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DECOUPLING AND POLE ASSIGNMENT BY CONSTANT OUTPUT FEEDBACK

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In this paper a system-theoretic approach is used to solve the decoupling in combination with the arbitrary pole assignment problem by constant output feedback and a constant nonsingular input transformation. Explicit necessary and sufficient conditions are given and a procedure is described for the determination of the control law.

1. INTRODUCTION

The problem of decoupling by constant output feedback has been studied in the past by Wolowich [10], Wang and Davison [8], Descusse [2], Howze [3], Bahey Argoun and Van de Vegte [1] and Parskevopoulos and Koumboulis [6]. In these papers necessary and sufficient conditions for the existence of constant output feedback to achieve noninteracting control have been obtained.

In the present work the problem of decoupling in combination with arbitrary pole assignment is studied. In particular, using a system-theoretic method, explicit necessary and sufficient conditions are derived for the existence of a constant output feedback and a constant nonsingular input transformation that solve the problem of decoupling with arbitrary pole assignment. It is proved that the above problem has a solution if and only if all the reachability indices and the observability indices of the open-loop system are equal to one. This is equivalent to saying that both the number of the inputs and the number of the outputs of the system are equal to the number of states and the output feedback becomes state feedback. Furthermore a constructive procedure is given for the computation of the control law that decouples the closed-loop system and assigns arbitrarily its poles.

2. PROBLEM STATEMENT

Let us consider a linear time-invariant, discrete-time, reachable and observable system S having as many inputs as outputs and described by the following equations

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned} \tag{2.1}$$

where $u(k) \in I\!R^m$, $y(k) \in I\!R^m$ and $x(k) \in I\!R^n$.

The transfer function matrix $T(z)$ of S is given by the relation

$$T(z) = C(Iz - A)^{-1}B = N_R(z)\, D_R^{-1}(z) \tag{2.2}$$

where the matrices $D_R(z)$ and $N_R(z)$ over $I\!R[z]$ (the ring of polynomials having real coefficients) form a standard right matrix fraction description of $T(z)$ [9].

Let the control law

$$u(k) = Fy(k) + Gv(k) \tag{2.3}$$

be applied to the system (2.1), where F and G are $m \times m$ constant matrices, G is nonsingular and $v(k)$ represents the $m \times 1$ reference input vector. Then the state space description of the compensated system is given by

$$\begin{aligned} x(k+1) &= [A + BFC]\, x(k) + BGv(k) \\ y(k) &= Cx(k). \end{aligned} \tag{2.4}$$

The problem of decoupling with arbitrary pole placement can be stated as follows. Find the control law, as in (2.3) so that for the compensated system:
The transfer function matrix $T_c(z)$ satisfies the relations

$$T_c(z) = C(Iz - A - BFC)^{-1}BG = \operatorname{diag}[b_1(z)/a_1(z), \ldots, b_m(z)/a_m(z)] \tag{2.5a}$$

$$\det[T_c(z)] \not\equiv 0 \tag{2.5b}$$

and the characteristic polynomial is

$$\det[Iz - A - BFC] = a(z) \tag{2.6}$$

where $a(z)$ is a monic polynomial of degree n having roots at the desired positions for the poles of the closed-loop system.

Since the control law (2.3) does not affect the reachability and observability properties for the compensated system, we have

$$a(z) = \prod_{i=1}^{m} a_i(z). \tag{2.7}$$

If the matrix fraction description is used for S, the transfer function matrix of the compensated system under the control law (2.3) is given by

$$T_c(z) = N_R(z)\left[G^{-1}D_R(z) + G^{-1}FN_R(z)\right]^{-1} \tag{2.8}$$

and the problem is reformulated as follows. Find the matrices F, G in (2.8) so that relations (2.5) – (2.7) are satisfied.

It is to be noted that $a(z)$ must have some real roots if its factorization, as in equation (2.7), has to take place. Furthermore, it is to be pointed out that since relation (2.5b) holds, the matrices B and C have full column and full row rank respectively and thus $m \leq n$.

3. BASIC CONCEPTS

Let us first introduce some notations that are used throughout the paper. For any $p \times q$ polynomial matrix $A(z)$ [5], we write $\deg_{ri} A(z) = w_i$ for the degree of the ith row of $A(z)$ and $\deg_{ci} A(z) = v_i$ for the degree of the ith column of $A(z)$. The $p \times q$ matrix $A(z)$ is defined *row reduced* if its highest row degree coefficient matrix $A_{hr} = \lim_{z\to\infty} \operatorname{diag}[z^{-w_1}, \dots, z^{-w_p}]\, A(z)$ has rank p, and is defined *column reduced* if its highest column-degree coefficient matrix $A_{hc} = \lim_{z\to\infty} A(z) \operatorname{diag}[z^{-v_1}, \dots, z^{-v_q}]$ has rank q. If the rows of $A(z)$ are arranged so that $\deg_{ri} A(z) \geq \deg_{rj}(z)$, for $i < j$, then $A(z)$ is called *row degree ordered* while if $\deg_{ci} A(z) \geq \deg_{cj} A(z)$, for $i < j$ is defined *column degree ordered.*

Two polynomial matrices $A(z)$ and $B(z)$ of respective size $p \times q$ and $p \times m$ are defined *relatively left prime* over $\mathbb{R}[z]$ it the matrix $[A(z),\ B(z)]$ does not lose rank for every $z \in C$. Similarly two polynomial matrices $A(z)$ and $B(z)$ of size $q \times m$ and $p \times m$ respectively are defined *relatively right prime* over $\mathbb{R}[z]$ if the matrix $[A^{\mathrm{T}}(z),\ B^{\mathrm{T}}(z)]^{\mathrm{T}}$ does not lose rank for every $z \in C$.

Definition 1. If $T(z)$ is a matrix with rational entries, the polynomial matrices $D_R(z)$ and $N_R(z)$ such that

(a) $T(z) = C(Iz - A)^{-1}B = N_R(z)\, D_R^{-1}(z)$,

(b) $D_R(z)$ and $N_R(z)$ are relatively right prime,

(c) $\begin{bmatrix} D_R(z) \\ N_R(z) \end{bmatrix}$ is column reduced and column degree ordered

are said to form a *standard right matrix fraction description* of $T(z)$.

Definition 2. Polynomial matrices $D_L(z)$ and $N_L(z)$ such that

(a) $T(z) = C(Iz - A)^{-1}B = D_L^{-1}(z)\, N_L(z)$,

(b) $D_L(z)$ and $N_L(z)$ are relatively left prime,

(c) $[D_L(z),\ N_L(z)]$ is row reduced and row degree ordered

are said to form a *standard left matrix fraction description* of $T(z)$.

The right minimal indices of $T(z)$ are defined to be the column degrees of any standard right matrix fraction description of $T(z)$. Since the system (2.1) is both reachable and observable, the right minimal indices of $T(z)$ are equal to the reachability indices of the system.

The left minimal indices of $T(z)$ are defined to be the row degrees of any standard left matrix fraction description of $T(z)$. Since the system (2.1) is both reachable and observable, the left minimal indices of $T(z)$ [4] are equal to the observability indices of the system.

The polynomial matrices $X(z)$ and $Y(z)$ are equivalent over $\mathbb{R}[z]$ if there exist unimodular matrices $U_1(z)$ and $U_2(z)$ such that $Y(z) = U_1(z)\, X(z)\, U_2(z)$. Since the system (2.1) is both reachable and observable, $N_R(z)$ and $N_L(z)$ can be easily proved to be equivalent over $\mathbb{R}[z]$ and their Smith form describes the position and the structure of the finite invariant zeros of the system (2.1).

4. PRELIMINARY RESULTS

This section contains results that are needed to prove the main theorem of this paper.

Lemma 4.1. Let $(D_R(z), N_R(z))$ and $(D_L(z), N_L(z))$ be a standard right matrix fraction description and a standard left matrix fraction description of the strictly proper rational matrix $T(z)$ respectively. Also let v_i for $i = 1, 2, \dots, m$ and w_j for $j = 1, 2, \dots, p$ be the minimal right and the minimal left indices of $T(z)$ respectively. Then for every $m \times p$ real matrix F and for every nonsingular $m \times m$ real matrix G we have

(a) The polynomial matrices $N_R(z)$ and $G^{-1}[D_R(z) + F N_R(z)]$ are relatively right prime.

(b) The polynomial matrices $N_L(z)G$ and $[D_L(z) + N_L(z)F]$ are relatively left prime.

(c) The numbers v_i for $i = 1, 2, \dots, m$ are the reachability indices of the closed-loop system (2.4).

(d) The numbers w_j for $j = 1, 2, \dots, p$ are the observability indices of the closed-loop system (2.4).

(e) The open-loop system (2.1) and the closed-loop system (2.4) have the same finite invariant zero structure.

Proof. See [7]. □

Lemma 4.2. Let $H(z)$ be a polynomial matrix with dimensions $2m \times m$ and column degrees v_i for $i = 1, 2, \dots, m$ ($\sum_{i=1}^{m} v_i = n$). Then the equation

$$x^{\mathrm{T}} H(z) = r^{\mathrm{T}}(z) \tag{4.1}$$

has a solution x^{T} over $\mathbb{R}$, for every $1 \times m$ polynomial vector $r^{\mathrm{T}}(z)$ over $\mathbb{R}[z]$ with $\deg_{ci} r^{\mathrm{T}}(z) = v_i \ \forall\, i = 1, \dots, m$, only if $n \le m$.

Proof. We assume that equation (4.1) as a solution for x^{T} over $\mathbb{R}$. Also let $h_i(z)$ for $i = 1, \dots, m$ be the columns of the polynomial matrix $H(z)$. Then we have

$$\deg h_i(z) = v_i \quad \text{for} \quad i = 1, \dots, m \tag{4.2}$$

$$x^{\mathrm{T}} h_i(z) = r_i(z) \quad \text{or} \quad x^{\mathrm{T}} \sum_{j=0}^{v_i} h_{ji} z^j = \sum_{j=0}^{v_i} r_{ji} z^j, \quad i = 1, \dots, m \tag{4.3}$$

where h_{ji} are real $2m \times 1$ vectors $i = 1, 2, \ldots, m$ and $r_{ji} \in \mathbb{R}$ are the coefficients of the ith element of the polynomial vector $r^{\mathrm{T}}(z)$. Equating coefficients of like powers of z in relationship (4.3) for $i = 1, \ldots, m$, we obtain the following set of linear algebraic equations

$$x^{\mathrm{T}}\left[h_{01}, \ldots, h_{v_1 1}, \ldots, h_{0m}, \ldots, h_{v_m m}\right] = \left[r_{01}, \ldots, r_{v_1 1}, \ldots, r_{0m}, \ldots, r_{v_m m}\right] = g. \tag{4.4}$$

The matrix in square brackets on the left side in (4.4) has dimension $2m \times (n + m)$. Since equation (4.1) has a solution for every polynomial vector $r^{\mathrm{T}}(z)$ with $\deg_{ci} r^{\mathrm{T}}(z) = v_i$ for $i = 1, \ldots, m$, equation (4.4) must have a solution for every real $1 \times (n+m)$ vector g. This is possible if, and only if the real matrix $[h_{01}, \ldots, h_{v_1 1}, \ldots, h_{0m}, \ldots, h_{v_m m}]$ has full column rank. This implies that $2m \geq n + m$ or equivalently $n \leq m$. □

5. BASIC RESULTS

The main results concerning the solvability of the decoupling problem defined in Section 2 are presented here.

Theorem 5.1. Let v_i, w_i, for $i = 1, 2, \ldots, m$ be the reachability and the observability indices respectively of open-loop system (2.1). Then a necessary condition for the solvability of the decoupling problem under the control law (2.3) is

$$v_i = w_i \neq 0 \quad \forall i = 1, 2, \ldots, m.$$

Proof. Let the problem of decoupling with arbitrary pole assignment have a solution.

Then we have that

$$T_c(z) = N_R(z)\left[G^{-1}D_R(z) + G^{-1}FN_R(z)\right]^{-1} = \operatorname{diag}\left[b_1(z)/a_1(z), \ldots, b_m(z)/a_m(z)\right] \tag{5.1}$$

where $b_i(z)$ and $a_i(z)$ are relatively prime for $i = 1, 2, \ldots, m$ and $a_i(z)$ is monic for $i = 1, 2, \ldots, m$. Without loss of generality we suppose that $\deg a_i(z) \geq \deg a_j(z)$ for $i > j$; for if not we can premultiply $T_c(z)$ by a permutation matrix P and postmultiply $T_c(z)$ by P^{T} to obtain it and this corresponds to a renumbering of the reference inputs and outputs.

Let us define the polynomial matrices

$$D_{CR}(z) = D_{CL}(z) = \operatorname{diag}\left[a_1(z), \ldots, a_m(z)\right], \tag{5.2}$$

$$N_{CR}(z) = N_{CL}(z) = \operatorname{diag}\left[b_1(z), \ldots, b_m(z)\right]. \tag{5.3}$$

Since $b_i(z)$ and $a_i(z)$ are relatively prime for $i = 1, 2, \ldots, m$, the matrix $\left[D_{CR}^{\mathrm{T}}(z), N_{CR}^{\mathrm{T}}(z)\right]^{\mathrm{T}}$ does not lose rank for every $z \in C$ and therefore $D_{CR}(z)$ and $N_{CR}(z)$ are relatively right prime. The highest order coefficient matrix of $D_{CR}(z)$ is equal to I_m and since the column degrees of $D_{CR}(z)$ are arranged in decreased order of magnitude, $D_{CR}(z)$, $N_{CR}(z)$ constitute a standard right matrix fraction description of the matrix $T_c(z)$. According to Lemma 4.1 $\deg a_i(z) = \ell_i$ is the ith reachability index of the compensated system. Since the reachability indices are invariant under the control law (2.3), we conclude that

$$\ell_i = v_i, \quad i = 1, 2, \ldots, m. \tag{5.4}$$

Similarly the matrices $D_{CL}(z)$ and $N_{CL}(z)$ are relatively left prime, the highest order coefficient matrix of $D_{CL}(z)$ is equal to I_m and since the column degrees of $D_{CL}(z)$ are arranged in decreased order to magnitude, $D_{CL}(z)$, $N_{CL}(z)$ constitute a standard left matrix fraction description of the matrix $T_c(z)$. According to Lemma 4.1 $\deg a_i(z) = \ell_i$ is the ith observability index of the compensated system. Since the observability indices are invariant under the control law (2.3), we conclude that

$$\ell_i = w_i, \quad i = 1, 2, \ldots, m. \tag{5.5}$$

Then theorem follows at once from (5.4) and (5.5). □

Theorem 5.1 ignores the requirement for the arbitrary placement of the poles of the compensated system. For the solvability of the decoupling problem, as defined in Section 2, we have.

Theorem 5.2. Let v_i, w_i for $i = 1, 2, \ldots, m$ be the reachability and the observability indices respectively of the system (2.1). Then the problem of decoupling in combination with arbitrary pole assignment has a solution by constant output feedback iff

$$v_i = w_i = 1 \quad \forall i = 1, 2, \ldots, m.$$

Proof. We assume that the problem of decoupling with arbitrary pole assignment under the control law (2.3) has a solution. Then the transfer function matrix of the closed-loop system is

$$T_c(z) = N_R(z)\left[G^{-1}D_R(z) + G^{-1}FN_R(z)\right]^{-1} = N_{CR}(z)\, D_{CR}^{-1}(z) \tag{5.6}$$

where $D_{CR}(z)$ and $N_{CR}(z)$ are as in equations (5.1) and (5.2).

By Lemma 4.1 $G^{-1}[D_R(z) + FN_R(z)]$ and $N_R(z)$ form a standard right matrix fraction description of $T_c(z)$. Also from Lemma 4.1 $D_{CR}(z)$ and $N_{CR}(z)$ form a standard right matrix fraction description of $T_c(z)$. Then there exists a unimodular matrix $U(z)$ so that

$$\begin{bmatrix} G^{-1}D_R(z) + G^{-1}FN_R(z) \\ N_R(z) \end{bmatrix} U(z) = \begin{bmatrix} D_{CR}(z) \\ N_{CR}(z) \end{bmatrix}. \tag{5.7}$$

Let us consider an $m \times 1$ real constant vector λ. From equation (5.7) we have

$$\begin{aligned} &\lambda^{\mathrm{T}}\left[G^{-1}D_R(z) + G^{-1}FN_R(z)\right] U(z) = \lambda^{\mathrm{T}}\left[G^{-1}, G^{-1}F\right] \begin{bmatrix} D_R(z)\,U(z) \\ N_R(z)\,U(z) \end{bmatrix} \\ = \;& \lambda^{\mathrm{T}} D_{CR}(z) = \operatorname{diag}\left[\lambda_1 a_1(z), \ldots, \lambda_m a_m(z)\right]. \end{aligned} \tag{5.8}$$

Since $\det D_{CR}(z) = \prod_{i=1}^{m} a_i(z)$ is the characteristic polynomial of the compensated system that is an arbitrary monic polynomial of degree n, its factors $a_i(z)$ can be considered as arbitrary monic polynomials of degrees v_i, $i = 1, 2, \ldots, m$ respectively. Since equation (5.8) has a constant solution $\lambda^{\mathrm{T}}[G^{-1}, G^{-1}F]$ for every polynomial vector $[\lambda_1 a_1(z), \ldots, \lambda_m a_m(z)]$, from Lemma 4.2 we deduce that

$$n \leq m. \tag{5.9}$$

Since $\det T_c(z) \not\equiv 0$ is as in equation (5.1), we deduce that

$$\operatorname{rank} T_c(z) = m \leq \min(n, m) \tag{5.10}$$

and therefore $(m = n)$ and the matrices B, C are both nonsingular. Then by the definition of the reachability indices

$$v_i = 1 \quad \text{for} \quad i = 1, 2, \ldots, m. \tag{5.11}$$

From Theorem 5.1 and relation (5.11) we deduce

$$w_i = v_i = 1 \quad \forall i = 1, 2, \ldots, m \tag{5.12}$$

and necessity has been proved.

To prove sufficiency we work as follows. Since the reachability indices are equal to one and the polynomials $a_i(z)$ have degrees $v_i = 1$ we conclude that all the poles $z = -\sigma_i$, $i = 1, 2, \ldots, m$, of the compensated system have to be real. We define

$$\Sigma = \operatorname{diag}[\sigma_i]. \tag{5.13}$$

Since $v_i = 1$ and $\sum_{i=1}^{m} v_i = n$ we conclude that $n = m$ and therefore the matrices B and C are nonsingular. Then $T(z) = C(Iz - A)^{-1}B$ is square and nonsingular for every $z \in C$ except for a finite number of $z_k \in C$ that are defined as the transmission zeros of $T(z)$. Since the transfer function matrix $T(z)$ of the system (2.1) is strictly proper,

$$\deg_{ci} D_R(z) = v_i = 1 > \deg_{ci} N_R(z) \quad \text{for } i = 1, \ldots, m. \tag{5.14}$$

Then the polynomial matrix $N_R(z)$ is a constant matrix and because $T(z)$ is nonsingular for almost every $z \in C$, $N_R(z)$ is nonsingular.

Because of (5.14), we can write

$$\begin{aligned} D_R(z) &= D_h z + D_{R0} & (5.15)\\ N_R(z) &= K. & (5.16) \end{aligned}$$

Since $D_R(z)$ is column reduced, D_h is nonsingular. From equation (5.7) taking

$$\begin{aligned} G &= D_h K^{-1} \Phi & (5.17)\\ F &= D_h K^{-1} \Sigma - D_{R0} K^{-1} & (5.18) \end{aligned}$$

where Φ is a real diagonal matrix with arbitrary nonzero elements, we have

$$\begin{aligned} N_{CR}(z) &= \Phi = \operatorname{diag}[\phi_1, \ldots, \phi_m] && (5.19)\\ D_{CR}(z) &= Iz + \Sigma && (5.20)\\ U(z) &= K^{-1}\Phi. && (5.21) \end{aligned}$$

Then

$$T_c(z) = N_{CR}(z)\, D_{CR}^{-1}(z) = \operatorname{diag}[\phi_1/(z+\sigma_1), \ldots, \phi_m/(z+\sigma_m)] \qquad (5.22)$$

i.e. it is a diagonal matrix and the poles of the compensated system are equal to $z_i = -\sigma_i$, for $i = 1, 2, \ldots, m$. □

Construction

Given A, B, C and Σ. Find F and G.

Step 1. Find the standard right matrix fraction description and the standard left matrix fraction description of the open-loop system (2.1)

$$T(z) = C(Iz - A)^{-1}B = N_R(z)\, D_R^{-1}(z) = D_L^{-1}(z)\, N_L(z).$$

The column degrees of the matrix $D_R(z)$ and the row degrees of the matrix $D_L(z)$ are the reachability indices and the observability indices of the open-loop system respectively.

Step 2. Check the condition of Theorem 5.2. If this condition is satisfied go to Step 3. If not go to Step 4.

Step 3. Find the constant solution over $\mathbb{R}$ for G^{-1} and $G^{-1}F$ of the linear diophantine equation (5.7). (This solution is given by (5.17) and (5.18).)

Step 4. Our problem has no solution.

6. CONCLUSIONS

In this paper a system-theoretic approach is developed for the solution of the problem of decoupling with arbitrary pole assignment by constant output feedback and a nonsingular input transformation. In particular it has been proved that the above problem has a solution if, and only if all the reachability indices and all the observability indices of the open-loop system are equal to one. Furthermore a procedure is given for the construction of the constant output feedback which solves the problem of decoupling with arbitrary pole assignment.

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REFERENCES

[1] M. Bahey Argoun and J. Van de Vegte: Output feedback decoupling in the frequency domain. Internat. J. Control *31* (1980), 665–675.
[2] J. Descusse: A necessary and sufficient condition for decoupling using output feedback. Internat. J. Control *31* (1980), 833–840.
[3] J. W. Howze: Necessary and sufficient condition for decoupling using output feedback. IEEE Trans. Automat. Control *AC-20* (1975), 833–840.
[4] T. G. Koussiouris: Controllability indices of a system, minimal indices of its transfer function matrix and their relations. Internat. J. Control *34* (1981), 613–622.
[5] V. Kučera: Analysis and Design of Discrete Linear Control Systems. Prentice Hall, Englewood Cliffs, N.J. 1991.
[6] P. N. Paraskevopoulos and F. N. Koumboulis: A new approach to the decoupling problem of linear time-invariant systems. J. Franklin Inst. *329* (1992), 347–369.
[7] A. C. Pugh and P. A. Ratcliffe: Infinite frequency interpretation of minimal bases. Internat. J. Control *32* (1980), 4, 581–588.
[8] S. Wang and E. J. Davison: Design of decoupled control systems: A frequency domain approach. Internat. J. Control *21* (1975), 4, 529–536.
[9] W. A. Wolovich: Linear Multivariable Systems. Springer, New York 1974.
[10] W. A. Wolovich: Output feedback decoupling. IEEE Trans. Automat. Control *AC-20* (1975), 651–659.

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