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MÖBIUS FITTING AGGREGATION OPERATORS

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Standard Möbius transform evaluation formula for the Choquet integral is associated with the **min**-aggregation. However, several other aggregation operators replacing **min** operator can be applied, which leads to a new construction method for aggregation operators. All binary operators applicable in this approach are characterized by the 1-Lipschitz property. Among ternary aggregation operators all 3-copulas are shown to be fitting and moreover, all fitting weighted means are characterized. This new method allows to construct aggregation operators from simpler ones.

1. INTRODUCTION

The aim of the paper is to present a new construction method of aggregation operators which is a generalization of the aggregation based on the Choquet integral.

Fix $n \in \mathbb{N}$. Let $(x_1, \ldots, x_n) \in [0,1]^n$ be an input vector of values to be aggregated. To this end, consider the space $\mathbb{X} = \{1, 2, \ldots, n\}$, define the function $f : \mathbb{X} \to [0,1]$, $f(i) = x_i$, representing the input system, and choose a fuzzy measure $m : \mathcal{P}(\mathbb{X}) \to [0,1]$, that is, a non-decreasing set function with $m(\emptyset) = 0$ and $m(\mathbb{X}) = 1$, [13, 15]. It is a well-known fact that the mapping $\mathbf{C}_m : [0,1]^n \to [0,1]$ given by

$$\mathbf{C}_m(x_1, \ldots, x_n) = C - \int_{\mathbb{X}} f \, \mathrm{d}m \tag{1}$$

where the integral on the right-hand side is the Choquet integral of f with respect to m [5], is an n-ary aggregation operator. Due to the properties of the Choquet integral, (1) can be written in the form

$$\mathbf{C}_m(x_1, \ldots, x_n) = \sum_{I \subseteq \mathbb{X}} M_m(I) \min_{i \in I} x_i \,, \tag{2}$$

where $M_m : \mathcal{P}(\mathbb{X}) \to \mathbb{R}$,

$$M_m(I) = \sum_{J \subseteq I} (-1)^{|I \setminus J|} m(J)$$

is the Möbius transform of the fuzzy measure m, see [4, 11].

If we considered in (1) the dual measure m^d to m, we should obtain the aggregation operator

$$\mathbf{C}_{m^d}(x_1,\dots,x_n)=\sum_{I\subseteq \mathbf{X}} M_m(I)\max_{i\in I} x_i\,. \tag{3}$$

For any fuzzy measure m also the mapping $\mathbf{P}_m:[0,1]^n\to[0,1]$,

$$\mathbf{P}_m(x_1,\dots,x_n)=\sum_{I\subseteq \mathbf{X}} M_m(I)\prod_{i\in I} x_i\,, \tag{4}$$

is an aggregation operator (so-called Lovász extension of fuzzy measure) see, e. g., [7].

The minimum, maximum and the product of values x_i in (2), (3) and (4), respectively, can be understood as the values of aggregation operators **min**, **max** and the product operator Π. This offers a possibility to construct a new aggregation operator by means of the Möbius transform of a fuzzy measure m and an aggregation operator $\mathbf{A}$. However, if we substitute any aggregation operator $\mathbf{A}$ instead of **min**, **max** or Π, the resulting operator defined in an analogous way as in (2) – (4), need not be an aggregation operator. Further, mention that all considered operators **min**, **max** and Π are symmetric aggregation operators. If we considered only symmetric aggregation operators $\mathbf{A}$ in the proposed construction method, some important aggregation operators would be immediately excluded from our considerations, for instance, weighted means. Therefore we will work with any aggregation operators, not necessarily symmetric, providing that the values will always be aggregated in the natural order of indexes, that is, we will define a new operator by

$$\mathbf{C}_{m,\mathbf{A}}(x_1,\dots,x_n)=\sum_{I\subseteq \mathbf{X}} M_m(I)\mathbf{A}(\mathbf{x}_I), \tag{5}$$

where $\mathbf{x}_I=(x_{i_1},\dots,x_{i_k})$, $I=\{i_1,\dots,i_k\}$, $i_1<\dots<i_k$.

The question is whether there are also some other aggregation operators different from **min**, **max** and Π, which combined with Möbius transform of any fuzzy measure as in (5) give an aggregation operator.

2. MÖBIUS–FITTING AGGREGATION OPERATORS

We start with recalling the definition of an aggregation operator [2, 8].

Definition 1. An aggregation operator is a mapping $\mathbf{A}:\bigcup_{n\in\mathbb{N}}[0,1]^n\to[0,1]$ with the following properties:

(i) $\mathbf{A}(\underbrace{0,\dots,0}_{n\text{-times}})=0\,,\ \mathbf{A}(\underbrace{1,\dots,1}_{n\text{-times}})=1$ for all $n\in\mathbb{N}$,

(ii) $\mathbf{A}(x_1,\dots,x_n)\le\mathbf{A}(y_1,\dots,y_n)$ for all $(x_1,\dots,x_n)\,,\ (y_1,\dots,y_n)\in[0,1]^n$, $n\in\mathbb{N}$, such that $x_i\le y_i$, $i=1,\dots,n$.

(iii) $\mathbf{A}(x) = x$ for each $x \in [0,1]$.

Note that an aggregation operator defined in this sense will be also called a global aggregation operator. An n-ary aggregation operator is defined only for a fixed number of inputs, that is, it is a $[0,1]^n \to [0,1]$ mapping with the properties as in Definition 1. A global aggregation operator corresponds to an infinite sequence of n-ary aggregation operators.

To obtain in (5) an n-ary aggregation operator $\mathbf{C}_{m,\mathbf{A}}$, on the right-hand side we need to know unary, binary, ... , n-ary forms of $\mathbf{A}$. That is, $\mathbf{A}$ there represents a finite sequence of p-ary aggregation operators $\left(\mathbf{A}_{(p)}\right)_{p=1}^{n}$. All of them will be denoted only by $\mathbf{A}$.

Definition 2. (i) Let $n \in \mathbb{N}$. We say that $\mathbf{A}$ is an (M,n)-fitting aggregation operator if the operator $\mathbf{C}_{m,\mathbf{A}}$ defined by (5) is an n-ary aggregation operator for each fuzzy measure m.

(ii) An aggregation operator $\mathbf{A}$ is said to be M-fitting if it is (M,n)-fitting for each $n \in \mathbb{N}$.

Note that M-fitting can be read as Möbius-fitting.

It is clear that in the part (i) of Definition 2 the aggregation operator $\mathbf{A}$ has to be known as p-ary aggregation operator for $p = 1, 2, \ldots, n$, while the second part of this definition has sense only for global aggregation operators.

For aggregation operators **min**, **max** and Π, the operator defined by (5) for an arbitrary number n of inputs is an n-ary aggregation operator, therefore **min**, **max** and Π are M-fitting aggregation operators.

Example 1. The Łukasievicz t-norm is $(M,2)$-fitting, but not M-fitting.

Let $\mathbf{A} = \mathbf{T}_L$, be the Łukasiewicz t-norm, which for $n \geq 2$ is defined by $\mathbf{T}_L(x_1, \ldots, x_n) = \max(0, \sum x_i - n + 1)$, $(x_1, \ldots, x_n) \in [0,1]^n$.

We first show that $\mathbf{T}_L$ is $(M,2)$-fitting. Let m be any fuzzy measure on $\mathbb{X} = \{1,2\}$, with $m(\{1\}) = a$, $m(\{2\}) = b$, for some a, $b \in [0,1]$. Then by (5)

$$\mathbf{C}_{m,\mathbf{T}_L}(x,y) = ax + by + (1-a-b)\max(0, x+y-1),$$

that is,

$$\mathbf{C}_{m,\mathbf{T}_L}(x,y) = \begin{cases} ax + by & \text{if } x + y \leq 1, \\ (1-b)x + (1-a)y - (1-a-b) & \text{if } x + y \geq 1. \end{cases} \tag{6}$$

The operator $\mathbf{C}_{m,\mathbf{T}_L}$ satisfies boundary conditions. Due to (6), it is evidently non-decreasing on the triangles $\Delta_1 = \{(x,y) \in [0,1]^2 \mid x + y \leq 1\}$ and $\Delta_2 = \{(x,y) \in [0,1]^2 \mid x+y \geq 1\}$. For points $(x_1,y_1) \in \Delta_1$, $(x_2,y_2) \in \Delta_2$ such that $x_1 \leq x_2$, $y_1 \leq y_2$ we have

$$\begin{aligned} \mathbf{C}_{m,\mathbf{T}_L}(x_2,y_2) - \mathbf{C}_{m,\mathbf{T}_L}(x_1,y_1) &= (\mathbf{C}_{m,\mathbf{T}_L}(x_2,y_2) - \mathbf{C}_{m,\mathbf{T}_L}(x_0,y_0)) \\ &\quad + (\mathbf{C}_{m,\mathbf{T}_L}(x_0,y_0) - \mathbf{C}_{m,\mathbf{T}_L}(x_1,y_1)) \geq 0, \end{aligned}$$

where $(x_0, y_0) = (x_0, 1 - x_0)$ is a cross point of the lines linking the points $(0,1)$ and $(1,0)$, and (x_1, y_1) and (x_2, y_2), respectively. Therefore the operator $\mathbf{C}_{m,\mathbf{T}_L}$ is non-decreasing on $[0,1]^2$, and by Definition 1 it is a binary aggregation operator, which means that $\mathbf{T}_L$ is $(M,2)$-fitting.

Now, put $n = 3$ and consider the maximal fuzzy measure $m^* : \mathcal{P}(\mathbb{X}) \to [0,1]$, defined by $m^*(I) = 1$ for all $I \neq \emptyset$ and $m^*(\emptyset) = 0$. According to (5) we obtain:

$$\begin{aligned} \mathbf{C}_{m^*,\mathbf{T}_L}(x,y,z) &= x + y + z - \max(0, x + y - 1) - \max(0, x + z - 1) \\ &\quad - \max(0, y + z - 1) + \max(0, x + y + z - 2). \end{aligned}$$

For $x = y = 0.5$ and $z = .0.6$ we have $\mathbf{C}_{m^*,\mathbf{T}_L}(0.5, 0.5, 0.6) = 1.4$ which means that $\mathbf{C}_{m^*,\mathbf{T}_L}$ is not an aggregation operator. So, $\mathbf{T}_L$ is not $(M,3)$-fitting, and consequently, it is not M-fitting. □

Similarly, it can be shown that also the projections to the first and last coordinates, $\mathbf{P}_F$ and $\mathbf{P}_L$, respectively, are M-fitting.

To show that a given aggregation operator $\mathbf{A}$ is (M,n)-fitting means to examine whether for all fuzzy measures (5) gives an n-ary aggregation operator. The problem can be simplified using the fact that each fuzzy measure m can be expressed as a convex combination of $\{0,1\}$-fuzzy measures m_i, $i = 1, \dots, k$, that is

$$m = \sum_{i=1}^{k} c_i m_i, \quad c_i \geq 0, \quad \sum_{i=1}^{k} c_i = 1,$$

see [14].

In general, if a fuzzy measure m is a convex combination of fuzzy measures m_i, then for its Möbius transform we have

$$\begin{aligned} M_m(I) &= \sum_{J \subseteq I} (-1)^{|I \setminus J|} \left(\sum_{i=1}^{k} c_i m_i(J) \right) = \sum_{i=1}^{k} c_i \left(\sum_{J \subseteq I} (-1)^{|I \setminus J|} m_i(J) \right) \\ &= \sum_{i=1}^{k} c_i M_{m_i}(I) \end{aligned}$$

and consequently, for the operator $\mathbf{C}_{m,\mathbf{A}}$ corresponding to the considered convex combination of measures we obtain

$$\begin{aligned} \mathbf{C}_{m,\mathbf{A}}(x_1, \dots, x_n) &= \sum_{I \subseteq \mathbf{X}} M_m(I) \mathbf{A}(\mathbf{x}_I) = \sum_{I \subseteq \mathbf{X}} \left(\sum_{i=1}^{k} c_i M_{m_i}(I) \right) \mathbf{A}(\mathbf{x}_I) \\ &= \sum_{i=1}^{k} c_i \mathbf{C}_{m_i,\mathbf{A}}(x_1, \dots, x_n). \end{aligned} \tag{7}$$

Moreover, each convex combination of aggregation operators is again an aggregation operator. Due to (7), for a given $\mathbf{A}$, the operator $\mathbf{C}_{m,\mathbf{A}}$ is an aggregation operator for each m if and only if all operators $\mathbf{C}_{m_i,\mathbf{A}}$ are aggregation operators. Note that the classes of all M-fitting and (M,n)-fitting aggregation operators are convex sets, therefore for example, the operator $\mathbf{A} = \lambda\,\mathbf{min} + (1-\lambda)\mathbf{max}$ for $\lambda \in [0,1]$ is M-fitting. Both classes are closed with respect to the standard duality for aggregation operators, therefore for example, the probabilistic sum $\mathbf{S}_P$ is also M-fitting.

3. $(M,2)$–FITTING AGGREGATION OPERATORS

In the next part let us consider that $n = 2$. Then there are four possible $\{0,1\}$-fuzzy measures, and due to the previous facts, an aggregation operator $\mathbf{A}$ is $(M,2)$-fitting if and only if the operators $\mathbf{C}_{m_i,\mathbf{A}}$, $i = 1,\dots,4$, are aggregation operators. The values of $\{0,1\}$-fuzzy measures for $n = 2$, together with values of their Möbius transform and the form of the corresponding operator $\mathbf{C}_{m_i,\mathbf{A}}$ are in Table 1.

Table 1.

I	$\emptyset$	$\{x\}$	$\{y\}$	$\{x,y\}$	$\mathbf{C}_{m_i,\mathbf{A}}$
m_1	0	1	1	1	
M_{m_1}	0	1	1	-1	$x + y - \mathbf{A}(x,y)$
m_2	0	0	0	1	
M_{m_2}	0	0	0	1	$\mathbf{A}(x,y)$
m_3	0	1	0	1	
M_{m_3}	0	1	0	0	x
m_4	0	0	1	1	
M_{m_4}	0	0	1	0	y

As we can see, the only case which has to be investigated is that for the maximal fuzzy measure $m^* = m_1$, because all other obtained operators are aggregation operators. Let us denote the operator $\mathbf{C}_{m_1,\mathbf{A}} = \mathbf{A}^*$, that is, $\mathbf{A}^*$ will be defined by

$$\mathbf{A}^*(x,y) = x + y - \mathbf{A}(x,y). \tag{8}$$

Before going further, recall that an n-ary aggregation operator has the *Lipschitz property* with constant $c \in]0,\infty[$ if

$\forall (x_1,\dots,x_n),\ (y_1,\dots,y_n) \in [0,1]^n$:

$$|\,\mathbf{A}(x_1,\dots,x_n) - \mathbf{A}(y_1,\dots,y_n)\,| \le c\sum_{i=1}^{n} |x_i - y_i|. \tag{9}$$

A global aggregation operator $\mathbf{A} : \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ is c-Lipschitz if (9) holds for all $n \in \mathbb{N}$.

Directly from definitions it follows that the Lipschitz property (with any c) ensures the continuity of $\mathbf{A}$ and that $\mathbf{A}$ is c-Lipschitz if and only if all its partial derivatives are bounded by c in all points where they exist. Note that no global aggregation operator can be c-Lipschitz with $c < 1$. The 1-Lipschitz property is important for the stability of aggregation operators [3].

From the well-known aggregation operators, for example, the operators $\mathbf{M}$, $\mathbf{\Pi}$, **max, min**, that is, the arithmetic mean, product, maximum and minimum (as global aggregation operators) are 1-Lipschitz. More details about Lipschitz aggregation operators can be found, e. g., in [2].

Let us denote the set of all 1-Lipschitz aggregation operators by ${}^1\mathcal{L}$ and the set of all binary 1-Lipschitz aggregation operators by ${}^1\mathcal{L}_{(2)}$.

The next theorem shows that just 1-Lipschitz property is a necessary and sufficient condition for an aggregation to be $(M, 2)$-fitting.

Proposition 1. The operator $\mathbf{A}^*$ defined by (8) is a binary aggregation operator if and only if $\mathbf{A}$ as a binary aggregation operator is 1-Lipschitz.

Proof. It is clear that the operator $\mathbf{A}^*$ satisfies the boundary conditions. We show that $\mathbf{A}^*$ is monotone if and only if the aggregation operator $\mathbf{A}$ has the 1-Lipschitz property. Since $\mathbf{A}$ is an aggregation operator, its partial derivatives exist almost everywhere and, due to (8), the same holds for $\mathbf{A}^*$. Moreover,

$$\frac{\partial \mathbf{A}^*}{\partial x}(x,y) = 1 - \frac{\partial \mathbf{A}}{\partial x}(x,y) \quad \text{and} \quad \frac{\partial \mathbf{A}^*}{\partial y}(x,y) = 1 - \frac{\partial \mathbf{A}}{\partial y}(x,y).$$

This means that

$$\frac{\partial \mathbf{A}^*}{\partial x}(x,y) \geq 0 \Leftrightarrow \frac{\partial \mathbf{A}}{\partial x}(x,y) \leq 1 \quad \text{and} \quad \frac{\partial \mathbf{A}^*}{\partial y}(x,y) \geq 0 \Leftrightarrow \frac{\partial \mathbf{A}}{\partial y}(x,y) \leq 1.$$

Therefore $\mathbf{A}^*$ is monotone if and only if the partial derivatives of the aggregation operator $\mathbf{A}$ in all points where they exist are bounded by the constant 1, which is true if and only if $\mathbf{A}$ has the 1-Lipschitz property. □

Remark 1. The equation (8) can be rewritten into the form

$$\mathbf{A}(x,y) + \mathbf{A}^*(x,y) = x + y. \tag{10}$$

The obtained equation is a generalization of the known Frank functional equation [6]. Due to Proposition 1 we have that each solution to the equation (10) in the framework of aggregation operators has to be 1-Lipschitz, and hence continuous. In [1] the solutions to the equation (10) for uninorms (and nullnorms) were discussed. The result obtained in [1] that there is no uninorm satisfying (10) here follows immediately, because no uninorm is continuous.

Corollary 1. The following claims are equivalent:

(i) An aggregation operator $\mathbf{A}$ is $(M, 2)$-fitting.

(ii) The mapping $\mathbf{A}^*$ defined by (8) is a binary aggregation operator.

(iii) The aggregation operator $\mathbf{A}$ is 1-Lipschitz.

By Corollary 1, for finding $(M, 2)$-fitting aggregation operators, binary 1-Lipschitz aggregation operators are important.

Example 2. Let a binary aggregation operator $\mathbf{A}$ be

(i) a weighted mean,

(ii) an OWA operator,

(iii) a Choquet integral-based aggregation operator,

(iv) a Sugeno integral-based aggregation operator,

(v) a 2-copula.

Then $\mathbf{A}$ is 1-Lipschitz.

The claims (i) and (ii) are evident, since weighted means $\mathbf{W}$ are defined by $\mathbf{W}(x,y) = w_1 x + w_2 y$, with $w_1 + w_2 = 1$ and $w_1, w_2 \geq 0$, and OWA operators $\mathbf{W}'$ are given by $\mathbf{W}'(x,y) = w_1 \min(x,y) + w_2 \max(x,y)$ (with the same requirements for weights w_1, w_2).

(iii) A binary form of the Choquet integral-based aggregation operator for $\mathbb{X} = \{1,2\}$ and a fuzzy measure m given by $m(\{1\}) = a$, $m(\{2\}) = b$, with a, $b \in [0,1]$, is by (2)

$$\mathbf{C}_m(x,y) = ax + by + (1-a-b)\min(x,y),$$

or

$$\mathbf{C}_m(x,y) = \begin{cases} (1-b)x + by & \text{if } x \leq y, \\ ax + (1-a)y & \text{if } x \geq y. \end{cases}$$

Since both partial derivatives are bounded by the constant 1, $\mathbf{C}_m$ is 1-Lipschitz.

(iv) Recall that the Sugeno integral-based aggregation operator is given by

$$\mathbf{S}_m(x_1, \ldots, x_n) = (S) - \int_{\mathbf{X}} f \, \mathrm{d}m,$$

that is, by the Sugeno integral of the function $f : \mathbf{X} = \{1, \ldots, n\} \to [0,1]$, $f(i) = x_i$, see, e. g., [2]. For a binary case as in (iii) we obtain

$$\mathbf{S}_m(x,y) = \begin{cases} x \vee (b \wedge y) & \text{if } x \leq y, \\ (a \wedge x) \vee y & \text{if } x \geq y. \end{cases} \tag{11}$$

For the same reasons as in (iii) $\mathbf{S}_m$ is a 1-Lipschitz aggregation operator.

(v) For the claim on the 1-Lipschitz property of 2-copulas see, e. g., [12, 9].

Very often used aggregation operators are triangular norms [9] (t-norms for short) or aggregation operators based on t-norms. It is well-known that in the class of t-norms the 1-Lipschitz property is equivalent to the moderate growth property of copulas, that is, a t-norm $\mathbf{T} \in {}^1\mathcal{L}_{(2)}$ if and only if $\mathbf{T}$ is an associative copula, that is, if it is an ordinal sum of continuous Archimedean t-norms with convex additive generators, compare [9, 12].

Proposition 2. Let $\mathbf{A}$ be a binary 1-Lipschitz aggregation operator. Then

(i) $\mathbf{A}^*$ is also a 1-Lipschitz aggregation operator.

(ii) $\mathbf{T}_L \leq \mathbf{A} \leq \mathbf{S}_L$,

where $\mathbf{T}_L$, $\mathbf{S}_L$ are the Łukasiewicz t-norm and t-conorm, respectively.

Proof. (i) If $\mathbf{A}$ is 1-Lipschitz, then $\mathbf{A}^*$ is an aggregation operator. Since

$$(\mathbf{A}^*)^*(x,y) = \mathbf{A}(x,y), \quad \text{for all } (\mathbf{x},\mathbf{y}) \in [0,1]^2,$$

$(\mathbf{A}^*)^*$ is an aggregation operator, and thus, by Proposition 1, $\mathbf{A}^*$ is 1-Lipschitz.

(ii) Since $(\mathbf{A}^*)^* = \mathbf{A}$, we have $\mathbf{A}(x,y) = x + y - \mathbf{A}^*(x,y)$ for all $x, y \in [0,1]$. $\mathbf{A}^*$ is by (i) an aggregation operator which means that $\mathbf{A}^*(x,y) \in [0,1]$, and therefore $x + y - 1 \leq \mathbf{A}(x,y) \leq x + y$. Because of $\mathbf{A}(x,y) \in [0,1]$, we obtain

$$\max(0, x + y - 1) \leq \mathbf{A}(x,y) \leq \min(1, x + y),$$

which is our claim. □

Proposition 2 (ii) gives only a necessary condition for $\mathbf{A}$ to be 1-Lipschitz.

Remark 2. As we can see, the $*$-operator is involutive, $(\mathbf{A}^*)^* = \mathbf{A}$, it is a type of a dual operator and we can compare the properties of the dual operator $\mathbf{A}^d$ to $\mathbf{A}$ defined by $\mathbf{A}^d(x,y) = 1 - \mathbf{A}(1-x, 1-y)$, $(x,y) \in [0,1]$ and the operator $\mathbf{A}^*$.

It can be easily shown that, if $\mathbf{A}$ is a 1-Lipschitz operator then also $\mathbf{A}^d$ is a 1-Lipschitz aggregation operator. Further,

$$\begin{aligned} (\mathbf{A}^d)^*(x,y) &= x + y - 1 + \mathbf{A}(1-x, 1-y) \\ &= 1 - \mathbf{A}^*(1-x, 1-y) \\ &= (\mathbf{A}^*)^d(x,y) \end{aligned}$$

for all $(x,y) \in [0,1]$, which means that the $*$-operator and d-operator commute. Further, for neutral elements e and annihilators a we can deduce $e_{\mathbf{A}} = a_{\mathbf{A}^*}$ and $a_{\mathbf{A}} = e_{\mathbf{A}^*}$, if $\mathbf{A}$ has the neutral element and/or annihilator. The dual operator $\mathbf{A}^d$ does not change the functions of a and e, but their values, $e_{\mathbf{A}^d} = 1 - e_{\mathbf{A}}$ and $a_{\mathbf{A}^d} = 1 - a_{\mathbf{A}}$ if $\mathbf{A}$ has the neutral element and/or annihilator. More about binary 1-Lipschitz aggregation operators and the convex structure of the set ${}^1\mathcal{L}_{(2)}$ can be found in [10].

4. $(M, 3)$–FITTING AGGREGATION OPERATORS

Example 3. Let $\mathbf{W}$ be a weighted mean whose weighting triangle consists of the weighting vectors $\mathbf{w}^{(2)} = (0.5, 0.5)$ and $\mathbf{w}^{(3)} = (0, 1, 0)$. By (5) for the maximal fuzzy measure m^* we obtain the operator

$$\mathbf{C}_{m^*,\mathbf{W}}(x,y,z) = x + y + z - \frac{x+y}{2} - \frac{x+z}{2} - \frac{y+z}{2} + y = y,$$

which is an aggregation operator.

However, if we take the fuzzy measure m with values $m(\{1\}) = m(\{2\}) = m(\{3\}) = 0$, and $m(\{1,2\}) = m(\{1,3\}) = m(\{2,3\}) = 1$, then by (5) we have

$$\mathbf{C}_{m,\mathbf{W}}(x,y,z) = \frac{x+y}{2} + \frac{x+z}{2} + \frac{y+z}{2} - 2y = x - y + z.$$

Evidently, $\mathbf{C}_{m,\mathbf{W}}$ is not an aggregation operator.

The example illustrates that the problem for $(M,3)$-fitting aggregation operators cannot be reduced to the examination of the operators $\mathbf{C}_{m^*,\mathbf{A}}$ corresponding to the maximal measure.

Table 2.

	$\emptyset$	1	2	3	1,2	2,3	1,3	$\mathbb{X}$	$\mathbf{C}_{m,\mathbf{A}}(x,y,z)$
m_1	0	1	1	1	1	1	1	1	
M_{m_1}	0	1	1	1	-1	-1	-1	1	$x+y+z-\mathbf{A}(x,y)-\mathbf{A}(y,z)-\mathbf{A}(x,z)+\mathbf{A}(x,y,z)$
m_2	0	0	0	0	0	0	0	1	
M_{m_2}	0	0	0	0	0	0	0	1	$\mathbf{A}(x,y,z)$
m_3	0	0	0	0	0	0	1	1	
M_{m_3}	0	0	0	0	0	0	1	0	$\mathbf{A}(x,z)$
m_4	0	0	0	0	0	1	0	1	
M_{m_4}	0	0	0	0	0	1	0	0	$\mathbf{A}(y,z)$
m_5	0	0	0	0	1	0	0	1	
M_{m_5}	0	0	0	0	1	0	0	0	$\mathbf{A}(x,y)$
m_6	0	0	0	0	1	1	0	1	
M_{m_6}	0	0	0	0	1	1	0	-1	$\mathbf{A}(x,y)+\mathbf{A}(y,z)-\mathbf{A}(x,y,z)$
m_7	0	0	0	0	1	0	1	1	
M_{m_7}	0	0	0	0	1	0	1	-1	$\mathbf{A}(x,y)+\mathbf{A}(x,z)-\mathbf{A}(x,y,z)$
m_8	0	0	0	0	0	1	1	1	
M_{m_8}	0	0	0	0	0	1	1	-1	$\mathbf{A}(y,z)+\mathbf{A}(x,z)-\mathbf{A}(x,y,z)$
m_9	0	0	0	0	1	1	1	1	
M_{m_9}	0	0	0	0	1	1	1	-2	$\mathbf{A}(x,y)+\mathbf{A}(y,z)+\mathbf{A}(x,z)-2\mathbf{A}(x,y,z)$
m_{10}	0	1	0	0	1	0	1	1	
$M_{m_{10}}$	0	1	0	0	0	0	0	0	x
m_{11}	0	1	0	0	1	1	1	1	
$M_{m_{11}}$	0	1	0	0	0	1	0	-1	$x+\mathbf{A}(y,z)-\mathbf{A}(x,y,z)$
m_{12}	0	0	1	0	1	1	0	1	
$M_{m_{12}}$	0	0	1	0	0	0	0	0	y
m_{13}	0	0	1	0	1	1	1	1	
$M_{m_{13}}$	0	0	1	0	0	0	1	-1	$y+\mathbf{A}(x,z)-\mathbf{A}(x,y,z)$
m_{14}	0	0	0	1	0	1	1	1	
$M_{m_{14}}$	0	0	0	1	0	0	0	0	z
m_{15}	0	0	0	1	1	1	1	1	
$M_{m_{15}}$	0	0	0	1	1	0	0	-1	$z+\mathbf{A}(x,y)-\mathbf{A}(x,y,z)$
m_{16}	0	1	1	0	1	1	1	1	
$M_{m_{16}}$	0	1	1	0	-1	0	0	0	$x+y-\mathbf{A}(x,y)$
m_{17}	0	1	0	1	1	1	1	1	
$M_{m_{17}}$	0	1	0	1	0	0	-1	0	$x+z-\mathbf{A}(x,z)$
m_{18}	0	0	1	1	1	1	1	1	
$M_{m_{18}}$	0	0	1	1	0	-1	0	0	$y+z-\mathbf{A}(y,z)$

As before, express a fuzzy measure m as a convex combination of $\{0,1\}$-fuzzy measures, $m = \sum_{i=1}^{k} c_i m_i$. For $n = 3$ there are exactly 18 $\{0,1\}$-fuzzy measures whose values together with values of the Möbius transform and the form of the corresponding operator $\mathbf{C}_{m_i,\mathbf{A}}$ are in Table 2.

The operators corresponding to fuzzy measures $m_2 - m_5$, m_{10}, m_{12}, m_{14} are aggregation operators for any $\mathbf{A}$. The operators obtained for fuzzy measures $m_{16} - m_{18}$ are aggregation operators if and only if $\mathbf{A}$ as a binary operator is 1-Lipschitz. In such a case only eight operators are to be examined. In spite of that, in general it can be difficult to solve the problem of $(M, 3)$-fitting aggregation operators. As it was mentioned in Example 2, 2-copulas and weighted means are $(M, 2)$-fitting aggregation operators. In the next part we will discuss the $(M, 3)$-fitting property of these types of aggregation operators and do some conclusions concerning M-fitting aggregation operators.

4.1. $(M, 3)$-fitting copulas

Let $\mathbf{K}$ be a 3-copula and let binary aggregation operators on the right-hand side of the formula (5) (again denoted by $\mathbf{K}$) be 2-copulas which are the corresponding 2-margins of the 3-copula $\mathbf{K}$, see below.

Recall that a function $\mathbf{K} : [0,1]^3 \to [0,1]$ is a 3-copula, if:

(i) For each $\mathbf{u} = (u_1, u_2, u_3) \in [0,1]^3$ we have $\mathbf{K}(\mathbf{u}) = 0$ if $u_i = 0$ at least for one $i \in \{1,2,3\}$,

and

if $u_i = 1$ for each $i \neq k$, then $\mathbf{K}(\mathbf{u}) = u_k$.

(ii) For each 3-box $B = [x_1, x_2] \times [y_1, y_2] \times [z_1, z_2] \subset [0,1]^3$,

$$V_{\mathbf{K}}(B) \geq 0,$$

where $V_{\mathbf{K}}(B)$ is a $\mathbf{K}$-volume of a box B defined by

$$\begin{aligned} V_{\mathbf{K}}(B) = {} & \mathbf{K}(x_2, y_1, z_1) - \mathbf{K}(x_2, y_2, z_1) + \mathbf{K}(x_1, y_2, z_1) - \mathbf{K}(x_1, y_1, z_1) \\ & - \mathbf{K}(x_2, y_1, z_2) + \mathbf{K}(x_2, y_2, z_2) - \mathbf{K}(x_1, y_2, z_2) + \mathbf{K}(x_1, y_1, z_2). \end{aligned}$$

By Example 2, 2-copulas are 1-Lipschitz aggregation operators, thus, for a given 3-copula $\mathbf{K}$ it remains to show for eight $\{0,1\}$-fuzzy measures (m_1, $m_6 - m_8$, m_9, m_{11}, m_{13}, m_{15}) that $\mathbf{C}_{m,\mathbf{K}}$ are aggregation operators.

Consider the maximal fuzzy measure $m_1 = m^*$. The operator $\mathbf{C}_{m^*,\mathbf{K}}$ satisfies boundary conditions. We show that it is non-decreasing in each component. It is only a matter of computation to show that for all points $[x, y, z] \in [0,1]^3$ and for each $\epsilon > 0$ such that $x + \epsilon \in [0,1]$:

$$\mathbf{C}_{m^*,\mathbf{K}}(x + \epsilon, y, z) - \mathbf{C}_{m^*,\mathbf{K}}(x, y, z) = V_{\mathbf{K}}(B_1) \geq 0,$$

where B_1 is a 3-box $B_1 = [x, x + \epsilon] \times [y, 1] \times [z, 1]$,

and further analogously,

$$\mathbf{C}_{m^*,\mathbf{K}}(x, y + \epsilon, z) - \mathbf{C}_{m^*,\mathbf{K}}(x, y, z) = V_{\mathbf{K}}(B_2) \geq 0,$$

where B_2 is a 3-box $B_2 = [x,1] \times [y, y+\epsilon] \times [z,1]$,
and

$$\mathbf{C}_{m^*,\mathbf{K}}(x,y,z+\epsilon) - \mathbf{C}_{m^*,\mathbf{K}}(x,y,z) = V_{\mathbf{K}}(B_3) \geq 0,$$

where B_3 is a 3-box $B_3 = [x,1] \times [y,1] \times [z, z+\epsilon]$. This means that the function $\mathbf{C}_{m^*,\mathbf{K}}$ is non-decreasing in the sense of Definition 1 and is a ternary aggregation operator.

In all other cases we can proceed in an analogous way as for m^*, but a simpler way is the proof by means of partial derivatives. It holds that $\frac{\partial \mathbf{K}}{\partial x}(x,y,z) \in [0,1]$ in all points where partial derivatives exist, and that $\frac{\partial \mathbf{K}}{\partial x}$ is a non-decreasing function with respect to variables y and z. Analogous claims are valid for other partial derivatives. Moreover, 2-copulas can be expressed as 2-margins of 3-copulas, e. g., $\mathbf{K}(x,y) = \mathbf{K}(x,y,1)$, etc.

Due to the mentioned properties, for example, for the fuzzy measure m_6, we obtain:

$$\mathbf{C}_{m_6,\mathbf{K}}(x,y,z) = \mathbf{K}(x,y) + \mathbf{K}(y,z) - \mathbf{K}(x,y,z) = \mathbf{K}(x,y,1) + \mathbf{K}(y,z) - \mathbf{K}(x,y,z),$$

and

$$\frac{\partial \mathbf{C}_{m_6,\mathbf{K}}}{\partial x}(x,y,z) = \frac{\partial \mathbf{K}}{\partial x}(x,y,1) - \frac{\partial \mathbf{K}}{\partial x}(x,y,z) \geq 0.$$

Analogously,

$$\frac{\partial \mathbf{C}_{m_6,\mathbf{K}}}{\partial y}(x,y,z) = \left(\frac{\partial \mathbf{K}}{\partial y}(x,y,1) - \frac{\partial \mathbf{K}}{\partial y}(x,y,z)\right) + \frac{\partial \mathbf{K}}{\partial y}(y,z) \geq 0,$$

and finally,

$$\frac{\partial \mathbf{C}_{m_6,\mathbf{K}}}{\partial z}(x,y,z) = \frac{\partial \mathbf{K}}{\partial z}(1,y,z) - \frac{\partial \mathbf{K}}{\partial y}(x,y,z) \geq 0.$$

We have proved that the operator $\mathbf{C}_{m_6,\mathbf{K}}$ is non-decreasing in each component, which also means that it is non-decreasing in the sense of Definition 1. This, together with boundary conditions, gives that $\mathbf{C}_{m_6,\mathbf{K}}$ is a ternary aggregation operator. All other cases can be proved similarly. Since all functions $\mathbf{C}_{m_i,\mathbf{K}}$, $i = 1, \ldots, 18$, are aggregation operators, all 3-copulas are $(M,3)$-fitting.

4.2. $(M,3)$-fitting weighted means

In what follows, we will look for all weighted means which are $(M,3)$-fitting.

Let $\mathbf{A}$ be a weighted mean $\mathbf{W}$ with a weighting triangle given by weighting vectors $\mathbf{w}^{(2)} = (u, 1-u)$ and $\mathbf{w}^{(3)} = (v, w, 1-v-w)$.

According to the previous discussion, and because weighted means are 1-Lipschitz aggregation operators, we have to examine only 8 operators. We gradually obtain the following restrictions for the weights of $(M,3)$-fitting weighted means.

- For the maximal measure $m_1 = m^*$ the operator is of the form

$$\mathbf{C}_{m^*,\mathbf{W}}(x,y,z) = (1-2u+v)\,x + w\,y + (2u-v-w)\,z,$$

and it is an aggregation operator iff all coefficients are non-negative, which leads to the following condition for the weights $u, v, w \in [0,1]$:

$$\frac{v+w}{2} \le u \le \frac{1+v}{2}. \tag{12}$$

Further,

- the operator

$$\mathbf{C}_{m_6,\mathbf{w}}(x,y,z) = (u-v)\,x + (1-w)\,y + (v+w-u)\,z,$$

  is an aggregation operator iff

$$v \le u \le v+w, \tag{13}$$

- the operator

$$\mathbf{C}_{m_7,\mathbf{w}}(x,y,z) = (2u-v)\,x + (1-u-w)\,y + (v+w-u)\,z,$$

  is an aggregation operator only under conditions

$$\frac{v}{2} \le u \le v+w \quad \text{and} \quad u+w \le 1, \tag{14}$$

- and the operator

$$\mathbf{C}_{m_8,\mathbf{w}}(x,y,z) = (u-v)\,x + (u-w)\,y + (1-2u+v+w)\,z,$$

  has the restrictions

$$v \le u \le \frac{1+v+w}{2} \quad \text{and} \quad w \le u. \tag{15}$$

- The operator

$$\mathbf{C}_{m_9,\mathbf{w}}(x,y,z) = (2u-2v)\,x + (1-2w)\,y + (2v+2w-2u)\,z,$$

  is an aggregation operator iff

$$v \le u \le v+w \quad \text{and} \quad w \le \frac{1}{2}, \tag{16}$$

- and

$$\mathbf{C}_{m_{11},\mathbf{w}}(x,y,z) = (1-v)\,x + (u-w)\,y + (v+w-u)\,z,$$

  is an aggregation operator iff

$$w \le u \le v+w, \tag{17}$$

– while the operator

$$\mathbf{C}_{m_{13},\mathbf{W}}(x,y,z) = (u-v)\,x + (1-w)\,y + (v+w-u)\,z,$$

brings the conditions

$$v \leq u \leq v+w, \tag{18}$$

– and finally, the operator

$$\mathbf{C}_{m_{15},\mathbf{W}}(x,y,z) = (u-v)\,x + (1-u-w)\,y + (v+w)\,z,$$

is an aggregation operator iff

$$v \leq u \leq 1-w. \tag{19}$$

The obtained inequalities are not independent, evidently

if $v \leq u$ and $w \leq u$ then also $\frac{v+w}{2} \leq u$,

if $u \leq v+w$ and $u \leq 1-w$ then $u \leq \frac{1+v}{2}$, and consequently also $u \leq \frac{1+v+w}{2}$,

if $w \leq 1-u$ and $w \leq u$ then also $w \leq \frac{1}{2}$, and

if $v \leq u$ then also $\frac{v}{2} \leq u$.

After omitting the redundant inequalities, we obtain the conditions

$$\begin{array}{rcl} v \leq & u & \leq v+w \\ w \leq & u & \leq 1-w, \end{array} \tag{20}$$

which determine the weights $u, v, w \in [0,1]$ of appropriate weighted means $\mathbf{W}$ which can be combined with any $\{0,1\}$-fuzzy measure, that is, for which all resulting operators $\mathbf{C}_{m_i,\mathbf{W}}$, $i = 1,\dots,18$, are aggregation operators. By the previous discussion inequalities in (20) determine weights for all $(M,3)$-fitting weighted means.

Geometrically, these inequalities determine in a 3-dimensional space $\mathbb{R}_{u,v,w}$ a convex set with the vertices $V_1 = (0,0,0)$, $V_2 = (1,1,0)$, $V_3 = (\frac{1}{2},0,\frac{1}{2})$ and $V_4 = (\frac{1}{2},\frac{1}{2},\frac{1}{2})$, which correspond to the weighted means $\mathbf{W}_1,\dots,\mathbf{W}_4$ with weighting triangles

$\mathbf{W}_1$	$\mathbf{W}_2$	$\mathbf{W}_3$	$\mathbf{W}_4$
1	1	1	1
0 1	1 0	$\frac{1}{2}$ $\frac{1}{2}$	$\frac{1}{2}$ $\frac{1}{2}$
0 0 1	1 0 0	0 $\frac{1}{2}$ $\frac{1}{2}$	$\frac{1}{2}$ $\frac{1}{2}$ 0.

The operator $\mathbf{W}_1$ corresponding to the vertex V_1 is the projection to the last component and the operator $\mathbf{W}_2$ corresponding to V_2 is the projection to the first component. Note that also the arithmetic mean is 3-fitting.

Theorem 1. A weighted mean $\mathbf{W}$ with weighting triangle given by weighting vectors $\mathbf{w}^{(2)} = (u, 1-u)$, $\mathbf{w}^{(3)} = (v, w, 1-v-w)$ is $(M,3)$-fitting aggregation operator if and only if the inequalities in (20) are satisfied.

The convex set described by (20) is the intersection of all convex sets determined by inequalities (12) – (19). In some cases when fitting only with respect to some special measures is required, the resulting conditions need not be so restrictive.

5. CONCLUSIONS

The Choquet integral extends the underlying fuzzy measure m to an aggregation operator by means of **min**-operator, based on the Möbius transform of m. We have generalized this approach replacing the **min**-operator by some other (fitting) aggregation operator, and proposed a new construction method for aggregation operators. While the binary Möbius fitting aggregation operators are characterized simply by the 1-Lipschitz property, for $n > 2$ the situation is more complicated. However, we expect that n-copulas will always be (M, n)-fitting (in the paper we have shown it for $n \leq 3$). In such a case a new operator extends the underlying fuzzy measure m. For the next investigations several interesting problems remain open. First, to characterize all (M, n)-fitting aggregation operators for $n > 2$, and especially, all M-fitting aggregation operators. Next, to characterize all Möbius fitting aggregation operators which lead to an extension of the starting fuzzy measure m, that is, fulfilling

$$\mathbf{C}_{m,\mathbf{A}}(1_E) = m(E)$$

for all crisp subsets $E \subset \mathbb{X}$. We expect several interesting applications of our results, especially in approximation of fuzzy measures m, as well as the Choquet integral with respect to them, by means of some other set functions and relevant aggregation operators, for example, by a probability measure and its Lebesgue integral, that is, by a weighted mean.

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