KYBERNETIKA — VOLUME 38 (2002), NUMBER 3, PAGES 245–257

AGGREGATION OPERATORS AND FUZZY MEASURES ON HYPOGRAPHS

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In a fuzzy measure space we study aggregation operators by means of the hypographs of the measurable functions. We extend the fuzzy measures associated to these operators to more general fuzzy measures and we study their properties.

0. INTRODUCTION

In a recent paper, Imaoka [8] introduced an integral by means of σ-additive measures, defined on the Borel sets of $[0,1[\times[0,1[$. This approach was extended to the fuzzy measure case in [10].

Following this idea, we recognize that to every aggregation operator there is uniquely associated a fuzzy measure defined on the set of the hypographs, see [2, 4, 5]. The properties of the aggregation operator can be viewed as properties of such fuzzy measure.

In particular, we study those aggregation operators which are *horizontal* $\oplus$-*additive* and we recognize that the fuzzy measure, defined on the set $\mathcal{U}$ of the hypographs, can be extended to the smallest σ-algebra containing $\mathcal{U}$. Such extension is unique if the fuzzy measure is *horizontal* $\oplus$-*additive* and *translation invariant*. The horizontal $\oplus$-additivity is a weaker condition with respect to comonotonicity: it is suitable to the comparison with the associated fuzzy measure.

Then, we examine those aggregation operators which are *vertical* $\oplus$-*additive* and we recognize that the fuzzy measure, defined on the set $\mathcal{U}$ of the hypographs, can be extended to the smallest σ-algebra containing $\mathcal{U}$ as a fuzzy measure which is *horizontal* and *vertical* $\oplus$-*additive*.

1. PRELIMINARIES

Let Ω be an abstract space, $\mathcal{C}$ a σ-algebra of subsets of Ω and $\mathcal{F}$ the family of all $\mathcal{C}$-measurable functions $f : \Omega \to [0, F]$, with $0 < F \leq +\infty$.

We suppose that the interval $[0, F]$ has the structure of I-semigroup defined by means of a pseudo-addition $\oplus$, cf. [12]. We recall that a binary operation $\oplus$:

$[0,F]^2 \to [0,F]$ is called *pseudo-addition* if it is commutative, associative, non-decreasing, continuous, with 0 as neutral element.

We shall use, as well, a *pseudo-difference* (cf. [5, 9]) defined by

$$y \ominus x = \inf\{\eta \in [0,F] \mid x \oplus \eta \geq y\}.$$

It is easy to see that

$$\begin{aligned} &x \oplus (y \ominus x) \geq y \\ &(y \oplus a) \ominus (x \oplus a) \leq \; y \ominus x \qquad \forall\, a \in [0,F]. \end{aligned} \tag{1}$$

The pseudo-addition $\oplus$ given in $[0,F]$ allows us to define a *translation* in $\Omega \times [0,F]$ in the following way (cf. [3]):

Definition 1.1. Given any set $D \subset \Omega \times [0,F]$ and a real number $a \in [0,F]$, we call *shifted set* the following

$$\tau_a(D) = \; a \oplus D \; = \; \{(\omega, a \oplus \eta) \mid (\omega,\eta) \in D\}. \tag{2}$$

Given $c \in]0,F]$ and $C \in \mathcal{C}$, we call *basic function* the following:

$$b(c,C)(\omega) = \begin{cases} c & \text{for} \quad \omega \in C, \\ 0 & \text{for} \quad \omega \notin C. \end{cases} \tag{3}$$

Definition 1.2. Given a function $f \in \mathcal{F}$, its *horizontal $\oplus$-decomposition* (cf. [2, 4]) is

$$f(\omega) = (f(\omega) \wedge a) \oplus (f(\omega) \ominus a) \quad \forall\, a \in [0,F],\ \forall\, \omega \in \Omega. \tag{4}$$

Definition 1.3. Given a function $f \in \mathcal{F}$, its *vertical $\oplus$-decomposition* (cf. [1, 4, 5]) obtained through a partition of Ω in two sets C and C^c is

$$f(\omega) = f_C(\omega) \vee f_{C^c}(\omega) = f_C(\omega) \oplus f_{C^c}(\omega) \quad \forall\, C \in \mathcal{C},\ \forall\, \omega \in \Omega, \tag{5}$$

where

$$f_C(\omega) = \begin{cases} f(\omega) & \text{for} \quad \omega \in C, \\ 0 & \text{for} \quad \omega \notin C. \end{cases}$$

2. HYPOGRAPHS OF FUNCTIONS AND AGGREGATION OPERATORS

We consider the family $\mathcal{R}$ of all rectangles $C \times]x,y]$, $C \in \mathcal{C}$, $x, y \in [0,F]$, $x < y$ and we denote with $\mathcal{R}_0$ the family of the *down-rectangles* $C \times]0,c] = \{(\omega,\xi) \mid 0 < \xi \leq c\}$. Moreover we indicate with $\mathcal{U}$ the minimal family containing $\mathcal{R}_0$ and closed with respect to the countable unions.

Following Imaoka's idea (cf. [8]) we give the following:

Definition 2.1. The *hypograph* H_f of the function $f \in \mathcal{F}$ is the following set:

$$H_f = \{(\omega, \xi) \in \Omega \times]0, F] \mid 0 < \xi \leq f(\omega)\}. \tag{6}$$

For example, the hypograph of a basic function (3), $b(c, C)$, is exactly the set $C \times]0, c]$.

Theorem 2.2. The hypograph H_f belongs to $\mathcal{U}$, if and only if the function f belongs to $\mathcal{F}$.

Proof. Let $\{c_j \in [0, F]\}_{j=1}^{\infty}$ be a dense subset in $[0, F]$. For any function $f \in \mathcal{F}$, putting $C_j := \{\omega \in \Omega \mid f(\omega) \geq c_j\}$, we get:

$$H_f = \bigcup_{j=1}^{\infty} (C_j \times]0, c_j]) \in \mathcal{U}.$$

Vice-versa, given $\bigcup_{j=1}^{\infty} (C_j \times]0, c_j]) \in \mathcal{U}$, observe that $C_j \times]0, c_j] = H_{b(c_j, C_j)}$. Then

$$f = \bigvee_{i=1}^{\infty} b(c_j, C_j) \tag{7}$$

is a measurable function from $\mathcal{F}$, and it is not difficult to check that $H_f = \bigcup_{j=1}^{\infty} (C_j \times]0, c_j])$. □

The expression (6) defines the map $h : \mathcal{F} \to \mathcal{U}$ which associates to every function $f \in \mathcal{F}$ its hypograph $H_f \in \mathcal{U}$:

$$h(f) = H_f. \tag{8}$$

The map (8) is reversible, that means that $h^{-1}(h(f)) = f$. The map $h^{-1} : \mathcal{U} \to \mathcal{F}$ can be built in the following way: the function $h^{-1}(U) = f_U$, $f_U : \Omega \to [0, F]$ is defined by

$$f_U(\omega) = \sup\{\xi / (\omega, \xi) \in U\}, \tag{9}$$

and its hypograph coincides exactly with $U : h(h^{-1}(U)) = U$.

Proposition 2.3. The application (8) satisfies the following properties for f, g and f_n in $\mathcal{F}$:

$$f(\omega) \leq g(\omega) \quad \forall \omega \in \Omega \leftrightarrow H_f \subset H_g, \tag{10}$$
$$f_n(\omega) \uparrow f(\omega) \Rightarrow H_{f_n} \uparrow H_f, \tag{11}$$
$$H_{f \vee g} = H_f \cup H_g, \tag{12}$$
$$H_{f \wedge g} = H_f \cap H_g. \tag{13}$$

Now, we define *a-horizontal cut*, for $a \in [0, F]$ and *C-vertical cut*, for $C \in \mathcal{C}$ ([1]).

Fix a real number $a \in [0, F]$ and a set $D \subset \mathcal{C} \times \mathcal{B}$, where $\mathcal{B}$ is the Borel σ-algebra on $]0, F]$. We consider the following sets:

$$D_a = \{(\omega, \xi) \in D \mid \xi \leq a\} \text{ and } D^a = \{(\omega, \xi) \in D \mid \xi > a\}. \tag{14}$$

In this way D is divided into two disjoint sets D_a and D^a with $D = D_a \cup D^a$. The operation which associates to every set D the parts D_a and D^a is called *a-horizontal cut*.

Fixed a set $C \in \mathcal{C}$ and $D \subset \mathcal{C} \times \mathcal{B}$ (cf. [1]), we consider the following sets:

$$D_C = D \cap \{C \times]0, F]\} \text{ and } D_{C^c} = D \cap \{C^c \times]0, F]\}. \tag{15}$$

In this way D is divided into two disjoint sets. The operation which associates to every set D the parts D_C and D_{C^c} is called *C-vertical cut* (with the partition C, C^c).

The following proposition describes the link between the $\oplus$-horizontal or $\oplus$-vertical decomposition of any function and the horizontal or vertical cut of its hypograph, respectively.

Proposition 2.4. Given a function $f \in \mathcal{F}$:

1. for every horizontal $\oplus$-decomposition of f, an a-horizontal cut of its hypograph corresponds $\forall\, a \in]0, F]$:

$$H_f = (H_f)_a \cup (H_f)^a = H_{f \wedge a} \cup \{a \oplus H_{f \ominus a}\} = H_{f \wedge a} \cup \tau_a(H_{f \ominus a}); \tag{16}$$

2. for every vertical $\oplus$-decomposition of f a C-vertical cut of its hypograph corresponds $\forall\, C \in \mathcal{C}$:

$$H_f = (H_f \cap C) \cup (H_f \cap C^c) = H_{f_C} \cup H_{f_{C^c}}. \tag{17}$$

Proof.

It is easy to see that

$$\begin{aligned} H_f &= \{(\omega, \xi) \mid 0 < \xi \leq f(\omega) \wedge a\} \cup \{(\omega, \xi) \mid a < \xi \leq f(\omega)\} \\ &= \{(\omega, \xi) \mid f(\omega) \wedge a \leq \xi\} \cup (a \oplus \{(\omega, \xi) \mid a \oplus \xi \leq f(\omega)\}). \end{aligned}$$

Therefore

$$H_f = H_{f \wedge a} \cup \{a \oplus \; H_{f_{\ominus a}}\}.$$

The relation (17) follows from (12). □

We are now considering the aggregation operators in the more general form, which takes into account only the most natural properties. We shall prove that, with every aggregation operator $\mathbf{A} : \mathcal{F} \to [0, F]$, a fuzzy measure $m_{\mathbf{A}} : \mathcal{U} \to [0, F]$ can be uniquely associated, which uniquely replaces $\mathbf{A}$.

Definition 2.5. An *aggregation operator* is a functional $\mathbf{A} : \mathcal{F} \to [0, F]$ satisfying the following properties:

(A1) $f(\omega) = c \quad \forall\, \omega \in \Omega \Rightarrow \mathbf{A}(c) = c$ (idempotency)

(A2) $f(\omega) \leq g(\omega) \;\; \forall\, \omega \in \Omega \Rightarrow \mathbf{A}(f) \leq \mathbf{A}(g)$ (monotonicity)

(A3) $f_n \nearrow f \Rightarrow \mathbf{A}(f_n) \nearrow \mathbf{A}(f)$ (continuity from below).

The properties (A1) – (A3) seem us to be natural for the aggregation operators, because they are evaluation models of averaging operations.

We recall that these operators have been introduced by Mesiar–Komorníková [11] and by Yager [15]. An overview of these operators can be found in [7]. Recently, Benvenuti, Vivona, Divari in [1, 2, 3, 4, 5] have studied these operators in a more general setting.

In the product σ-algebra $\mathcal{C} \times \mathcal{B}$, we shall use a fuzzy measure $m : \mathcal{C} \times \mathcal{B} \to R^+$ i. e. a non-decreasing set function, continuous from below, with $m(\emptyset) = 0$, ([14]).

Proposition 2.6. If $m : \mathcal{C} \times B \to [0, F]$ is a fuzzy measure, which satisfies the condition:

$$m(\Omega \times]0, c]) = c, \quad \forall\, c \in]0, F], \tag{18}$$

the functional $\mathbf{A}_m : \mathcal{F} \to [0, F]$ defined by

$$\mathbf{A}_m(f) = m(H_f) \quad \forall\, f \in \mathcal{F}, \tag{19}$$

is an aggregation operator (H_f is the *hypograph* of f).

P r o o f. (A1) is a consequence of the hypothesis (18) on m;
(A2) follows from (10) and the monotonicity of the fuzzy measure m;
(A3) follows from the continuity from below of the measure m. □

Proposition 2.7. If $\mathbf{A} : \mathcal{F} \to [0, F]$ is an aggregation operator, the function $m_{\mathbf{A}} : \mathcal{U} \to [0, F]$ *defined by*

$$m_{\mathbf{A}}(U) = \mathbf{A}(f_U) \quad \forall\, U \in \mathcal{U}, \tag{20}$$

is a fuzzy measure fulfilling (18) (f_U is the function defined by (9)).

P r o o f. For (A1) we have $m_{\mathbf{A}}(\emptyset) = 0$; the properties (A2) and (A3) ensure the monotonicity and the continuity from below of $m_{\mathbf{A}}$. Therefore $m_{\mathbf{A}}$ is a fuzzy measure. (18) follows from the idempotency of $\mathbf{A}$. □

As consequence of Propositions 2.5 and 2.6, we give:

Proposition 2.8. The correspondences individuated by (19) and (20) are inverse to each other:

$$m_{\mathbf{A}_m} = m, \quad \text{and} \quad \mathbf{A}_{m_\mathbf{A}} = \mathbf{A}.$$

Remark 2.9. Given an aggregation operator $\mathbf{A}$, we denote with $\mathcal{M}_\mathbf{A}$ the family of all fuzzy measures μ on $\mathcal{C} \times \mathcal{B}$ generating $\mathbf{A}$:

$$\mathcal{M}_\mathbf{A} = \{\mu : \mathcal{C} \times \mathcal{B} \to [0, F]/\mu \text{ is a fuzzy measure such that } \mu_{|\mathcal{U}} = m_\mathbf{A}\}. \tag{21}$$

If μ_1 and $\mu_2 \in \mathcal{M}_\mathbf{A}$, they coincide on the family $\mathcal{U}$:

$$\mu_1(U) = \mu_2(U) \quad \forall\, U \in \mathcal{U}.$$

Now we put:

$$\underline{m_\mathbf{A}}(D) = \sup\{m_\mathbf{A}(U)/U \in \mathcal{U}, U \subset D\}, \tag{22}$$
$$\overline{m_\mathbf{A}}(D) = \inf\{m_\mathbf{A}(U)/U \in \mathcal{U}, U \supset D\}. \tag{23}$$

$\underline{m_\mathbf{A}}$ and $\overline{m_\mathbf{A}}$ are fuzzy measures, in general $\underline{m_\mathbf{A}} \neq \overline{m_\mathbf{A}}$, more $\underline{m_\mathbf{A}} \leq \overline{m_\mathbf{A}}$.

Moreover, the measures $\underline{m_\mathbf{A}}$ and $\overline{m_\mathbf{A}}$ are equal on $\mathcal{U}$, but possibly different on $\mathcal{C} \times \mathcal{B}$.

In fact, put $\mathcal{C} = \{\emptyset, \Omega\}$, then $\mathcal{U} = \{H_{b(c,\Omega)} \mid c \in [0, F]\}$. Obviously, $\mathbf{A}$ is a trivial aggregation operator assigning to constant functions their constant value. However, evidently for all D, $\underline{m_\mathbf{A}}(D) = \inf\{\xi \in]0, F] \mid (\omega, \xi) \in D\}$, while $\overline{m_\mathbf{A}}(D) = \sup\{\xi \in]0, F] \mid (\omega, \xi) \in D\}$.

Then we have the following:

Theorem 2.10. Let $\mathbf{A}$ be an aggregation operator. A fuzzy measure μ is an element of the family $\mathcal{M}_\mathbf{A}$ defined by (21) if and only if

$$\underline{m_\mathbf{A}} \leq \mu \leq \overline{m_\mathbf{A}}, \tag{24}$$

where $\underline{m_\mathbf{A}}$ and $\overline{m_\mathbf{A}}$ are the fuzzy measures defined by (22) and (23), respectively.

Proof. Let μ be an element of $\mathcal{M}_\mathbf{A}$; so the restriction of μ to the family $\mathcal{U}$ coincides with $m_\mathbf{A}$. Given $D \in \mathcal{C} \times \mathcal{B}$, we consider two sets U_1 and U_2 of $\mathcal{U}$ such that $U_1 \subset D$ and $U_2 \supset D$. As μ is monotone, we have

$$\underline{m_\mathbf{A}}(U_1) = \mu(U_1) \leq \mu(D) \leq \mu(U_2) = \overline{m_\mathbf{A}}(U_2) \quad \forall\, U_1, U_2.$$

The equality (24) easily follows.

Conversely, in particular for $U \in \mathcal{U}$, we obtain from (24):

$$\underline{m_\mathbf{A}}(U) = \mu(U) = \overline{m_\mathbf{A}}(U),$$

that means that μ is an element of $\mathcal{M}_\mathbf{A}$. □

3. HORIZONTAL $\oplus$–ADDITIVITY

In this section we want to extend a fuzzy measure m defined on $\mathcal{U}$ to the σ-algebra $\mathcal{C} \times \mathcal{B}$ by means of horizontal cuts.

We say that $\mathbf{A}$ is *horizontal $\oplus$-additive* if it is additive along a–horizontal cuts for every $a \in [0, F]$:

(A4) $\qquad \mathbf{A}(f) = \mathbf{A}(f \wedge a) \oplus \mathbf{A}(f \ominus a), \quad f \in F,\ a \in]0, F],$

where $f = (f \wedge a) \oplus (f \ominus a)$ as in (4).

Now we give the following definitions.

Definition 3.1. Given any set $D \subset \Omega \times]0, F]$ and a real positive number a we say that a fuzzy measure m is *translation invariant* if

$$m(\tau_a(D)) = m(D) \quad \forall D \text{ and } a \in [0, F \ominus d[, \tag{25}$$

where $d = \sup\{\xi \in]0, F] / (\omega, \xi) \in D\}$.

Definition 3.2. A fuzzy measure m, defined on $\mathcal{C} \times \mathcal{B}$, is called *horizontal $\oplus$-additive* if it is additive along a-horizontal cuts for every $a \in]0, F]$:

$$m(D) = m(D_a) \oplus m(D^a) \quad \forall D \in \mathcal{C} \times \mathcal{B}. \tag{26}$$

Proposition 3.3. If the fuzzy measure m is horizontal $\oplus$-additive and translation invariant, the corresponding aggregation operator $\mathbf{A}_m$ *(defined by (19))* is horizontal $\oplus$-additive.

The proof is immediate.

Proposition 3.4. If the aggregation operator $\mathbf{A}$ is horizontal $\oplus$-additive, the corresponding fuzzy measure $m_{\mathbf{A}}$ *(defined by (20))* satisfies the following equation:

$$m_{\mathbf{A}}(C \times]0, x]) \oplus m_{\mathbf{A}}(C \times]0, y \ominus x]) = m_{\mathbf{A}}(C \times]0, y]) \ \forall x, y \in [0, F],\ x < y,\ C \in \mathcal{C}. \tag{27}$$

Proof. Given $C \in \mathcal{C}$ and $x, y \in [0, F], x < y$, we consider the function $f(\omega) = b(y, C)(\omega)\ \forall \omega \in \Omega$. The equation (27) is a consequence of the Proposition 3.3 as $b(y, C) \wedge x = b(x, C)$ and $(b(y, C))_x = b(y \ominus x, C)$. □

We know that several fuzzy measures may generate the same aggregation operator (see Remark 2.9), but the properties: *horizontal $\oplus$-additivity* and *translation invariance* make unique their extension.

Now we give the first main theorem.

Theorem 3.5. (H-Extension Theorem.) Let $\mathbf{A}$ be a horizontal $\oplus$-additive aggregation operator on $\mathcal{F}$. Then there exists a unique fuzzy measure m^*, defined on $\mathcal{C} \times \mathcal{B}$ such that:

m^* is horizontal $\oplus$-additive;

m^* is translation invariant;

m^* generates $\mathbf{A}$.

Proof. We proceed by steps. Let $m_{\mathbf{A}}$ be the associated fuzzy measure defined on $\mathcal{U}$ by (20). First we prove the assertion for the rectangles of the type $C\times]x, y]$, then in the general case. As we want that m^* should be translation invariant, we are led to put

$$m^*(C\times]x,y]) = m_{\mathbf{A}}(C\times]0, y \ominus x]). \tag{28}$$

From (2) and (1) we recognize that

$$\begin{aligned} & m^*(\tau_a(C\times]x,y])) = m^*(a \oplus (C\times]x,y])) = m^*(C\times]a \oplus x, a \oplus y]) \\ = \quad & m_{\mathbf{A}}(C\times]0,(a \oplus y) \ominus (a \oplus x)]) = m_{\mathbf{A}}(C\times]0, y \ominus x]) = m^*(C\times]x,y]), \\ & \forall x, y \in [0,F],\ x < y,\ C \in \mathcal{C}, \text{ and } a \in [0, F \ominus y]. \end{aligned}$$

Because of properties (27) and (28) we have

$$m^*(C\times]0,x]) \oplus m^*(C\times]0, y\ominus x]) = m^*(C\times]0,y]) \quad \forall\, x,\, y,\ x<y, C \in \mathcal{C}.$$

That means that m^* is horizontal $\oplus$-additive on the rectangles of the type $C\times]0, y]$.
So we have proved that m^* satisfies the properties on the rectangles of the type $C\times]0, x]$.

Let X be a finite subset of $]0, F]$ and assume the elements of X in increasing order: $0 \le c_0 < c_1 < \ldots < c_n \le F$. Moreover, we consider the sets $C_1, C_2, \ldots, C_n$ and we denote with $R = \bigcup_{i=1}^n (C_i\times]c_{i-1}, c_i])$ a horizontal plurirectangle. As we want that m^* is horizontal $\oplus$-additive, we are led to put

$$m^*(R) = \oplus_{i=1}^n m^*(C_i\times]c_{i-1}, c_i]). \tag{29}$$

Finally, we consider the general case. Let D be any set in $\mathcal{C} \times \mathcal{B}$ and consider a horizontal plurirectangle $R = \bigcup_{i=1}^n (C_i\times]c_{i-1}, c_i])$ contained in D. We denote by $\mathcal{R}(D)$ the family of the plurirectangles R contained in D.

But m^* had to be continuous from below, so it is natural to set

$$m^*(D) = \sup_{\mathcal{R}(D)} \oplus_{i=1}^n m^*(C_i\times]c_{i-1}, c_i]). \tag{30}$$

Now we prove that m^* defined by (30) is horizontal $\oplus$-additive, i.e.

$$m^*(D) = m^*(D_a) \oplus m^*(D^a)\ \forall\, D \in \mathcal{R}(D),\ a \in [0,F],$$

where D_a and D^a are those subsets of D obtained with horizontal cuts (12). We cut D_a into the levels j, $1 < j < r$, and D^a into the levels k, $1 < k < p$. Then

$$\begin{aligned} m^*(D_a) &= \sup_{\mathcal{R}(D_a)} \{\oplus_{j=1}^{r} m^*(C_j \times]c_{j-1}, c_j])\} = \sup_{\mathcal{R}(D_a)} \left\{ m^*(\textstyle\bigcup_{j=1}^{r} (C_j \times]c_{j-1}, c_j])) \right\}, \\ m^*(D^a) &= \sup_{\mathcal{R}(D^a)} \{\oplus_{k=1}^{p} m^*(C_k \times]c_{k-1}, c_k])\} = \sup_{\mathcal{R}(D^a)} \{ m^*(\textstyle\bigcup_{k=1}^{p} (C_k \times]c_{k-1}, c_k])) \}, \end{aligned}$$

and then

$$\begin{aligned} & m^*(D_a) \oplus m^*(D^a) \\ = & \left(\sup_{\mathcal{R}(D_a)} \{\oplus_{j=1}^{r} m^*(C_j \times]c_{j-1}, c_j])\} \right) \oplus \left(\sup_{\mathcal{R}(D^a)} \{\oplus_{k=1}^{p} m^*(C_k \times]c_{k-1}, c_k])\} \right) \\ = & \sup_{\mathcal{R}(D_a) \cup \mathcal{R}(D^a)} \left\{ m^*(\textstyle\bigcup_{j=1}^{r} C_j \times]c_{j-1}, c_j]) \cup (m^*(\textstyle\bigcup_{k=1}^{p} C_k \times]c_{k-1}, c_k])) \right\} \\ = & \sup_{\mathcal{R}(D_a) \cup \mathcal{R}(D^a)} \left\{ (\oplus_{j=1}^{r} m^*(C_j \times]c_{j-1}, c_j]) \oplus (\oplus_{k=1}^{p} m^*(C_k \times]c_{k-1}, c_k]) \right\} \\ = & \sup_{\mathcal{R}(D)} \{\oplus_{i=1}^{n} m^*(C_i \times]c_{i-1}, c_i])\} \leq m^*(D). \end{aligned}$$

To see the opposite inequality, it is enough to observe that any R in $\mathcal{R}(D)$ can be expressed as a disjoint union $R = R_1 \cup R_2$ where $R_1 \in \mathcal{R}(D_a)$ and $R_2 \in \mathcal{R}(D^a)$.

It remains to prove that m^* is translation invariant.

For every $a \in]0, F[$ such that $\sup\{\xi : (\omega, \xi) \in D\} \leq F \ominus a$ it holds:

$$\begin{aligned} & \tau_a(m^*(D)) = \tau_a \left(\sup_{\mathcal{R}(D)} \oplus_{i=1}^{n} m^*(C_i \times]c_{i-1}, c_i] \right) \\ = & \sup_{\mathcal{R}(D)} (\tau_a \oplus_{i=1}^{n} m^*(C_i \times]c_{i-1}, c_i])) = \sup_{\mathcal{R}(D)} (\oplus_{i=1}^{n} m^*(C_i \times]c_{i-1}, c_i])) = m^*(D), \end{aligned}$$

because for the first part of this proof the summands $m^*(C_i \times]x_i, y_i])$ are translation invariant.

Finally, this measure m^* coincides with $m_{\mathbf{A}}$ on $\mathcal{U}$: in fact $m^*(\emptyset) = 0, m^*(C \times]0, y]) = m_{\mathbf{A}}(C \times]0, y])$, as $y \ominus 0 = 0$, $m^*(R) = \oplus_{i=1}^{n} m^*(C_i \times]c_{i-1}, c_i]) = m_{\mathbf{A}}(R)$, therefore m^* generates the given $\mathbf{A}$.

The construction of the measure m^* (see (28), (29), (30)) is unique and the only possible, because it satisfies the requested properties. □

4. VERTICAL $\oplus$–ADDITIVITY

In this section we want to extend a fuzzy measure m defined on $\mathcal{U}$ to the algebra $\mathcal{C} \times \mathcal{B}$, by means of vertical cuts.

We say that an aggregation operator $\mathbf{A}$ defined on $\mathcal{F}$ is *vertical $\oplus$-additive* if

(A5) $\qquad \mathbf{A}(f) = \mathbf{A}(f_C) \oplus \mathbf{A}(f_{C^c}) \quad f \in \mathcal{F}, C \in \mathcal{C} \times \mathcal{B},$

where $f = f_C \oplus f_{C^c}$ as in (5).

Definition 4.1. A fuzzy measure m defined on $\mathcal{C} \times \mathcal{B}$ is called *vertical $\oplus$-additive* if it is additive along C-vertical cuts for every $C \in \mathcal{C}$

$$m(D) = m(D_C) \oplus m(D_{C^c}) \quad \forall\, D \in \mathcal{C} \times \mathcal{B}, \tag{31}$$

where D_C and D_{C^c} are defined by (13).

We get the following result:

Proposition 4.2. If the fuzzy measure m is vertical $\oplus$-additive on $\mathcal{C} \times \mathcal{B}$ then the corresponding aggregation operator $\mathbf{A}_m$ defined on $\mathcal{U}$ by (18) is vertical $\oplus$-additive.

Now we recall the definition of a *valuation* (cf. [6, 13]).

Definition 4.3. Given a lattice $(S, \vee, \wedge)$, with a pseudo-addition $\oplus : [0, F]^2 \to [0, F]$, a function $s : S \to S$ is *$\oplus$-valuation* (with respect to $\prec$) if

$$s(\alpha \vee \beta) \oplus s(\alpha \wedge \beta) = s(\alpha) \oplus s(\beta) \quad \forall\, \alpha, \beta \in S.$$

Proposition 4.4. If the aggregation operation $\mathbf{A}$ is vertical $\oplus$-additive, then it is a $\oplus$-valuation ($\vee$ is common maximum of real functions):

$$\mathbf{A}(f \vee g) \oplus \mathbf{A}(f \wedge g) = \mathbf{A}(f) \oplus \mathbf{A}(g) \quad \forall\, f,\, g \in \mathcal{F}.$$

Proof. Given $f,\, g \in \mathcal{F}$, we consider the following set:

$$C = \{\omega \in \Omega / f(\omega) \leq g(\omega)\} \quad \text{and} \quad C^c = \{\omega \in \Omega / f(\omega) > g(\omega)\},$$

and the vertical $\oplus$-decomposition of the functions $f \vee g$, $f \wedge g$, f and g. From (5) it is easy to see that

$$\begin{aligned} f \vee g &= (f \vee g)_C \oplus (f \vee g)_{C^c} = g_C \oplus f_{C^c}, \\ f \wedge g &= (f \wedge g)_C \oplus (f \wedge g)_{C^c} = f_C \oplus g_{C^c}. \end{aligned}$$

As $\mathbf{A}$ is vertical $\oplus$-additive, we get

$$\mathbf{A}(f \vee g) \oplus \mathbf{A}(f \wedge g) = \mathbf{A}(g_C) \oplus \mathbf{A}(f_{C^c}) \oplus \mathbf{A}(f_C) \oplus \mathbf{A}(g_{C^c}) = \mathbf{A}(f) \oplus \mathbf{A}(g). \quad \square$$

As consequence of Proposition 2.3, equalities (12), (13), we have the following:

Proposition 4.5. An aggregation operation $\mathbf{A}$ on $\mathcal{F}$ is a $\oplus$-valuation if and only if its associated fuzzy measure $m_{\mathbf{A}}$ defined on $\mathcal{U}$ is a $\oplus$-valuation (with respect to the lattice $(\mathcal{U}, \cup, \cap)$):

$$m_{\mathbf{A}}(H_f \cup H_g) \oplus m_{\mathbf{A}}(H_f \cap H_g) = m_{\mathbf{A}}(H_f) \oplus m_{\mathbf{A}}(H_g) \quad \forall\, f,\, g \in \mathcal{F}.$$

Proposition 4.6. If the aggregation operation **A** is vertical $\oplus$-additive, the corresponding fuzzy measure $m_{\mathbf{A}}$ defined by (17) satisfies the following:

$$m_{\mathbf{A}}((C_1 \cup C_2)\times]0,y]) = m_{\mathbf{A}}(C_1\times]0,y]) \oplus m_{\mathbf{A}}(C_2\times]0,y]) \qquad (32)$$
$$\forall\, C_1,\, C_2 \in \mathcal{C}, \text{ with } C_1 \cap C_2 = \emptyset.$$

Proof. Let $b(y,C)$ be a basic function with $C = C_1 \cup C_2$ and $C = C_1 \cap C_2 = \emptyset$. It is $b(y,C) = b(y,C_1) \oplus b(y,C_2)$; as **A** is vertical $\oplus$-additive, it holds

$$m_{\mathbf{A}}(C\times]0,y]) = m_{\mathbf{A}}(C_1\times]0,y]) \oplus m_{\mathbf{A}}(C_2\times]0,y]). \qquad \square$$

Now we are ready to give the second main theorem.

Theorem 4.7. (V-Extension Theorem.) Let **A** be a vertical $\oplus$-additive aggregation operator on $\mathcal{F}$.

Then there exists a unique fuzzy measure m^* defined on $\mathcal{C} \times \mathcal{B}$ such that:

m^* is vertical $\oplus$-additive;

m^* generates **A**.

Proof. We proceed by steps. First we prove the assertion for the rectangles of the type $C\times]x,y]$, then in the general case.

Fixed $C \in \mathcal{C}$ and $x \in [0,F], x < y$, we build the unique $\oplus$-additive measure $m_C : \mathcal{B} \to [0,F]$, such that

$$m_C(]0,x]) = \mathbf{A}(b(x,C)), \qquad (33)$$

where $b(x,C)$ is the basic function defined in (3). Then it is natural to put, for every $C \in \mathcal{C}$ and $x,y \in [0,F], x < y$,

$$m_C(]x,y]) = m_C(]0,y]) \ominus m_C(]0,x]) = m_{\mathbf{A}}(C\times]0,y]) \ominus m_{\mathbf{A}}(C\times]0,x]). \qquad (34)$$

Let X be a finite subset of $[0,F]$; we assume the elements of X in increasing order: $0 \le c_0 < c_1 < \ldots < c_n \le F$. Moreover, we consider the sets $C_1, C_2, \ldots, C_n \subset \mathcal{C}$ and we denote with $R = \bigcup_{i=1}^{n}(C_i\times]c_{i-1},c_i])$ a horizontal plurirectangle. As we want that m^* is vertical $\oplus$-additive, we are led to set

$$m^*(R) = \oplus_{i=1}^{n} m_{C_i}(]c_{i-1},c_i]). \qquad (35)$$

Finally, let D be any set in $\mathcal{C} \times \mathcal{B}$ and we cut D into vertical slices.

We consider the plurirectangle $R = \bigcup_{i=1}^{n}(C_i\times]c_{i-1},c_i])$ contained in D and $\mathcal{R}(D)$ the family of the plurirectangles R.

As m^* is continuous from below, it is natural to set

$$m^*(D) = \sup_{\mathcal{R}(D)} \oplus_{i=1}^{n} m_{C_i}(]c_{i-1},c_i]). \qquad (36)$$

Now we prove that m^* is vertical $\oplus$-additive.

Fixed $C \in \mathcal{C}$, we decompose D in two sets D_C and D_{C^c}, and so

$$\begin{aligned} m^*(D_C) &= \sup_{\mathcal{R}(D_C)} \left\{ \oplus_{i=1}^{l} m^*((C_i \cap D)\times]c_{i-1}, c_i]) \right\}, \\ m^*(D_{C^c}) &= \sup_{\mathcal{R}(D_{C^c})} \left\{ \oplus_{j=l+1}^{n} m^*((C_j \cap D^c)\times]c_{j-1}, c_j])) \right\}. \end{aligned}$$

Therefore

$$\begin{aligned} & m^*(D_C) \oplus m^*(D_{C^c}) = \sup_{\mathcal{R}(D_C)} \left\{ \oplus_{i=1}^{l} m^*((C_i \cap D)\times]c_{i-1}, c_i]) \right\} \\ & \quad \oplus \sup_{\mathcal{R}(D_{C^c})} \left\{ \oplus_{j=l+1}^{n} m^*((C_j \cap D^c)\times]c_{j-1}, c_j]) \right\} \\ = & \sup_{\mathcal{R}(D_C)\cup\mathcal{R}(D_{C^c})} \left\{ (\oplus_{i=1}^{l} m^*((C_i \cap D)\times]c_{i-1}, c_i])) \right. \\ & \quad \left. \oplus(\oplus_{j=l+1}^{n} m^*(C_j \cap D^c)\times]c_{j-1}, c_j])) \right\} \\ = & \sup_{\mathcal{R}(D_C\cup D_{C^c})} \left\{ (\oplus_{i=1}^{n} m^*((C_i \cap D)\times]c_{i-1}, c_i])) \oplus (\oplus_{i=1}^{n} m^*((C_i \cap D^c)\times]c_{i-1}, c_i])) \right\} \\ = & \sup_{\mathcal{R}(D_C\cup D_{C^c})} \left\{ \bigcup_{i=1}^{n} m^*((C_i \cap D)\times]c_{i-1}, c_i])) \cup (\bigcup_{i=1}^{n} m^*((C_i \cap D^c)\times]c_{i-1}, c_i])) \right\} \\ = & \sup_{\mathcal{R}(D_C\cup D_{C^c})} \left\{ m^*(\cup_{i=1}^{n}((C_i \cap D) \cup (C_i \cap D^c))\times]c_{i-1}, c_i])) \right\} \\ = & \sup_{\mathcal{R}(D_C\cup D_{C^c})} \left\{ m^*(\bigcup_{i=1}^{n}(C_i \times]c_{i-1}, c_i])) \right\} = m^*(D). \end{aligned}$$

Finally, this measure m^* coincides with $m_{\mathbf{A}}$ on $\mathcal{U}$: in fact from (34) for every $C \in \mathcal{C}$ and $x \in]0, F]$, $m_C(]0, x]) = m_{\mathbf{A}}\ (C\times]0, x])$, where $m_{\mathbf{A}}$ is the fuzzy measure defined on $\mathcal{U}$ associated to the given $\mathbf{A}$ by (18); moreover $m^*(R) = m_{\mathbf{A}}(R)$ for all R in $\mathcal{R}$. Therefore m^* generates $\mathbf{A}$. The construction of the measure m^* (see (35), (36), (37)) is unique and the only possible, because it should satisfy the requested properties.□

(Received January 30, 2002.)

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