

ROTARY INVERTED PENDULUM: TRAJECTORY TRACKING VIA NONLINEAR CONTROL TECHNIQUES

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The nonlinear control techniques are applied to the model of rotary inverted pendulum. The model has two degrees of freedom and is not exactly linearizable. The goal is to control output trajectory of the rotary inverted pendulum asymptotically along a desired reference. Moreover, the designed controller should be robust with respect to specified perturbations and parameters uncertainties. A combination of techniques based on nonlinear normal forms, output regulation and sliding mode approach is used here. As a specific feature, the approximate solution of the so-called *regulator equation* is used. The reason is that its exact analytic solution can not be, in general, expressed in the closed form. Though the approximate solution does not give asymptotically decaying tracking error, it provides reasonable bounded error. The performance of the designed feedback regulator is successfully tested via computer simulations.

1. INTRODUCTION

The problem of the output regulation of a laboratory model of the rotary inverted pendulum is considered here. This plant is similar to the classical inverted pendulum, the difference is that its base is moving along a circular trajectory. The problem of the output regulation (or the regulator problem) has received a lot of attention and, especially during the last decade, its nonlinear theory has been intensively developed [2, 4, 6]. The purpose of the output regulation is to achieve asymptotic tracking of desired output trajectory and/or reject the influence of undesired disturbances. The regulator problem for nonlinear systems has been introduced in the pioneering works [4, 6]. These initial results were not analyzed with respect to their robustness under uncertainties to the system. Basically, the necessary and sufficient conditions to solve the regulator problem [4, 6] consist in certain observability and stabilizability properties together with solvability of the set of algebraic and partial differential equations. This nonlinear analogue of the well-known Sylvester equation, called as the *regulator equation*, is difficult to solve analytically and a natural

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idea is to use approximate solution obtained via series expansions. That yields in general, a nondecaying tracking error which, nevertheless may be limited by a certain sufficiently small maximal bound, [5]. Starting with [3, 4] several methods have been proposed for reducing the influence of unknown parameters in the nonlinear regulator scheme. In particular, a combination of the nonlinear regulation theory and the sliding mode approach is used in [1, 8, 9, 10]. Such an approach allows to compensate, under certain assumptions, the effects of unknown parameters on the system, while keeping the tracking error within an acceptable small margin. More precisely, if the upper bound of the uncertain terms is known, the sliding mode based controller is able to perform the so-called approximate tracking, i. e. the controller may be adjusted in order to force the output tracking error remain less than any prescribed tolerance margin. The present paper develops and applies this approach to the rotary inverted pendulum model to obtain a control law which guarantees bounded output tracking error in the presence of variation of some parameters of the system.

The paper is organized as follows. Some basic results from nonlinear control theory are repeated in Section 2. In Section 3 the nonlinear model of the rotary inverted pendulum is presented while the control laws to assure the tracking reference signal are obtained in Section 4. Simulation results are discussed in Section 5. Finally, some conclusions are drawn in Section 6.

2. BASIC FACTS FROM NONLINEAR CONTROL THEORY

2.1. Nonlinear regulation

Consider the following nonlinear system

$$\dot{x} = f(x) + g(x)u + p(x)w \quad (1)$$

$$\dot{w} = s(w) \quad (2)$$

$$e = h(x) - r(w), \quad (3)$$

where equation (1) describes the plant dynamics with state $x \in \mathbb{R}^n$ and input $u \in \mathbb{R}$. Here, $f(\cdot)$, $s(\cdot)$, $g(\cdot)$ and the columns of $p(\cdot)$ are smooth vector fields, while $h(\cdot)$ and $r(\cdot)$ are smooth functions. The second equation describes the so-called exogenous system (*exosystem*) with the state $w \in \mathbb{R}^s$ which is assumed to generate both disturbances and reference signals. The exosystem is assumed to be neutrally stable, i. e. the point $w = 0$ is a stable equilibrium (in the usual Lyapunov's sense) and there exists an open neighborhood of the point $w = 0$ in which every point is Poisson stable.⁴ The last equation represents the output tracking error $e \in \mathbb{R}^p$ being the difference between the given output of the plant and another output of exosystem. The output regulation scheme is illustrated by Figure 1.

⁴Roughly saying, the point w_0 is said to be Poisson stable if the trajectory starting at w_0 returns to its arbitrarily small neighborhood after sufficiently large time. Typical example is a linear oscillator.

Remark 1. By virtue of the First Lyapunov’s Method, the hypothesis of neutral stability implies that the matrix $S = \left[\frac{\partial s(w)}{\partial w} \right]_{w=0}$ which characterizes the linear approximation of the vector field $s(w)$ at $w = 0$, has all its eigenvalues on the imaginary axis. Actually, the existence of a positive real part eigenvalue implies the existence of unstable solution, thereby violating the Lyapunov’s stability. On the other hand, the existence of a negative real part eigenvalue implies the existence of nontrivial solution converging to the origin what contradicts the Poisson stability property.

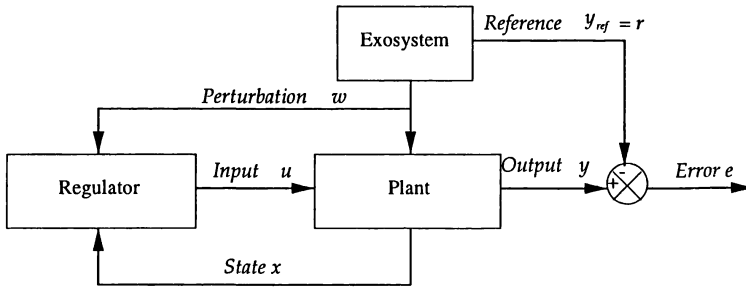


Fig. 1. Output regulation scheme.

It is also assumed that $(x, w, u) = (0, 0, 0)$ is an equilibrium state for the system (1-2), i.e. $f(0) = 0$, $s(0) = 0$ and $h(0) = r(0) = 0$.

Definition 1. The State Feedback Regulator Problem (SFRP) consists in finding a state feedback controller $u = \gamma(x, w)$ where $\gamma(\cdot, \cdot)$ is a $C^k (k \geq 2)$ mapping, with $\gamma(0, 0) = 0$ such that:

- S1. The equilibrium $x = 0$ of the so-called *disconnected system* $\dot{x} = f(x) + g(x)\gamma(x, 0)$ is asymptotically stable in the first approximation.
- R1. There exists a neighborhood $U \subset \mathbb{R}^n \times \mathbb{R}^s$ of $(0, 0)$ such that, for each initial condition on U , the system (1-3) with $u = \gamma(x, w)$ satisfies that

$$\lim_{t \rightarrow \infty} (h(x(t)) - r(w(t))) = 0.$$

Theorem 1. [6] Suppose:

- A1. The exosystem (2) is neutrally stable.
- A2. The pair $\left[\frac{\partial f}{\partial x}(0), g(0) \right]$ is asymptotically stabilizable.

The State Feedback Regulator Problem is solvable if and only if there exist $C^r (r \geq 2)$ mappings $x = \pi(w)$ and $u = c(w)$ with $\pi(0) = 0$ and $c(0) = 0$, both defined

in a neighborhood $W^0 \subset \mathbb{R}^q$ of 0, satisfying the conditions

$$\frac{\partial \pi(w)}{\partial w} s(w) = f(\pi(w)) + g(\pi(w))c(w) \quad (4)$$

$$0 = h(\pi(w)) - r(w). \quad (5)$$

Remark 2. Suppose (A2) holds, then there exists $K = [K_1, \dots, K_n]$ such that the matrix $\frac{\partial f}{\partial x}(0) + g(0)K$ is Hurwitz (i. e. asymptotically stable). It is straightforward to check that the choice of $\gamma(x, w) = c(w) + K(x - \pi(w))$ satisfies (S1), since $\gamma(x, 0) = Kx = K_1x_1 + \dots + K_nx_n$ stabilizes the linear approximation of $f(x) + g(x)Kx$. Moreover, it is shown in [6] that such a choice also satisfies (R1).

In other words, conditions (4)–(5), usually referred to as the *regulator equation*, are crucial for the solvability of the State Feedback Regulation Problem. That is, in general, difficult task, however, approximate solutions may be used, see e. g. [4, 5]. In particular, the first paper improved the stabilization results and addressed the issue of finding a power series expansion of the solution $\pi(w)$ of the regulator equation. The second paper obtained the property that if the solution in question is determined only up a certain degree of accuracy, then the output regulation can be secured up to a steady-state error of the same degree. Issues related to polynomial approximation and/or power series expansions for the determination of the solution of (4)–(5) were also considered in [7].

2.2. Input-output linearization and the normal forms of nonlinear systems

Consider the nonlinear system of the form

$$\begin{aligned} \dot{\tilde{x}} &= f(\tilde{x}) + g(\tilde{x})u \\ y &= h(\tilde{x}) \end{aligned} \quad (6)$$

with output $y \in \mathbb{R}^p$. Differentiating y with respect to time, we have

$$\dot{y} = \frac{dh}{d\tilde{x}} f(\tilde{x}) + \frac{dh}{d\tilde{x}} g(\tilde{x})u := L_f h(\tilde{x}) + L_g h(\tilde{x})u \quad (7)$$

where $L_f h(\tilde{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$ and $L_g h(\tilde{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$ stand for the Lie derivatives of h with respect to f and g , respectively. If $L_g h(\tilde{x}) \neq 0 \forall \tilde{x} \in B(\tilde{x}_0)$, an open ball centered at \tilde{x}_0 , then state feedback transformation of the form $u = \alpha(\tilde{x}) + \beta(\tilde{x})v$, introducing a new input variable v as follows

$$u = \frac{1}{L_g h(\tilde{x})} (-L_f h(\tilde{x}) + v) \quad (8)$$

yields the linear system

$$\dot{y} = v. \quad (9)$$

The control (8) renders $n-1$ of the states of (6) unobservable. If $L_g h(\tilde{x}) = 0 \forall \tilde{x} \in B(\tilde{x}_0)$, one differentiates (7) further to get

$$\ddot{y} = L_f^2 h(\tilde{x}) + L_g L_f h(\tilde{x})u \quad (10)$$

where $L_f^2 h(\tilde{x})$ stands for $L_f(L_f h)(\tilde{x})$ and $L_g L_f h(\tilde{x})$ for $L_g(L_f h)(\tilde{x})$. As before, if $L_g h(\tilde{x}) \neq 0 \forall \tilde{x} \in B(\tilde{x}_0)$, the control law

$$u = \frac{1}{L_g L_f h(\tilde{x})} (-L_f^2 h(\tilde{x}) + v) \quad (11)$$

yields the linear system

$$\ddot{y} = v. \quad (12)$$

More generally, if r is the smallest integer such that

$$L_g L_f^i h(\tilde{x}) = 0 \quad \forall \tilde{x} \in B(\tilde{x}_0) \quad i = 0, \dots, r-2 \quad (13)$$

and $L_g h(\tilde{x}) \neq 0 \forall \tilde{x} \in B(\tilde{x}_0)$, then the control law

$$u = \frac{1}{L_g L_f^{r-1} h(\tilde{x})} (-L_f^r h(\tilde{x}) + v) \quad (14)$$

yields

$$y^{(r)} = v. \quad (15)$$

In this case we say that a system has *relative degree* r .⁵ Suppose that the procedure terminates for some $r \leq n$ and $\dim\{dh(\tilde{x}), dL_f h(\tilde{x}), \dots, dL_f^{r-1} h(\tilde{x})\} = r$ for each $\tilde{x} \in B(\tilde{x}_0)$. For $r \leq n$, set

$$\phi_i(\tilde{x}) = L_f^{i-1} h(\tilde{x}), \quad i = 0, 1, \dots, r. \quad (16)$$

If r is strictly less than n , it is easy to verify that at each $\tilde{x} \in B(\tilde{x}_0)$ there exists a neighborhood U_0 of \tilde{x} where it is always possible to find $n-r$ more functions $\phi_{r+1}(\tilde{x}), \dots, \phi_n(\tilde{x})$ such that the mapping

$$\phi : U_0 \rightarrow \mathbb{R}^n \quad (17)$$

defined as

$$\phi(\tilde{x}) = \begin{pmatrix} \phi_1(\tilde{x}) \\ \vdots \\ \phi_r(\tilde{x}) \\ \phi_{r+1}(\tilde{x}) \\ \vdots \\ \phi_n(\tilde{x}) \end{pmatrix} := \begin{pmatrix} z_1 \\ \vdots \\ z_r \\ \eta_1 \\ \vdots \\ \eta_{n-r} \end{pmatrix} = \begin{pmatrix} h(\tilde{x}) \\ \vdots \\ L_f^{r-1} h(\tilde{x}) \\ \phi_{r+1}(\tilde{x}) \\ \vdots \\ \phi_n(\tilde{x}) \end{pmatrix} \quad (18)$$

is a diffeomorphism. Without going into the details, it can be shown that $\phi_{r+1}, \dots, \phi_n$ can be chosen so that $L_g \phi_i = 0$ for all $\tilde{x} \in B(\tilde{x}_0)$, $i = r+1, \dots, n$. If we set $\eta = (\phi_{r+1}, \dots, \phi_n)$, it follows that the equations (6) may be written in the *normal form* as

$$\begin{aligned} \dot{z}_i &= z_{i+1}, \quad i = 1, \dots, r-1, \\ \dot{z}_r &= b(z, \eta) + a(z, \eta)u \end{aligned} \quad (19)$$

$$\begin{aligned} \dot{\eta} &= q(z, \eta) \\ y &= z_1. \end{aligned} \quad (20)$$

⁵The theory is considerably more complicated if $L_g L_f^{r-1} h(\tilde{x}_0) = 0$, but is not identically zero on any neighborhood of \tilde{x}_0 .

In (19) $b(z, \eta)$ represents $L_f^r h(\tilde{x})$ and $a(z, \eta)$ represents $L_g L_f^{r-1} h(\tilde{x})$. The linearizing control is

$$u = \frac{1}{a(z, \eta)} (-b(z, \eta) + v). \quad (21)$$

so that the state variables $\eta = (\phi_{r+1}, \dots, \phi_n)$ are rendered unobservable from the output z_1 . The feedback law (21) is referred to as the *static state feedback input-output linearizing control law*. Now, if $\tilde{x} = 0$ is an equilibrium point of the system (6), i. e. $f(0) = 0, h(0) = 0$, then $z = 0, \eta = 0$ is an equilibrium point of the undriven system (19) i. e. $b(0, 0) = 0, a(0, 0) = 0$. A crucial property of the linearizing control (21) with $v = 0$ applied to (19) is that it keeps the submanifold

$$\begin{aligned} M_h &= \{\tilde{x} \in U_0 : h(\tilde{x}) = L_f h(\tilde{x}) = \dots = L_f^{r-1} h(\tilde{x}) = 0\} \\ &= \{\tilde{x} \in U_0 : z_1 = z_2 = \dots = z_r = 0\} \end{aligned}$$

invariant with respect to the closed-loop system.

The unforced dynamics of η on this invariant submanifold is referred to as the *zero-dynamics*, in other words the *zero-dynamics* is defined as

$$\dot{\eta} = q(0, \eta). \quad (22)$$

If the relative degree $r = n$, then η should be skipped in (19) and the full state equation (19) may be linearized by feedback (21). This is referred to as the *full state linearization* via coordinate transformation and static state feedback.

Remark 3. The dynamics (22) are referred to as the zero dynamics since it is the maximal dynamics always producing zero output made unobservable by static state feedback (21). It might help to the reader to note that the linearizing state feedback law is nonlinear analogue of placing some of the closed-loop poles at the zeros of the system, thereby rendering them unobservable.

Definition 2. The nonlinear system (6) is said to be locally minimum phase if \tilde{x}_0 is a locally asymptotically stable equilibrium of the zero-dynamics. In particular, a locally minimum phase system is said to be:

- (a) hyperbolically minimum phase if the Jacobian matrix $\frac{\partial q(0, \eta)}{\partial \eta} |_{\eta=0}$ has all its eigenvalues with negative real part,
- (b) critically minimum phase if the Jacobian matrix $\frac{\partial q(0, \eta)}{\partial \eta} |_{\eta=0}$ has an eigenvalue with zero real part.

An interesting application of the notion of the normal form is the following one. Consider (1)–(3) as the system (6) with the state $\tilde{x} = (x^T, w^T)^T$. Assuming that its relative degree r with respect to input u is well defined, the state space coordinate transformation $(z, \eta, w) = \Phi_e(x, w)$, where $z = \text{col}(z_1, \dots, z_r)$ and η are independent

coordinates, can be chosen in such a way that the description of the system (1-3) in the new coordinates becomes

$$\dot{z}_i = z_{i+1}, \quad i = 1, \dots, r-1, \quad (23)$$

$$\dot{z}_r = b(z, \eta, w) + a(z, \eta, w)u$$

$$\dot{\eta} = q(z, \eta, w, u) \quad (24)$$

$$\dot{w} = s(w) \quad (25)$$

$$e(t) = z_1. \quad (26)$$

Since the zero tracking error is desired, that implies $z_1 = 0$ and successive substitutions in (23) yield $z_i = 0, i = 1, \dots, r$. Thus, the input which keeps the error constrained to zero has the form

$$u = \alpha^*(0, \eta, w) = -a^{-1}(0, \eta, w)b(0, \eta, w). \quad (27)$$

The zero dynamics of the above system is in this case given by

$$\dot{\eta} = q(0, \eta, w, \alpha^*(0, \eta, w)) := q_0(\eta, w) \quad (28)$$

$$\dot{w} = s(w) \quad (29)$$

and $q_0(\eta, 0)$ coincides with the zero dynamics of the system (6). Theorem 1 then implies that, under its assumptions A1, A2, the SFRP has a solution for (1)–(3) if and only if there exists the smooth map $\pi_\eta(w) : \mathbb{R}^s \mapsto \mathbb{R}^{n-r}$ satisfying

$$\frac{\partial \pi_\eta(w)}{\partial w} s(w) = q(0, \pi_\eta(w), w, \alpha^*(0, \pi_\eta(w), w)), \quad \pi_\eta(0) = 0. \quad (30)$$

In other words, the regulator equation, being the set of n partial differential equations and single algebraic equation, is in the above normal form reduced to $n - r$ partial differential equations only. If (30) has the solution $\pi_\eta(w)$, it may be seen that the choice of the controller

$$u = a^{-1}(z, \eta, w)\{-b(z, \eta, w) + \tilde{K}_1 z + \tilde{K}_2(\eta - \pi_\eta)\} = \alpha_s(z, \eta, w), \quad (31)$$

solves the SFRP, where $\tilde{K} = [\tilde{K}_1 \quad \tilde{K}_2]$ is chosen such that $\tilde{K}_1 z + \tilde{K}_2 \eta$ stabilizes the linear approximation of equations (23)–(24).

Moreover, the normal form (23)–(26) provides significantly simpler and constructive sufficient condition to solve the SFRP. Namely, the Center Manifold Theorem shows that (30) is solvable if the zero dynamics $\dot{\eta} = q_0(\eta, 0)$ have an hyperbolic equilibrium at $\eta = 0$. Therefore, if the system in question is hyperbolic minimum phase, the SFRP is obviously solved via state feedback (31) with $\tilde{K}_2 = 0$ and \tilde{K}_1 being the vector of gains stabilizing the chain of integrators (23). Notice, that in this case the knowledge of the solution $\pi_\eta(w)$ of (30) is not required for the controller design.

2.3. Sliding mode technique

Since the control law (31) is designed based on the exact cancellation of nonlinear terms, it is clear, that if some additional perturbations occur in the system (due to parameter changes in the plant, for example), then this control law is not able to guarantee the zero output tracking error. The sliding mode technique is proposed here to deal with this situation.

When combining the variable structure control with the exact (partial) linearization, the essential idea is to exploit linearizability by the smooth state feedback in order to reduce the problem to one which is solvable by available methods. It is presumed that the output set is given and the associated zero dynamics are asymptotically stable. However, the results therein provide a straightforward construction for the switching surface and do not depend on a local explicit representation of the zero dynamics. Therefore, we use a scheme which is based on the nonlinear regulation theory and sliding mode approach. The last point is important in some applications, including rotary inverted pendulum studied by the present paper.

Remark 4. The connection between the zero dynamics and the variable structure control is straightforward since the constrained motion is analogous to the *sliding motion* of variable structure control. When the concepts of variable structure control are combined with the idea of partial linearization and the zero dynamics for nonlinear dynamical systems, we obtain an elegant characterization of control systems of this type. It will be seen that the *equivalent control* of variable structure theory is precisely the feedback (partial) linearizing and stabilizing control.

To proceed with the exposition of the sliding mode technique, let $H(z, \eta, w, t)$ be a smooth function that we refer in the sequel as the *switching function*. The set

$$M_H = \{(z, \eta, w) | H(z, \eta, w, t) = 0\} \quad (32)$$

is known as the *switching surface* and is called a *sliding surface* if it attracts all motion starting in a neighborhood of it. The motion of the system constrained on such a surface is known as *sliding mode*. Here, we choose $H(z, \eta, w, t)$ as

$$H(z, \eta, w, t) = z_r. \quad (33)$$

During the sliding modes, $H = \dot{H} = 0$, and the state trajectory of the nominal system is constrained to evolve on the sliding surface by the so-called *equivalent control* $u = u_{eq}$, which can be determined from the equation

$$\dot{H} = \dot{z}_r = b(z, \eta, w) + a(z, \eta, w)u_{eq} = 0 \quad (34)$$

for all (z, η, w) belonging to a neighborhood of $(0, 0, 0)$. On the other hand, $z = 0$ and $\eta = \pi_\eta(w)$ on M_H , thus, on this surface, the equivalent control coincides with the control law (31) that solves the regulator problem for the nominal system, i. e. $u_{eq} = \alpha_s(0, \pi_\eta(w), w)$. So, if the state trajectory of the system is attracted to the sliding surface, the equivalent control makes invariant the sliding surface. Note that

the surface $M_e = \{(z, \eta, w) | z = 0, \eta - \pi_\eta = 0\}$, which is precisely the steady state zero error submanifold, is contained in the switching surface M_H .

From [12] it is known that if an initial point (z_0, η_0, w_0) does not belong to M_H , the sliding condition

$$\dot{H}^T H \leq -\beta, \quad \beta > 0 \tag{35}$$

must be satisfied in a neighborhood of M_H so that this surface becomes attractive, as illustrated in Figure 2. A control law, which allows to reach the sliding surface, can be obtained from the equation $\dot{H} = -F(z, \eta, w)$, known as the *attractiveness equation*, where $F(\cdot)$ is, in general, a discontinuous vector function of its arguments. The control needed is thus given by

$$u = -a^{-1}(z, \eta, w) \{F(z, \eta, w) + b(z, \eta, w)\} = u_{slid}(z, \eta, w). \tag{36}$$

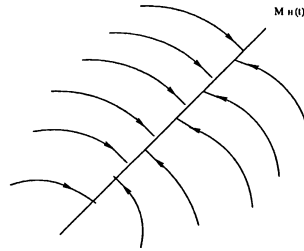


Fig. 2. Illustrating the sliding condition.

To guarantee the validity of the equation (35) and to have $u_{slid} = u_{eq}$ on the sliding surface, we choose the function F to be

$$F(z, \eta, w) = \gamma \text{sign}(H) - [\tilde{K}_1 z + \tilde{K}_2(\eta - \pi_\eta)] \tag{37}$$

where $\gamma > 0$, by definition $\text{sign}(0) = 0$ and \tilde{K}_1 and \tilde{K}_2 are calculated for the system (23)–(24). The proof that the control law (36)–(37) achieves bounded tracking error with parameters perturbations is given in [1].

Remark 5. Note that M_e is a submanifold of M_h which, in turn, is submanifold of M_H . The relationship between M_h , M_H and output zeroing manifold $H = 0$ is illustrated in Figure 3.

3. THE ROTARY INVERTED PENDULUM

The rotary inverted pendulum is an underactuated system shown in Figure 4. For our purpose, we assume that it has a planar motion without friction. The equation of

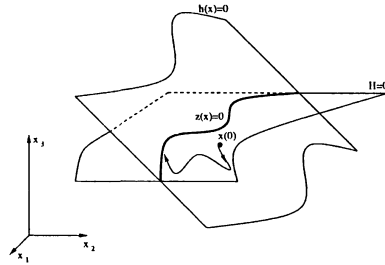


Fig. 3. The relationship between the output constraint manifold, the sliding manifold and the zero dynamics manifold in a three dimensional state space.

motion for the rotary inverted pendulum can be described by the standard equation for general mechanical system with several links [11]

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \tau \tag{38}$$

where q is the vector of joint variables (generalized coordinates), $D = \{d_{ij}\}$ is the $n \times n$ inertia matrix, $C = \{c_{ij}\}$ is the matrix of Coriolis and centripetal torques, $G = \{g_{ij}\}$ are the gravitational terms, and τ is the vector of input torques. If only m joints are actuated, vector q can be partitioned, without loss of generality as (q_1, q_2) , where q_1 represents the actuated joints, and q_2 represents the unactuated joints. For the rotary inverted pendulum system, the dynamic model (38) is further specified as

$$\begin{bmatrix} \tau_1 \\ 0 \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} \ddot{\alpha} \\ \ddot{\beta} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix},$$

where $d_{11} = m_2 l_1^2 + J$, $d_{12} = d_{21} = m_2 l_1 l_2 \cos(\beta)$, $d_{22} = m_2 l_2^2$, $c_{11} = c_{21} = c_{22} = 0$, $c_{12} = -m_2 l_1 l_2 \sin(\beta)\dot{\beta}$, $g_1 = 0$, $g_2 = -m_2 l_2 g \sin(\beta)$ and J is the moment of inertia of the base, g is the gravity acceleration, l_1 , α are respectively the length and the rotation angle of link 1, τ is the torque of the motor and m_2 , l_2 , β are respectively the mass, length and the rotation angle of pendulum.

The dynamics of the motor is given by

$$\tau_1 = T_1 v - T_2 \dot{\beta}, \quad T_1 = \frac{k_m k_g}{R_m}, \quad T_2 = \frac{k_m^2 k_g^2}{R_m},$$

and k_g is the gear ratio and k_m , R_m , and v are respectively the gain, resistance and the voltage input to motor.

Choosing as the state vector $x = (\alpha, \beta, \dot{\alpha}, \dot{\beta})^T$, as the input $u = (\tau_1 + T_2 \dot{\beta})T_1^{-1}$ and $y = \beta$ as the output, the description of the system can be given in state space form as:

$$\dot{x} = \begin{bmatrix} x_3 \\ x_4 \\ f_{31}(x) \\ f_{41}(x) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ g_{31}(x) \\ g_{41}(x) \end{bmatrix} u, \quad y = x_2, \tag{39}$$

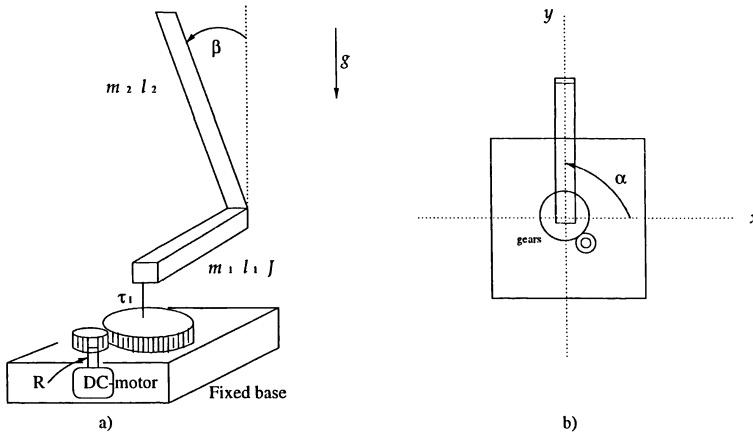


Fig. 4. Rotary inverted pendulum system.

$$\begin{aligned}
 f_{31} &= \frac{(T_2 + c_{12})x_4 - d_{12}g_2}{d_{12}^2 - d_{11}d_{22}}, & f_{41} &= \frac{d_{11}g_2 - (T_2 + c_{12})x_4}{d_{12}^2 - d_{11}d_{22}} \\
 g_{31} &= -\frac{T_1d_{22}}{d_{12}^2 - d_{11}d_{22}}, & g_{41} &= \frac{T_1d_{12}}{d_{12}^2 - d_{11}d_{22}}.
 \end{aligned}$$

This model will be used to compute the control law necessary for achieving the tracking of a reference signal.

4. CONTROL SCHEMES

Suppose we are interested in tracking a reference signal given by $y_{ref}(t) = A \sin(\lambda t + \rho)$. Therefore, we consider an exosystem given by

$$\dot{w} = \begin{bmatrix} \lambda w_2 \\ -\lambda w_1 \end{bmatrix}, \quad \text{with } w(0) = [w_1(0) \quad w_2(0)]^T.$$

4.1. State feedback regulator

Setting $e = x_2 - w_2$ as the output tracking error, equations (4-5) for system (39) take the form:

$$\frac{\partial \pi_1}{\partial w} s(w) = \pi_3 \tag{40}$$

$$\frac{\partial \pi_2}{\partial w} s(w) = \pi_4 \tag{41}$$

$$\frac{\partial \pi_3}{\partial w} s(w) = f_{31}(w) + g_{31}(w)c(w) \tag{42}$$

$$\frac{\partial \pi_4}{\partial w} s(w) = f_{41}(w) + g_{41}(w)c(w) \quad (43)$$

$$\pi_2 - w_2 = 0. \quad (44)$$

From the equation (44) we obtain directly $\pi_2 = w_2$, straightforward computations give $\pi_4 = -\lambda w_1$ and $c(w)$ may be obtained from (43). Nevertheless, as the system in question fails to be hyperbolic minimum phase, it is necessary to calculate the solutions $\pi_1(w)$ and $\pi_3(w)$ solving (40) and (42). As a matter of fact, (40) and (42) are exactly the reduced regulator equation (30) applied to the case of the rotary inverted pendulum model in normal form. Since solving these equations analytically is, in general, a difficult task, we propose an approximate solution. The price for such a simplification would be approximate tracking instead of the asymptotic one, as explained earlier, see for more details [5]. We thus assume a solution for π_1 of the form

$$\begin{aligned} \pi_1(w) = & a_1 w_1 + a_2 w_2 + a_3 w_1^2 + a_4 w_1 w_2 + a_5 w_2^2 + a_6 w_1^3 \\ & + a_7 w_1^2 w_2 + a_8 w_1 w_2^2 + a_9 w_2^3 + \theta(|w^4|). \end{aligned}$$

Differentiating this equation twice with respect to time, substituting in (41), and performing the expansion of the right side of the same equation of w^k to obtain a set of equations from which the values of a_j ($j = 1, \dots, 9$) are calculated. For the case of $\lambda = 1$, the solutions are given by

$$\pi_1(w) = -15.33w_2 - 4.31w_1w_2 - 4.78w_2^2$$

$$\pi_2(w) = 45.99w_1 + 12.3w_1^3 + 18w_1w_2^2.$$

Finally, the control law is given by

$$\gamma(x, w) = -\frac{w_2 + f_{41}(w)}{g_{41}(w)} + K(x - \pi(w)).$$

4.2. Control law using sliding modes

Setting $e = x_2 - w_2 = z_1$ as the output tracking error, the equations (23–25) for system (39), take the form:

$$\dot{z}_1 = z_2 \quad (45)$$

$$\dot{z}_2 = f_{41}(z, \eta, w) + g_{41}(z, \eta, w)u \quad (46)$$

$$\dot{\eta}_1 = \eta_2 \quad (47)$$

$$\dot{\eta}_2 = f_{31}(z, \eta, w) + g_{31}(z, \eta, w)u \quad (48)$$

$$\dot{w} = s(w), \quad (49)$$

where

$$f_{31}(z, \eta, w) = \frac{(T_2 + c_{12}(z, \eta, w))(z_2 - \lambda w_1) - d_{12}(z, \eta, w)g_2(z, \eta, w)}{p(z, \eta, w)},$$

$$f_{41}(z, \eta, w) = -\frac{(T_2 + c_{12}(z, \eta, w))(z_2 - \lambda w_1) + d_{11}g_2(z, \eta, w)}{p(z, \eta, w)},$$

$$\begin{aligned}
g_{31}(z, \eta, w) &= -\frac{T_1 d_{22}}{p(z, \eta, w)}, & g_{41}(z, \eta, w) &= \frac{T_1 d_{12}}{p(z, \eta, w)}, \\
p(z, \eta, w) &= d_{12}^2(z, \eta, w) - d_{11} d_{22}(z, \eta, w), & d_{12}(z, \eta, w) &= m_2 l_1 l_2 \cos(z_1 + w_2), \\
c_{12}(z, \eta, w) &= -m_2 l_1 l_2 \sin(z_1 + w_2)(z_2 - \lambda w_1), & g_2(z, \eta, w) &= -m_2 l_2 g \sin(z_1 + w_2).
\end{aligned}$$

After verifying all requirements, the robust controller (36)–(37) can be obtained, with

$$a(\cdot) = g_{41}(z, \eta, w), \quad b(\cdot) = f_{41}(z, \eta, w). \quad (50)$$

The mapping $\pi_\eta(w)$ must be obtained by solving the equations

$$\frac{\partial \pi_{\eta_1}}{\partial w} s(w) = \pi_{\eta_2} \quad (51)$$

$$\frac{\partial \pi_{\eta_2}}{\partial w} s(w) = f_{31}(0, \eta, w) + g_{31}(0, \eta, w)u. \quad (52)$$

As before, we propose an approximated solution for $\pi_{\eta_1}(w)$ of the form

$$\begin{aligned}
\pi_{\eta_1}(w) &= a_1 w_1 + a_2 w_2 + a_3 w_1^2 + a_4 w_1 w_2 + a_5 w_2^2 + a_6 w_1^3 \\
&\quad + a_7 w_1^2 w_2 + a_8 w_1 w_2^2 + a_9 w_2^3 + \theta(|w^4|).
\end{aligned}$$

For the case of $\lambda = 1$, the solution is given by

$$\begin{aligned}
\pi_{\eta_1}(w) &= 530.92w_1 - 81.23w_2 - 147.97w_1^3 + 73.80w_1^2 w_2 - 221.95w_1 w_2^2 + 58.55w_2^3 \\
\pi_{\eta_2}(w) &= 81.23w_1 + 530.92w_2 - 73.80w_1^3 + 28.04w_1 w_2^2 - 221.95w_2^3.
\end{aligned}$$

Finally, the control law is given by

$$u = -\frac{1}{g_{41}(z, \eta, w)} \{f_{41}(z, \eta, w) + \gamma \operatorname{sign}(H) - \tilde{K}_1 z - \tilde{K}_2(\eta - \pi_\eta)\}. \quad (53)$$

To avoid the discontinuities at $H = 0$ of the function $\gamma \operatorname{sign}(H)$, the following saturation function was implemented:

$$\gamma \operatorname{sign}(H) = \begin{cases} \gamma, & H > \epsilon \\ \frac{\gamma H}{\epsilon}, & -\epsilon \leq H \leq \epsilon \\ -\gamma, & H < -\epsilon \end{cases} \quad (54)$$

where ϵ is a suitable positive number.

5. SIMULATION RESULTS

For the simulation purposes the parameters shown in Table 1 were used. Simulations plots are presented in Figures 5 and 6.

The rotary inverted pendulum started at initial time moment from the state $\alpha = 0$ and $\beta = 0.5$, i. e. the initial condition of the system was near the equilibrium point. The reference signal to be tracked was taken as

$$y_{\text{ref}}(t) = 0.1 \sin(t).$$

Table 1. Parameters of the rotary inverted pendulum.

m_2 [kg]	l_1 [m]	l_2 [m]	J [kg m ²]	$g[\frac{m}{s^2}]$	k_g [cm]	k_m	$R_m[\Omega]$
0.55	0.13	0.75	0.005	9.81	5.5	0.104	1.9

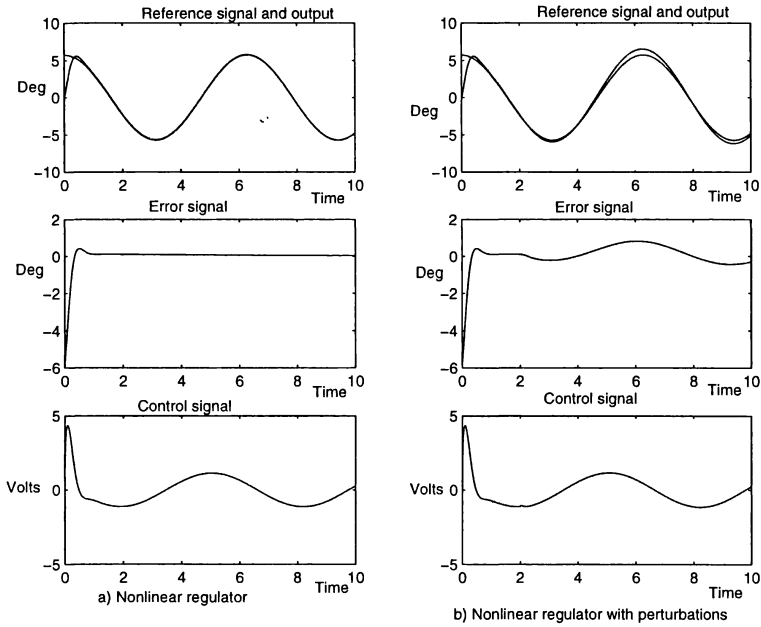


Fig. 5. Simulation results: nonlinear regulator.

The matrix K introduced in Section 4 was chosen as the matrix that stabilizes the linear approximation of the system (39). It was obtained by solving a LQR problem as

$$K = \begin{bmatrix} -0.66 & -43.25 & -1.33 & -13.5 \end{bmatrix}$$

and the matrix $\tilde{K} = [\tilde{K}_1 \quad \tilde{K}_2]$ is chosen such that $\tilde{K}_1 z + \tilde{K}_2 \eta$ stabilizes the linear approximation of equations (23)–(24). It was obtained by solving a LQR problem as

$$\tilde{K} = \begin{bmatrix} 94.42 & 361.06 & 77.731 & 15.064 \end{bmatrix}.$$

For comparison purposes, we have first tested the controller described in subsection 4.1, based on the state feedback computed via the approximation of the regulator equation. As expected by the theory, the controller provides a good performance resulting in a small tracking error for the initial condition chosen, see left column of

Figure 5. Nevertheless , it does not compensate unknown variations of the nominal values of l_1 , l_2 and m_2 , cf. the right column of the same figure.

Secondly, Figure 6 illustrates the simulations results using sliding mode based robust controller developed in the subsection 4.2. In this case, the corresponding controller compensates the effects of perturbations and/or uncertainties on the system. Actually, to show the robustness of the sliding mode based controller scheme, a modification up to 20% of the nominal values of the parameters l_1 , l_2 and m_2 has been introduced on the time interval [2,10] and $\epsilon = 0.1$. Despite these modifications, the sliding mode controller designed based on the nominal parameters knowledge performs well, resulting in the output tracking with a prescribed tracking error. Notice the typical sliding mode controller plots. For the real-time implementation, a suitable filter is to be used to produce an implementable control signal.

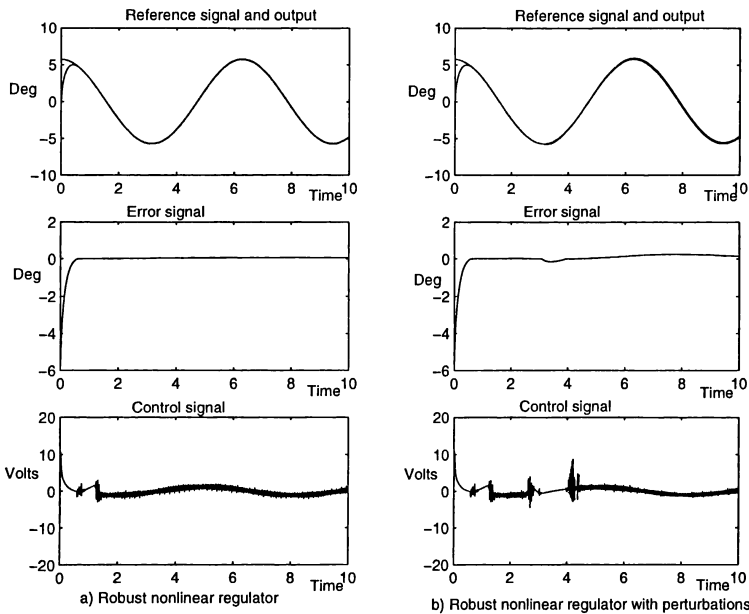


Fig. 6. Simulation results: nonlinear sliding regulator.

6. CONCLUSIONS

Nonlinear control techniques, applicable to the output regulation problem, have been studied in this paper. The rotary inverted pendulum model has been investigated in detail to illustrate how the combination of the normal forms and sliding mode control may be used in order to design controllers robust with respect to the changes

of the system parameters. Normal forms were used to reduce the order of the regulator equation and the corresponding reduced equation (30) has been solved approximately. On the other hand, the sliding surface has been designed in such a way that the corresponding sliding mode equivalent control coincides with the input-output linearizing state feedback, thereby handling impossibility to compute linearizing feedback in the presence of uncertainties. The computer simulations showed effectiveness of the above approach for the rotary inverted pendulum model subjected to parameter variations up to 20 %.

The future research will be devoted both to further aspects of real-time implementation (e.g. filtering of sliding mode control signal) and further theoretical interpretation of robust properties of the scheme based on sliding modes.

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