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EXISTENCE OF POLE–ZERO STRUCTURES IN A RATIONAL MATRIX EQUATION ARISING IN A DECENTRALIZED STABILIZATION OF EXPANDING SYSTEMS

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A necessary and sufficient condition for the existence of pole and zero structures in a proper rational matrix equation $T_2X = T_1$ is developed. This condition provides a new interpretation of sufficient conditions which ensure decentralized stabilizability of an expanded system. A numerical example illustrate the theoretical results.

NOTATION

— C_+ denotes the closed right half plane and C_{+e} denotes the extended right half plane (i. e., $C_{+e} = C_+U\{\infty\}$).

— R_p denotes the ring of proper rational functions.

— R_s denotes the ring of stable rational functions (i. e., those with no poles in C_+).

— R_{ps} denotes the ring of proper stable rational functions with real coefficients.

— $M(R_{ps}(s))$ denotes the set of matrices whose entries are in $R_{ps}(s)$.

— A matrix $\mathcal{M} \in M(R_{ps}(s))$ is called $R_{ps}(s)$-unimodular iff $\mathcal{M}^{-1} \in M(R_{ps}(s))$.

1. INTRODUCTION

In spite of the active research carried out in decentralized control in last two decades, there has been little attention paid to the stable solution of a rational matrix that arises in a decentralized controller for interconnected systems. A systematic design approach that solves the decentralized problem using a concurrent synthesis approach or a sequential stable synthesis approach is discussed by Davision and Chang [1]. The methods of implementing overall stable interconnected control systems, reported so far [3], are mostly those by state feedback or observer based state feedback. A different method for such stabilization using the proper stable factorization approach has drawn considerable interest in the literature [7, 9] and the significant result of this approach can be used to parametrically characterize all stabilizing controllers. This approach can easily be employed to implement a local or decentralized stabilizing controller with an unspecified parameter which can further be tuned to ensure the composite closed-loop system connectively stable.

The problem considered in this paper is an extension of the solution of stable exact model matching problem in order to design local dynamic controller for each subsystem of interconnected systems.

2. PROBLEM FORMULATION

Let us consider an interconnected system and each subsystem is described by its input-output decentralized form [5]

$$S_i: \quad \dot{X}_i(t) = A_iX_i(t) + B_iU_i(t) + G_iV_i \tag{1}$$

$$Y_i(t) = C_iX_i(t) \tag{2}$$

$$W_i = H_iX_i(t), \qquad i = 1, 2, \ldots, N \tag{3}$$

where $X_i(t)$ is the state, $U_i(t)$ is the control input, $Y_i(t)$ is the measured output, $V_i(t)$ is the interconnection input, and $W_i(t)$ is the interconnection output of the subsystem S_i. The matrices A_i, B_i, C_i, G_i, H_i are constant and of appropriate dimensions. It is assumed that the pair (A_i, B_i) is stabilizable and (C_i, A_i) is detectable. Figure 1 below shows the basic expanding structure of interconnected systems.

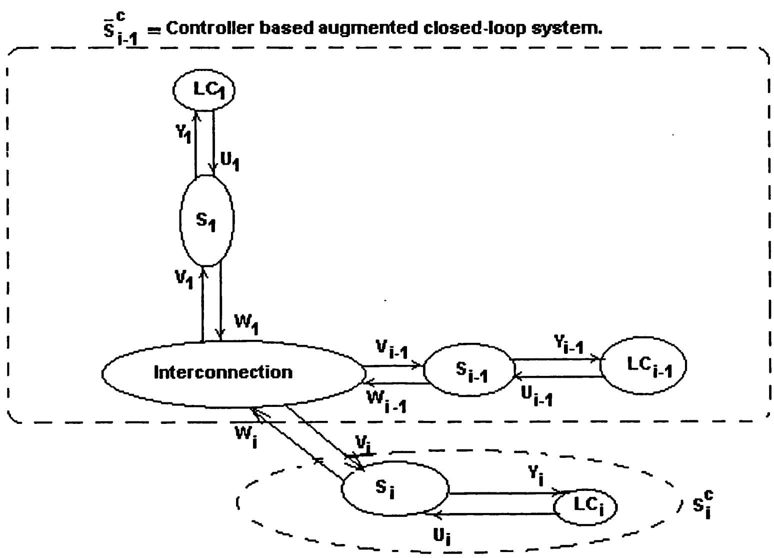

Fig. 1. Expanding construction.

A systematic design procedure to find a set of decentralized stabilizing controllers LC_i based on proper stable factorization approach [9] is addressed in this paper and each subsystem described by equations (1) – (3) is now represented by its equivalent transfer function matrix as

$$S_i : \begin{bmatrix} W_i(s) \\ Y_i(s) \end{bmatrix} = \begin{bmatrix} Z_{i11} & Z_{i12} \\ Z_{i21} & Z_{i22} \end{bmatrix} \begin{bmatrix} V_i(s) \\ U_i(s) \end{bmatrix} \tag{4}$$

where Z_{ijk} $(j,k = 1,2)$ are defined as

$$\begin{aligned} Z_{i11} &= H_i(sI - A_i)^{-1}G_i\,; \quad Z_{i12} = H_i(sI - A_i)^{-1}B_i \\ Z_{i21} &= C_i(sI - A_i)^{-1}G_i\,; \quad Z_{i22} = C_i(sI - A_i)^{-1}B_i \end{aligned} \tag{5}$$

It can be noted that the transfer function matrix from the control input U_i to the measured output Y_i is Z_{i22}, which is stricly proper. To find the set of stabilizing controllers, Z_{i22} is factorized as

$$Z_{i22} = N_i D_i^{-1} = \bar{D}_i^{-1} \bar{N}_i$$

where $(N_i, D_i)$and$(\bar{N}_i, \bar{D}_i)$ are the right and left coprime factorization of Z_{i22} and $N_i, D_i, \bar{N}_i, \bar{D}_i \in M(R_{ps})$.Then there exist matrices, $P_i, Q_i, \bar{P}_i$, and $\bar{Q}_i \in M(R_{ps})$ such that

$$\begin{bmatrix} Q_i & P_i \\ -\bar{N}_i & \bar{D}_i \end{bmatrix} \begin{bmatrix} D_i & -\bar{P}_i \\ N_i & \bar{Q}_i \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{6}$$

and this expression is known as forward *Bezout identity*.

The set of all stabilizing controllers for the subsystem S_i of equations (1) – (3) is given by

$$U_i = -(\bar{P}_i + D_i R_i)(\bar{Q}_i - N_i R_i)^{-1} Y_i = K_i Y_i \tag{7}$$

where $R_i \in M(R_{ps})$ is the free parameter [9]. Left and right factorization of Z_{i22} and the solution of *Bezout identity* equation (6) are combined to provide a set of stabilizing controllers and subsequently the minimal realization of equation of (7) can be obtained by using the techniques described in [4]. With a specified R_i, (1) – (3) is a system to be controlled. To establish the transfer function matrix $T_i(R_i)$ from the interconnection input V_i, to the interconnection output W_i for subsystem S_i, we consider the equation (4)

$$\begin{aligned} W_i(s) &= Z_{i11} V_i(s) + Z_{i12} U_i(s) \\ &= [Z_{i11} + Z_{i12} K_i(R_i) Y_i(s)] \\ &= [Z_{i11} + Z_{i12} K_i(R_i)(I - Z_{i22} K_i(R_i))^{-1} Z_{i21}] V_i(s) \\ &= [Z_{i11} - Z_{i12}(\bar{P}_i + D_i R_i)\bar{D}_i Z_{i21}] V_i(s) \end{aligned} \tag{8}$$

where, $K_i(R_i)(I - Z_{i22}K_i(R_i))^{-1} = -(\bar{P}_i + D_i R_i)\bar{D}_i$. Equation (8) can be written as

$$\begin{aligned} W_i(s) &= [Z_{i11} - Z_{i12}\bar{P}_i\bar{D}_i Z_{i21} - Z_{i12} D_i R_i \bar{D}_i Z_{i21}] V_i(s) \\ &= [T_{i1} - T_{12} R_i T_{i3}] V_i(s) \\ &= T_i(R_i) V_i(s) \end{aligned} \tag{9}$$

where

$$\begin{aligned} T_{i1} &= Z_{i11} - Z_{i12}\bar{P}_i\bar{D}_i Z_{i21} \\ T_{i2} &= Z_{i12} D_i \\ T_{i3} &= \bar{D}_i Z_{i21}. \end{aligned} \tag{10}$$

The main motivation for considering this problem comes from sufficient conditions for decentralized stabilizability of expanding system due to Tan and Ikeda [6]. These conditions are stated below.

Condition 1. There exits a local controller for which both the closed-loop subsystem and the expanded system are stable if the equations

$$\begin{aligned} T_{i2}L_i &= T_{i1} \\ N_i T_{i3} &= T_{i1} \end{aligned} \tag{11}$$

have solutions L_i and $N_i \in M(R_{ps})$ where T_{i1}, T_{i2}, and $T_{i3} \in M(R_{ps})$.

Condition 2. Closed-loop and expanded systems can be made stable by a local controller provided the following equations

$$\begin{aligned} \begin{bmatrix} Z_{i12} \\ Z_{i22} \end{bmatrix} L_i &= \begin{bmatrix} Z_{i11} \\ Z_{i21} \end{bmatrix} \\ N_i \begin{bmatrix} Z_{i21} & Z_{i22} \end{bmatrix} &= \begin{bmatrix} Z_{i11} & Z_{i12} \end{bmatrix} \end{aligned} \tag{12}$$

have solutions L_i and $N_i \in M(R_s)$.

The contribution of the present paper is the development of a tractable necessary and sufficient condition for a stable solution of the above matrix equations. The approach here is an extension of the solution of the *stable exact model matching problem*

$$T_2 X = T_1 \tag{13}$$

where $T_2 \in R_p(s)^{p\times m}$ and $T_1 \in R_p(s)^{p\times q}$ are two proper rational matrices and the solution X is required to be both *proper and stable.* It was shown by Vardulakis and Karcanias [8] that there exists a proper, stable solution $X \in R_{ps}(s)^{m\times q}$ for the matrix equation (13) if and only if the matrices T_2 and $[T_2 \quad T_1]$ have the same pole structure in C_+ and the same zero structure in C_{+e}. The main result of this paper is that a stable but not necessarily proper solution $X \in R_s(s)^{m\times q}$ exists if and only if the pole and zero structures of these matrices are the same in C_+, implying that $[T_2 \quad T_1]$ need not have the same zero structure as T_2 at infinity. Note that two matrices are called equivalent at C_+ if there exist in R_s unimodular matrices T_L and T_R such that $T_L T_1 T_R = T_2$.

3. MAIN RESULT

Suppose that $T_2 \in R_p(s)^{p\times m}$ has rank r; then there exit $R_{ps}(s)$-unimodular matrices $U \in R_{ps}(s)^{p\times p}, V \in R_{ps}(s)^{m\times m}$ such that

$$U T_2 V = \left[\begin{array}{ccc:c} \epsilon_1 \lambda_1/\psi_1 & & & \\ & \ddots & & 0_{r\times(m-r)} \\ & & \epsilon_r \lambda_r/\psi_r & \\ \hdashline & 0_{(p-r)\times r} & & 0_{(p-r)\times(m-r)} \end{array}\right]$$

$$= \ : S_2 \tag{14}$$

where, for $i = 1, \cdots, r$, ϵ_i, $\psi_i \in R_{ps}(s)$ are not strictly proper and $\lambda_i \in R_{ps}(s)$ is possibly strictly proper with no zeros in C_+. The matrix S_2 is the Smith–McMillan form of T_2 with respect to $R_ps(s)$ [4]. Let $\Lambda \in R_{ps}(s)^{m\times m}$ be defined as

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \lambda_r & \\ & & & I_{m-r} \end{bmatrix}; \tag{15}$$

i.e., Λ reflects the zero structure of T_2 at infinity. Then $S_2\,\Lambda^{-1} \in R_{ps}(s)^{p\times m}$ is not strictly proper and it reflects the pole and zero structure of T_2 in C_+.

Theorem 3.1. Let $T_2 \in R_p(s)^{p\times m}$ and $T_1 \in R_p(s)^{p\times q}$; there exists a solution $X \in R_s(s)^{m\times q}$ for the matrix equation $T_2X = T_1$ if and only if the matrices T_2 and $[T_2 \quad T_1]$ have the same pole and zero structure in C_+.

Proof. *Sufficiency*: Consider the Smith–McMillan form (14) of T_2. By assumption, the Smith–McMillan form of $[T_2 \quad T_1]$ is $[S_2\Lambda^{-1} \quad 0_{p\times q}]$. Therefore, there exists an $R_{ps}(s)$-unimodular matrix M such that

$$\begin{aligned} U[T_2 \quad T_1]\begin{bmatrix} V & 0 \\ 0 & I \end{bmatrix} &= [S_2 \quad U\,T_1] \\ &= [S_2\,\Lambda^{-1} \quad 0_{p\times q}]\,M. \end{aligned} \tag{16}$$

Let $M_{12} = M[0 \quad I]^T$; then (15) implies that

$$\begin{aligned} T_1 &= (U^{-1}\,S_2\,V^{-1})(V\,\Lambda^{-1}\,M_{12}) \\ &= T_2\,X \end{aligned}$$

where $X = V\,\Lambda^{-1}\,M_{12}$ is stable but not necessarily proper.

Necessity: If $T_1 = T_2\,X$, then

$$\begin{aligned} U[T_2 \quad T_1]\begin{bmatrix} V & 0 \\ 0 & I \end{bmatrix} &= [S_2\,\Lambda^{-1}\,\Lambda \quad S_2\,\Lambda^{-1}\,\Lambda\,V^{-1}\,X] \\ &= S_2\,\Lambda^{-1}[\Lambda \quad \Lambda\,V^{-1}\,X]. \end{aligned} \tag{17}$$

Since U, V are $R_{ps}(s)$-unimodular, the pole and zero structure of $[T_2 \quad T_1]$ does not change in C_{+e} when multiplied by U and V. Furthermore, since Λ is full rank in C_+, the stable matrix $[\Lambda \quad \Lambda\,V^{-1}\,X]$ has no zeros in C_+ and hence by (16), the pole and zero structure of $[T_2 \quad T_1]$ in C_+ is the same as $S_2\,\Lambda^{-1}$, which is the same as that of T_2 in C_+. □

3.1. Comments

(a) If T_2 and T_1 are proper stable matrices, then it is clear that a stable solution X exists for the matrix equation $T_2 X = T_1$ if and only if the matrices T_2 and $[T_2 \quad T_1]$ have the same zero structure in C_+ since they have no poles in C_+.

(b) With T_{i1}, T_{i2} and $T_{i3} \in M\,(R_{ps}(s))$, we can apply the results of the Theorem 3.1 in equation (11) to obtain the new decentralized stabilizability condition of the expanded system. There exists a local controller for which both the closed-loop subsystem and the expanded system are stable if

   i. T_{i2} and $[T_{i2} \quad T_{i1}]$ have the same zero structure at C_+ and

   ii. T_{i3}^T and $[T_{i3}^T \quad T_{i1}^T]$ have the same zero structure at C_+.

(c) The equivalent decentralized stalizability condition given in equation (12) can also be stated in terms of the pole and zero structures of the appropriate matrices in C_+. There exists a local controller for which both the closed loop subsystem and the expanded system are stable if

   i.

$$\begin{bmatrix} Z_{i12} \\ Z_{i22} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} Z_{i12} & Z_{i11} \\ Z_{i22} & Z_{i21} \end{bmatrix}$$

   ii.

$$\begin{bmatrix} Z_{i21}^T \\ Z_{i22}^T \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} Z_{i21}^T & Z_{i11}^T \\ Z_{i22}^T & Z_{i12}^T \end{bmatrix}$$

have the same pole and zero structures in C_+.

Note that in this case, the matrices Z_{N11}, Z_{N12}, Z_{N21} and Z_{N22} [6] are not stable and hence, the pole structures at C_+ need to be checked as well as the zero structures. Existence of decentralized stabilizing controller is discussed in (b) and (c) and it can be checked by finding the pole-zero structures at C_+.These poles and zeros can be computed using elementary row and column operation on the matrices in (b) and (c) to obtain Smith–McMillan form of transfer function matrices. A numerically reliable stable computations of zeros using unitary transformation method can be found in [2].

4. AN EXAMPLE

The first open loop subsystem is described by

$$\begin{aligned} S_1: \qquad \dot{X}_1(t) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} X_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U_1 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} V_1 \\ Y_1 &= \begin{bmatrix} 1 & 2 \end{bmatrix} X_1 \\ W_1 &= \begin{bmatrix} 1 & 1 \end{bmatrix} X_1. \end{aligned} \tag{18}$$

As this is the only system to be stabilized, it is not required to satisfy any of the established conditions discussed in Section 3.2.

Applying static output feedback $U_1 = -Y_1$, we obtain the closed-loop stable system:

$$\begin{aligned} \bar{S}_1^C = S_1^C: \qquad \dot{X}_1(t) &= \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} X_1 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} V_1 \\ W_1 &= \begin{bmatrix} 1 & 1 \end{bmatrix} X_1. \end{aligned} \tag{19}$$

Now, we connect the second subsystem given by the following equation:

$$\begin{aligned} S_2: \qquad \dot{X}_2(t) &= \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} X_2 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U_2 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} V_2 \\ Y_2 &= \begin{bmatrix} -1 & 1 \end{bmatrix} X_2 \\ W_2 &= \begin{bmatrix} -1 & 1 \end{bmatrix} X_2 \end{aligned} \tag{20}$$

with interconnection

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}. \tag{21}$$

Using the expressions given in (5), we obtain

$$Z_{2,11} = Z_{2,12} = Z_{2,21} = Z_{2,22} = \frac{s-1}{s^2+s+1}.$$

We factorize $Z_{2,22}$ as follows:

$$\begin{aligned} N_2 = \bar{N}_2 = Z_{2,22} &= \frac{s-1}{s^2+s+1} \\ D_2 &= \bar{D}_2 = 1.0 \\ P_2 &= \bar{P}_2 = 0.0 \\ Q_2 &= \bar{Q}_2 = 1.0. \end{aligned}$$

This implies that the class of all stabilizing controllers is characterized as (see equation (7))

$$U_2 = \frac{-R_2(s^2+s+1)}{s^2+s(1-R_2)+(1+R_2)} Y_2$$

where, $R_2 \in R_{ps}$. So, we have

$$T_{2,1} = T_{2,2} = T_{2,3} = \frac{s-1}{s^2+s+1}.$$

It is straight forward to check that the conditions given in Section 3.1 (see (a) and (b)) are satisfied. So, the expanded system $\bar{S}_2^C$ is decentrally stabilizable. This fact is clearly supported by the observation that the two systems, connected only via inputs and outputs, are individually stabilizable implying the absence of any (unstable) fixed modes.

5. CONCLUSIONS

The subtle point of the main result of this paper is that it provides a condition which avoids the computation of the finite transmission zeros of some transfer function matrices. A proof of the main theorem using Smith–McMillan form of transfer function matrix is given. Explicit expression for computing the solutions are also given which can then be used in equation (7) to design a decentralized controller for the expanded system.

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