, y_1) \tag{2}$$

is always non-negative.

Observe that each copula is necessarily a continuous binary aggregation operator in the spirit of [10, 11] and that it is always a 1-Lipschitz function. If a copula is associative then it is also commutative and, subsequently, a t-norm [7, 16, 17].

In the framework of copulas, two types of dualities have been introduced [16]: For each copula C, its *dual* $\tilde{C} : [0,1]^2 \longrightarrow [0,1]$ is defined by

$$\tilde{C}(x, y) = x + y - C(x, y), \tag{3}$$

while its *co-copula* $C' : [0,1]^2 \longrightarrow [0,1]$ is given by

$$C'(x, y) = 1 - C(1 - x, 1 - y). \tag{4}$$

Hence the co-copula C' is the dual of the aggregation operator C in the terminology of aggregation operators [1], while the dual $\tilde{C}$ is an operator which is adjoint to C in the spirit of [12] (for more details see [16]).

Further, the dual $\tilde{C}$ or the co-copula C' of a given copula C is never a copula (e. g., if C is also a t-norm then $\tilde{C}$ is the corresponding dual t-conorm and, therefore, not a copula).

Applying consecutively formulae (3) and (4) to a copula C we obtain the binary operation $\hat{C} : [0,1]^2 \longrightarrow [0,1]$ given by

$$\hat{C}(x, y) = x + y - 1 + C(1 - x, 1 - y). \tag{5}$$

Note that the following relations always hold:

$$\hat{C}(x,y) = x + y - C'(x,y),$$
$$\hat{C}(x,y) = 1 - \tilde{C}(1-x, 1-y).$$

Moreover, it is not difficult to check that $\hat{C}$ is always a copula and that the operator $\hat{}$ is involutive, i. e., $(\hat{C})\hat{} = C$. Because of its natural applications in reliability theory, $\hat{C}$ is often called a *survival copula* (see [16]), it also plays an important role in the copula-based integration with respect to fuzzy measures as discussed in [8].

Recall that, for each copula C and each point $(a,b) \in [0,1]^2$, we can define a copula $C_{a,b}$ putting

$$C_{a,b}(x,y) = P_C([a(1-x), x + a(1-x)] \times [b(1-y), y + b(1-y)])$$

(compare [16, Exercise 2.6]), where P_C is defined in (2). In this terminology we obviously have $C_{0,0} = C$ and $C_{1,1} = \hat{C}$ for each copula C.

Two other important involutive operators on the class of all copulas correspond to $C_{0,1}$ and $C_{1,0}$ (see also [4, 7]) given by, respectively,

$$C_{0,1}(x,y) = x - C(x, 1-y), \tag{6}$$
$$C_{1,0}(x,y) = y - C(1-x, y). \tag{7}$$

Observe also that for all $i,j,k,l \in \{0,1\}$ and for each copula C we have

$$(C_{i,j})_{k,l} = C_{i\oplus k, j\oplus l}, \tag{8}$$

where $1 \oplus 1 = 0 \oplus 0 = 0$ and $1 \oplus 0 = 0 \oplus 1 = 1$. Note also that for arbitrary copulas C, D and for all $\lambda \in [0,1]$ we have

$$(\lambda C + (1-\lambda)D)_{i,j} = \lambda C_{i,j} + (1-\lambda)D_{i,j}. \tag{9}$$

Furthermore, for each copula C and for all $(x,y) \in [0,1]^2$

$$C_{0,1}(x,y) = C_{1,0}(y,x). \tag{10}$$

Example 2.2. Consider the copula $C_{\mathbf{H}} : [0,1]^2 \longrightarrow [0,1]$ which, for $(x,y) \neq (0,0)$, is given by

$$C_{\mathbf{H}}(x,y) = \frac{xy}{x + y - xy}$$

and which is called the Hamacher product [7] (it occurs also in connection with the Gumbel bivariate logistic distribution [16]). Then we obtain (supposing $(x,y) \neq$

$(0,0))$:

$$
\begin{aligned}
(C_{\mathbf{H}})\hat{}(x,y) &= \frac{x^2(1-y)+y^2(1-x)+xy}{x+y-xy}, \\
(C_{\mathbf{H}})'(x,y) &= \frac{x+y-2xy}{1-xy}, \\
(C_{\mathbf{H}})_{1,1}(x,y) &= \frac{xy(2-x-y)}{1-xy}, \\
(C_{\mathbf{H}})_{0,1}(x,y) &= \frac{x^2y}{1-y+xy}, \\
(C_{\mathbf{H}})_{1,0}(x,y) &= \frac{xy^2}{1-x+xy}.
\end{aligned}
$$

Observe that $C_{\mathbf{H}}$ is a t-norm, i. e., an associative copula, and that $(C_{\mathbf{H}})_{1,1}$ is commutative but not associative. Obviously, both $(C_{\mathbf{H}})_{0,1}$ and $(C_{\mathbf{H}})_{1,0}$ are not commutative (and, hence, not associative).

3. INVARIANT COPULAS

Much of the usefulness of copulas in nonparametric statistics comes from the fact that, when applying a strictly monotone transformation to random variables, the copulas linking the marginal and joint distributions are either invariant or change in a predictable way [16, Theorems 2.4.3 and 2.4.4]:

Theorem 3.1. Let X and Y be random variables with continuous distribution functions F_X and F_Y, respectively, and corresponding copula C_{XY}, and assume that $\varphi, \psi\colon [0,1] \longrightarrow [0,1]$ are strictly monotone functions. Then we have:

$$
C_{\varphi\circ X,\psi\circ Y} = \begin{cases} (C_{XY})_{0,0} & \text{if } \varphi \text{ and } \psi \text{ are both increasing,} \\ (C_{XY})_{0,1} & \text{if } \varphi \text{ is increasing and } \psi \text{ is decreasing,} \\ (C_{XY})_{1,0} & \text{if } \varphi \text{ is decreasing and } \psi \text{ is increasing,} \\ (C_{XY})_{1,1} & \text{if } \varphi \text{ and } \psi \text{ are both decreasing.} \end{cases}
$$

The coincidence of a copula and the corresponding survival copula is also interesting in the study of symmetric distributions [16, Theorem 2.7.3]:

Theorem 3.2. Let X and Y be continuous random variables with marginal distribution functions F_X and F_Y, respectively, and corresponding copula C_{XY}, and assume that X and Y are symmetric with respect to a and b, respectively, i. e., $F_{X-a} = F_{a-X}$ and $F_{Y-b} = F_{b-Y}$. Then the random vector (X,Y) is radially symmetric with respect to (a,b), i. e., $F_{(X-a,Y-b)} = F_{(a-X,b-Y)}$, if and only if $C_{XY} = (C_{XY})\hat{}$.

In the rest of this paper, we shall investigate the invariance of copulas with respect to three involutive operators introduced in (5) – (7), i. e., to the operators associating with a copula C the copulas $\hat{C}$, $C_{0,1}$ and $C_{1,0}$, respectively. In other words, we are interested in copulas C satisfying $\hat{C} = C$ and/or $C_{0,1} = C$ and/or $C_{1,0} = C$.

Denote by $\mathcal{C}$ the class of all copulas, by $\mathcal{T}$ the class of all associative copulas (i. e., the class of all copulas which are also t-norms), and by $\mathcal{S}$ the class of all commutative (i. e., symmetric) copulas. Moreover, let $\overline{\mathcal{T}}$ be the convex hull of $\mathcal{T}$. Then obviously the following strict inclusions hold:

$$\mathcal{T} \subset \overline{\mathcal{T}} \subset \mathcal{S} \subset \mathcal{C}.$$

Furthermore, for each pair $(i, j) \in \{0, 1\}^2$ let $\mathcal{C}_{i,j}$ be the class of all copulas which are invariant under the corresponding involutive transformation, i. e.,

$$\mathcal{C}_{i,j} = \{C \in \mathcal{C} \mid C_{i,j} = C\}.$$

It is trivial that $\mathcal{C}_{0,0} = \mathcal{C}$.

Theorem 3.3. Let $C \in \mathcal{C}$ be a copula and $(i, j) \in \{0, 1\}^2$. Then we have $C \in \mathcal{C}_{i,j}$ if and only if there is a $D \in \mathcal{C}$ such that $D_{(i,j)} = C$, where

$$D_{(i,j)} = \frac{D + D_{i,j}}{2}.$$

P r o o f. If such a $D \in \mathcal{C}$ exists then, applying (8) and (9), we obtain

$$C_{i,j} = \left(\frac{D + D_{i,j}}{2}\right)_{i,j} = \frac{D_{i,j} + D_{0,0}}{2} = C.$$

Conversely, if $C = C_{i,j}$ then we may put $D = C$ because of $C = \frac{C+C_{i,j}}{2}$. □

It is immediate that, for each $(i, j) \in \{0, 1\}^2$, the class $\mathcal{C}_{i,j}$ is a convex subset of $C([0, 1]^2)$ (denoting the class of continuous functions on $[0, 1]^2$) which is also closed with respect to the uniform topology, while the class $\mathcal{T}$ is not convex. Recall also that the Łukasiewicz t-norm $T_{\mathbf{L}}$ (which is given by $T_{\mathbf{L}}(x, y) = \max(0, x + y - 1)$) and the minimum $T_{\mathbf{M}}$ (which are also called the Fréchet–Hoeffding bounds) are the smallest and the greatest element, respectively, of each of the classes $\mathcal{C}, \mathcal{C}_{1,1}$ and $\mathcal{T}$. However, because of $(T_{\mathbf{L}})_{0,1} = (T_{\mathbf{L}})_{1,0} = T_{\mathbf{M}}$, neither $T_{\mathbf{L}}$ nor $T_{\mathbf{M}}$ are contained in $\mathcal{C}_{1,0}$ or in $\mathcal{C}_{0,1}$. By virtue of Theorem 3.3, the copula

$$K = \frac{T_{\mathbf{M}} + T_{\mathbf{L}}}{2} \tag{11}$$

is contained in $\mathcal{C}_{i,j}$ for each $(i, j) \in \{0, 1\}^2$, but not in $\mathcal{T}$. Observe also that, as a consequence of (10), we have $\mathcal{S} \cap \mathcal{C}_{0,1} = \mathcal{S} \cap \mathcal{C}_{1,0}$.

Example 3.4. We consider again the copula $C_{\mathbf{H}}$ as in Example 2.2. Then we have

$$(C_{\mathbf{H}})_{(i,j)} = \frac{C_{\mathbf{H}} + (C_{\mathbf{H}})_{i,j}}{2} \in \mathcal{C}_{i,j},$$

where

$$\begin{aligned}
(C_{\mathbf{H}})_{(1,1)}(x,y) &= \frac{xy[2(1-xy) + (1-x)(1-y)(x+y-1)]}{2(1-xy)(x+y-xy)},\\
(C_{\mathbf{H}})_{(0,1)}(x,y) &= \frac{xy(1-y+2xy+x^2-x^2y)}{2(x+y-xy)(1-y+xy)},\\
(C_{\mathbf{H}})_{(1,0)}(x,y) &= \frac{xy(1-x+2xy+y^2-xy^2)}{2(x+y-xy)(1-x+xy)}.
\end{aligned}$$

Observe that the commutativity of a copula C implies the commutativity of $C_{1,1}$, and consequently we have $\frac{C+C_{1,1}}{2} \in \mathcal{S} \cap \mathcal{C}_{1,1}$ for each $C \in \mathcal{S}$. However, the opposite is not true in general:

Example 3.5. For each $\alpha \in [1, \infty[$ the function $C_\alpha : [0,1]^2 \longrightarrow [0,1]$ given by

$$C_\alpha(x,y) = xy + x^\alpha y(1-x)(1-y) \tag{12}$$

is a copula, which is commutative only if $\alpha = 1$. However, for $\alpha = 2$ the copula $(C_2)_{(1,1)} \in \mathcal{C}_{1,1}$ given by

$$(C_2)_{(1,1)} = \frac{T_{\mathbf{P}} + C_1}{2}$$

is commutative.

Next, we want to investigate the class of copulas which are invariant with respect to all three involutive operators (5)–(7), i. e., the class

$$\mathcal{C}^* = \mathcal{C}_{0,1} \cap \mathcal{C}_{1,0} \cap \mathcal{C}_{1,1}.$$

The following result is a consequence of (8).

Lemma 3.6. A copula C is invariant with respect to two of the operators defined in (5), (6) and (7) if and only if C is invariant with respect to all three operators (5)–(7), i. e.,

$$\mathcal{C}^* = \mathcal{C}_{0,1} \cap \mathcal{C}_{1,0} = \mathcal{C}_{0,1} \cap \mathcal{C}_{1,1} = \mathcal{C}_{1,0} \cap \mathcal{C}_{1,1}.$$

Moreover, we have

$$\mathcal{C}_{0,1} \cap \mathcal{S} = \mathcal{C}_{1,0} \cap \mathcal{S} \subset \mathcal{C}^*.$$

Theorem 3.7. Let C be a copula. Then we have $C \in \mathcal{C}^*$ if and only if there is a copula $D \in \mathcal{C}$ such that $D^* = C$, where

$$D^* = \frac{D + D_{0,1} + D_{1,0} + D_{1,1}}{4}.$$

Proof. The necessity is obvious. In order to show the sufficiency assume first $(i,j) = (0,1)$, in which case (8) and (9) imply

$$(D^*)_{0,1} = \Big(\frac{D + D_{0,1} + D_{1,0} + D_{1,1}}{4}\Big)_{0,1} = \frac{D_{0,1} + D_{0,0} + D_{1,1} + D_{1,0}}{4} = D^*.$$

The other cases can be handled analogously. □

Evidently, the copula $K = \frac{T_{\mathbf{M}}+T_{\mathbf{L}}}{2}$ considered in (11) belongs to $\mathcal{C}^*$, and we also have $T_{\mathbf{P}} \in \mathcal{C}^*$. Moreover, for all $C \in \mathcal{C}$ we have

$$C^* = (C^*)_{0,1} = (C^*)_{1,0} = (C^*)_{1,1}.$$

Example 3.8. We consider again the family of copulas $(C_\alpha)_{\alpha \in [1,\infty[}$ as introduced in (12).

(i) Putting

$$(C_\alpha)^* = \frac{C_\alpha + (C_\alpha)_{0,1} + (C_\alpha)_{1,0} + (C_\alpha)_{1,1}}{4},$$

then for for each $\alpha \in [1,\infty[$ we have $(C_\alpha)^* = T_{\mathbf{P}}$.

(ii) The class $\mathcal{C}^*$ also contains non-commutative copulas: for the ordinal sum (see [7]) $D = (\langle 0, 0.5, C_2 \rangle)$ given by

$$D(x,y) = \begin{cases} 2xy + 4x^2y(1-2x)(1-2y) & \text{if } (x,y) \in [0,0.5]^2, \\ \min(x,y) & \text{otherwise,} \end{cases}$$

we have $D^* \in \mathcal{C}^*$ but, e.g., $D^*(0.2, 0.4) \neq D^*(0.4, 0.2)$.

Remark 3.9. For copulas C_1 and C_2 the *$*$-product* $C_1 * C_2 \colon [0,1]^2 \longrightarrow [0,1]$ (see [2, 16]) is defined by

$$C_1 * C_2(x,y) = \int_0^1 \frac{\partial C_1(x,t)}{\partial t} \cdot \frac{\partial C_2(t,y)}{\partial t}\, \mathrm{d}t.$$

The function $C_1 * C_2$ is well-defined since the partial derivatives exist almost everywhere, and it is always a copula, i.e., the $*$-product is a binary operation on $\mathcal{C}$. Moreover, $(\mathcal{C}, *)$ is a non-commutative semigroup whose annihilator is the product t-norm $T_{\mathbf{P}}$ and whose neutral element is the minimum $T_{\mathbf{M}}$ [13]. We also have $T_{\mathbf{L}} * T_{\mathbf{L}} = T_{\mathbf{M}}$, and for each copula C it is possible to show that $T_{\mathbf{L}} * C = C_{1,0}$, $C * T_{\mathbf{L}} = C_{0,1}$ and $T_{\mathbf{L}} * C * T_{\mathbf{L}} = C_{1,1}$, and that $C \in \mathcal{C}_{1,1}$ if and only if $T_{\mathbf{L}} * C = C * T_{\mathbf{L}}$.

This allows us to give another representation of the copulas $D_{(i,j)}$ and D^* used in Theorems 3.3, 3.7: $D_{(1,0)} = K * D$, $D_{(0,1)} = D * K$, and $D^* = K * D * K$, where the copula K is given by (11). In other words, $\mathcal{C}_{1,0}$ is a right ideal and $\mathcal{C}_{0,1}$ is a left ideal of the semigroup $(\mathcal{C}, *)$.

4. INVARIANT ASSOCIATIVE COPULAS

Theorems 3.3 and 3.7 give a hint how to construct invariant copulas. Unfortunately, neither for the class $\mathcal{C}$ of all copulas nor for the class of all invariant copulas a full characterization is known so far. However, associative copulas, i. e., elements of the class $\mathcal{T}$, are characterized as ordinal sums of t-norms generated by convex additive generators [7, 16]. Therefore, in this section we focus on the characterization of the classes $\mathcal{T}_{i,j} = \mathcal{T} \cap \mathcal{C}_{i,j}$. Because of the convexity of $\mathcal{C}_{i,j}$ then obviously also the convex closure $\overline{\mathcal{T}_{i,j}}$ of $\mathcal{T}_{i,j}$ is a subset of $\mathcal{C}_{i,j}$.

Considering first the case $(i,j) = (1,1)$ we have the following result:

Lemma 4.1. For each copula C we have:

(i) $C \in \mathcal{C}_{1,1}$ if and only if $\tilde{C} = C'$.

(ii) C is associative if and only if C' is associative.

Recall the family of Frank t-norms $(T^{\mathbf{F}}_\lambda)_{\lambda \in [0,\infty]}$ (all of them are copulas too) which, together with their ordinal sums, were shown in [5] to be the only t-norms T solving the functional equation

$$T(x,y) + S(x,y) = x + y, \tag{13}$$

for some t-conorm S (not necessarily the one which is dual to T in the sense of (4)). As a consequence of this result we immediately get the following:

Proposition 4.2. Let C be a copula. Then we have $C \in \mathcal{T}_{1,1}$ if and only if there is a $\lambda \in [0,\infty]$ such that $C = T^{\mathbf{F}}_\lambda$ or if C is an ordinal sum of Frank t-norms of the form

$$C = (\langle a_k, b_k, T^{\mathbf{F}}_{\lambda_k} \rangle)_{k \in K},$$

where for each $k \in K$ there is a $k' \in K$ such that $\lambda_k = \lambda_{k'}$ and $a_k + b_{k'} = b_k + a_{k'} = 1$.

Proof. If C is an associative copula, then C is a t-norm and C' is a t-conorm, and (5) can be rewritten as

$$\hat{C}(x,y) + C'(x,y) = x + y. \tag{14}$$

Then $\hat{C} = C$ if and only if C solves the functional equation (13) and $\tilde{C} = C'$, and the result follows (see also [7]). □

Denote by $\mathcal{F}$ the class of Frank copulas and their ordinal sums which are contained in $\mathcal{T}_{1,1}$, as described in Proposition 4.2, and by $\overline{\mathcal{F}}$ the convex hull of $\mathcal{F}$. Because of the convexity of $\mathcal{C}_{1,1}$, $\overline{\mathcal{F}}$ is a subclass of $\mathcal{C}_{1,1}$. For example, the family $(C^{\mathbf{M}}_\lambda)_{\lambda \in [-1,1]}$ of Mardia copulas [16] given by

$$C^{\mathbf{M}}_\lambda = \tfrac{\lambda^2(1+\lambda)}{2} \cdot T_{\mathbf{M}} + (1-\lambda^2) \cdot T_{\mathbf{P}} + \tfrac{\lambda^2(1-\lambda)}{2} \cdot T_{\mathbf{L}}.$$

is a subfamily of $\overline{\mathcal{F}}$. However, there are copulas in $\mathcal{C}_{1,1}$ which do not belong to $\overline{\mathcal{F}}$, i. e., $\overline{\mathcal{F}}$ is a proper subset of $\mathcal{C}_{1,1}$.

As a consequence of (10), we immediately obtain the following lemma.

Lemma 4.3. $\mathcal{T}_{0,1} = \mathcal{T}_{1,0}$.

Proof. For each $T \in \mathcal{T}_{0,1}$, T is an associative and, subsequently, commutative copula satisfying $T = T_{0,1}$. Then for all $(x, y) \in [0,1]^2$ we have

$$T(x,y) = T(y,x) = T_{0,1}(y,x) = T_{1,0}(x,y),$$

i.e., $T \in \mathcal{T}_{1,0}$. The converse is straightforward. □

Next, given a t-norm T and an increasing bijection $\varphi : [0,1] \longrightarrow [0,1]$, the transformed t-norm $T_\varphi : [0,1]^2 \longrightarrow [0,1]$ is defined by

$$T_\varphi(x,y) = \varphi^{-1}(T(\varphi(x), \varphi(y))).$$

Starting from the family of Frank t-norms $(T_\lambda^{\mathbf{F}})_{\lambda \in [0,\infty]}$, define for each $\lambda \in [0, \infty]$ (using the standard conventions $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$)

$$F_{(\lambda)} = \frac{T_\lambda^{\mathbf{F}} + T_{\frac{1}{\lambda}}^{\mathbf{F}}}{2}. \tag{15}$$

The following important result was proved in [4].

Proposition 4.4. Let T_1, T_2 be two continuous t-norms. Then the equality

$$T_1(x,y) + T_2(x, 1-y) = x$$

holds for all $(x, y) \in [0,1]^2$ if and only if there exist a number $\lambda \in [0, \infty]$ and an increasing bijection $\varphi : [0,1] \longrightarrow [0,1]$ such that $\varphi(x) + \varphi(1-x) = 1$ for all $x \in [0,1]$ and $T_1 = (T_\lambda^{\mathbf{F}})_\varphi$ and $T_2 = \left(T_{\frac{1}{\lambda}}^{\mathbf{F}}\right)_\varphi$.

Observe that this means that the t-norms T_1, T_2 in Proposition 4.4 are nearly Frank t-norms as introduced in [15] (see also [14]).

Now we are able to characterize the class $\mathcal{T}_{0,1} (= \mathcal{T}_{1,0})$.

Theorem 4.5. $\mathcal{T}_{0,1} = \mathcal{T}_{1,0} = \{T_{\mathbf{P}}\}$.

Proof. Because of Lemmas 3.6 and 4.3 we have

$$\mathcal{T}_{0,1} = \mathcal{T}_{1,0} = \mathcal{T}_{0,1} \cap \mathcal{T}_{1,0} = \mathcal{T}_{0,1} \cap \mathcal{T}_{1,1}.$$

Combining Propositions 4.2 and 4.4 we see that the only copula contained in both $\mathcal{T}_{1,1}$ and $\mathcal{T}_{0,1}$ is just the product $T_{\mathbf{P}}$. □

Observe that, because of Proposition 4.4, for each $\lambda \in [0, \infty]$ we have

$$T_\lambda^{\mathbf{F}}(x,y) = x - T_{\frac{1}{\lambda}}^{\mathbf{F}}(x, 1-y),$$

i.e., $(T_\lambda^{\mathbf{F}})_{0,1} = T_{\frac{1}{\lambda}}^{\mathbf{F}}$. Recalling the definition of $F_{(\lambda)}$ given in (15), this means $F_{(\lambda)} \in \mathcal{C}_{0,1}$ for each $\lambda \in [0, 1]$. Due to the convexity of the class $\mathcal{C}_{1,1}$ we also get $F_{(\lambda)} \in \mathcal{C}_{1,1}$. Therefore we obtain the following subset of $\mathcal{C}^*$.

Corollary 4.6. The convex hull of the set $\{F_{(\lambda)} \mid \lambda \in [0,1]\}$ is a subset of $\mathcal{C}^*$.

Example 4.7.

(i) Observe first that $F_{(\lambda)} = F_{(\frac{1}{\lambda})}$ for all $\lambda \in [0,\infty]$ and, therefore, $F_{(0)} = F_{(\infty)} = K$, a copula already known as being invariant with respect to all three operators (5) - (7).

(ii) The ordinal sum $C = (\langle 0, 0.5, T_{\mathbf{L}} \rangle)$ solves the functional equation (13). However, we have $C \notin \mathcal{C}_{1,1} \cup \mathcal{C}_{0,1} \cup \mathcal{C}_{1,0}$ since for all $(x,y) \in [0,1]^2$ (compare Figure 1)

$$\begin{aligned}
C(x,y) &= \max(0, \min(x, y, x+y-0.5)),\\
C_{1,0}(x,y) &= \min(y, \max(0, x+y-1, x-0.5)),\\
C_{1,1}(x,y) &= \max(x+y-1, \min(x, y, 0.5)),\\
C_{0,1}(x,y) &= \min(x, \max(0, x+y-1, y-0.5)).
\end{aligned}$$

This shows that there are copulas C solving the functional equation (13) which are not invariant in the sense of this note. That is, where the copulas C, $C_{1,0}$, $C_{1,1}$, and $C_{0,1}$ are pairwise distinct.

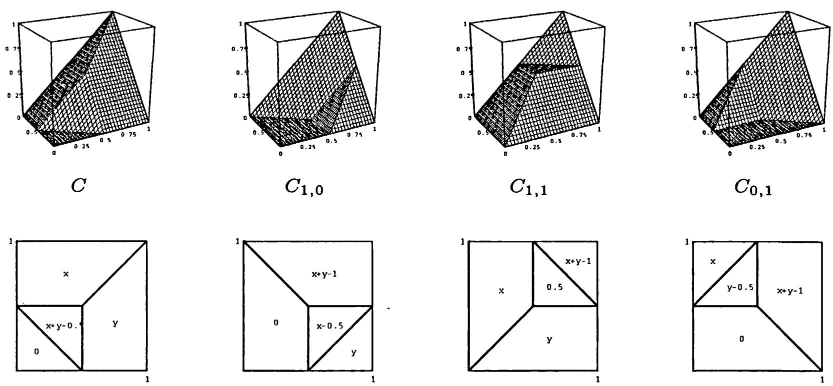

Fig. 1. 3D plots, domains and values of the copulas C, $C_{1,0}$, $C_{1,1}$, and $C_{0,1}$ in Example 4.7 (ii).

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