KYBERNETIKA — VOLUME 37 (2001), NUMBER 6, PAGES 725–735

NEW COPRIME POLYNOMIAL FRACTION REPRESENTATION OF TRANSFER FUNCTION MATRIX

YELENA M. SMAGINA

A new form of the coprime polynomial fraction $C(s)\,F(s)^{-1}$ of a transfer function matrix $G(s)$ is presented where the polynomial matrices $C(s)$ and $F(s)$ have the form of a matrix (or generalized matrix) polynomials with the structure defined directly by the controllability characteristics of a state-space model and Markov matrices HB, $HAB, \dots$

1. INTRODUCTION

In many practical control problems it is desirable to have the right or left coprime polynomial matrix fraction (or matrix fraction descriptions (MFD)) of the transfer matrix $G(s) = C(s)\,F(s)^{-1}$ where $C(s)$ and $F(s)$ are polynomial matrices in the Laplace operation s and $F(s)$ is a nonsingular matrix. For example, decomposition of this type plays the key role in the methods of H^{∞} problem [7] and model reduction techniques [9]. Polynomial matrix descriptions are widely used in the design of state estimators and regulators [6, 8, 14, 18].

In this paper we obtain a MFD formula for the transfer function matrix of a multi-input multi-output (MIMO) control system in the state-space. This formula has a special structure that is different from the existing matrix fractions [4, 8, 10, 17]: the new MFD $C(s)\,F(s)^{-1}$ includes the matrix polynomials $C(s)$ and $F(s)$ of the order depended on controllability characteristics of state-space system. Block coefficients of the 'numerator' $C(s)$ are expressed in the terms of the Markov matrices $HB, HAB, HA^2B, \dots$ So, a new analytical expression for the polynomial matrix fraction introduced in the paper can be considered as a generalization of the classic representation of transfer functions (TF) studied in ([10], p. 38).

It is known that important properties of TF for a classic single-input single-output case are related to Markov parameters (the number of a finite and infinite zeros, invertibility, the relative degree of a control system [6] etc.). Certain relationships between the matrices HA^iB and MIMO system properties have already been studied in the works [5, 11, 15]. New MFD form presented in the paper reveals the direct connection between TFM of MIMO system and Markov matrices that allows to predict some TFM properties (e. g. degeneracy [3], invertibility, existence and number of transmission zeros etc.) without performing complex calculations. These properties can be obtained without the MFD computation by evaluating Markov matrices that can be calculated from the state-space model. The method proposed in this paper is a development of the result presented in the IFAC Conference, Belford, France, May 1997 [13].

2. PRELIMINARY NOTIONS AND PROBLEM STATEMENT

Let a linear multivariable time-invariant system be described in the state-space by

$$\dot{x} = Ax + Bu \tag{1}$$

$$y = Hx \tag{2}$$

where x is a state n-vector, u is an input r-vector, y is an output l-vector, A, B, H are real constant matrices of appropriate sizes, $r,\, l \leq n$. It is assumed that $\operatorname{rank} B = r$ and the pair (A, B) is controllable.

Consider the transfer function matrix $G(s)$ of system (1), (2)

$$G(s) \;=\; H(sI - A)^{-1}B \tag{3}$$

that is the $l \times r$ matrix with elements presented by strictly proper rational functions of complex element s with real coefficients. It is known [16, 17] that any $l \times r$ rational matrix $G(s)$ can be always (nonuniquely) represented as the product

$$G(s) \;=\; C(s)\,F(s)^{-1} \tag{4}$$

where $C(s)$ and $F(s)$ are relatively right prime polynomial matrices of sizes $l \times r$ and $r \times r$ respectively with $F(s)$ is nonsingular and column proper matrix [17].

Introduce polynomial matrices with an ordered structure. By a matrix polynomial of the order m and the size $l \times r$ we understand a polynomial matrix of the form [2]

$$F(s) \;=\; F_0 + F_1 s + \cdots + F_{m-1}s^{m-1} + F_m s^m \tag{5}$$

where F_i, $i = 0, 1, \ldots, m$ are $l \times r$ real constant matrices.

By a generalized matrix polynomial we shall understand the following polynomial matrix [12]

$$\begin{aligned} F(s) \;=\;& F_0 + F_1 \operatorname{diag}(I_{l_m - l_1}, I_{l_1}s) + F_2 \operatorname{diag}(I_{l_m - l_2}, I_{l_2 - l_1}s, I_{l_1}s^2) + \cdots \\ & \cdots + F_{m-1}\operatorname{diag}(I_{l_m - l_{m-1}}, I_{l_{m-1} - l_{m-2}}s, \cdots, I_{l_1}s^{m-1}) \\ & + F_m \operatorname{diag}(I_{l_m - l_{m-1}}s, I_{l_{m-1} - l_{m-2}}s^2, \cdots, I_{l_1}s^m) \end{aligned} \tag{6}$$

where l_i, $i = 1, 2, \ldots, m$ are some integers that satisfy the inequalities: $l_1 \leq l_2 \leq \cdots \leq l_m$.

It is obvious that form (6) is a generalization of the matrix polynomial (5) structure represented as: $F(s) \;=\; F_0 + F_1(I_r s) + F_2(I_r s^2) + \cdots + F_m(I_r s^m)$.

Using these notions we will define the right coprime factorization (4) with the matrices $C(s)$ and $F(s)$ of structure (5) or (6).

3. FACTORIZATION OF TRANSFER FUNCTION MATRIX

Assertion 1. The transfer function matrix of the system (1), (2) with $n \le r\nu$ can be always factored into the product

$$G(s) = C(s)\,F(s)^{-1}Q^* \tag{7}$$

where $Q^* = QM^T$ is a constant matrix, the polynomial $l \times r$ and $r \times r$ matrices $C(s)$ and $F(s)$ are the following generalized matrix polynomials

$$\begin{aligned} C(s) &= [0, C_1] + [0, C_2]\mathrm{diag}(I_{r-l_1}, I_{l_1}s) + [0, C_3]\mathrm{diag}(I_{r-l_2}, I_{l_2-l_1}s, I_{l_1}s^2) + \dots \\ &\quad \dots + C_\nu \mathrm{diag}(I_{r-l_{\nu-1}}, I_{l_{\nu-1}-l_{\nu-2}}s, \cdots, I_{l_1}s^{\nu-1}) \end{aligned} \tag{8}$$

$$\begin{aligned} F(s) &= [0, F_1] + [0, F_2]\mathrm{diag}(I_{r-l_1}, I_{l_1}s) + \cdots \\ &\quad \cdots + [0, F_{\nu-1}]\mathrm{diag}(I_{r-l_{\nu-2}}, I_{l_{\nu-2}-l_{\nu-1}}s, \cdots \\ &\quad \cdots, I_{l_1}s^{\nu-2}) + F_\nu \mathrm{diag}(I_{r-l_{\nu-1}}, I_{l_{\nu-1}-l_{\nu-2}}s, \cdots, I_{l_1}s^{\nu-1}) \\ &\quad + I_r \mathrm{diag}(I_{r-l_{\nu-1}}s, I_{l_{\nu-1}-l_{\nu-2}}s^2, \cdots, I_{l_1}s^\nu). \end{aligned} \tag{9}$$

If the pair matrices (A, H) is observable than the polynomial matrices $C(s)$ and $F(s)$ are right coprime.

In (8), (9) the matrices $F_1, F_2, \dots, F_\nu$ are defined from the Yokoyama canonical form [19] for the controllable pair (A, B) with the transformed matrices

$$\bar{A} = NAN^{-1} = \begin{bmatrix} 0 & [0, I_{l_1}] & 0 & \cdots & 0 \\ 0 & 0 & [0, I_{l_2}] & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & [0, I_{l_{\nu-1}}] \\ -F_1 & -F_2 & -F_3 & \cdots & -F_\nu \end{bmatrix}, \quad \bar{B} = NBM = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ Q \end{bmatrix} \tag{10}$$

where N is a transformation $n \times n$ matrix. In (10) the permutation $r \times r$ matrix M rearranges the columns of the matrix B such that the last columns of the matrices $A^iBM, i = 1, 2, \dots, \nu - 1$ are linearly independent from the first columns. The controllability index ν is the smallest integer that satisfies the equation $\mathrm{rank}[B, AB, \dots, A^{\nu-1}B] = n$. The integers $l_1, l_2, \dots, l_\nu$ can be derived from the relations:

$$\begin{aligned} l_i &= \mathrm{rank}[B, AB, \dots, A^{\nu-i}B] - \mathrm{rank}[B, AB, \dots, A^{\nu-i-1}B], \quad i = 1, \dots, \nu - 1, \\ l_\nu &= \mathrm{rank}\, B = r \end{aligned}$$

and they satisfy the inequalities: $l_1 \le l_2 \le \cdots \le l_\nu$, $l_1 + l_2 + \cdots + l_\nu = n$. The $l_\nu \times l_i$ blocks $F_i, i = 1, 2, \dots, \nu$ have no special structure, $[0, I_{l_i}]$ – $l_i \times l_{i+1}$ submatrices, I_{l_i} – the identity matrices of the order l_i and Q is a nonsingular low triangular $r \times r$ matrix with unit elements on the principal diagonal. In Section 5 we describe a new recurrent algorithm for the matrices Q and $F_1, F_2, \dots, F_\nu$ calculations.

In (8) C_i are $l \times l_i$ blocks of the matrix

$$C = HN^{-1} = [C_1, C_2, \dots, C_\nu]. \tag{11}$$

The proof of Assertion 1 is based on the calculation of TFM for canonical system (10), (11). The details are discussed in [12].

Consider a particular case when $n = r\nu$, $l_1 = l_2 = \dots = l_\nu = r$ and the pair (A, B) is reduced into Asseo's canonical form [1] with matrices

$$\bar{A} = \begin{bmatrix} 0 \quad I_{n-r} \\ \dots\dots\dots\dots\dots\dots \\ -F_1, -F_2 \dots, -F_\nu \end{bmatrix}, \qquad \bar{B} = \begin{bmatrix} 0 \\ I_r \end{bmatrix}$$

where F_i, $i = 1, 2, \dots, \nu$ are $r \times r$ blocks. It follows immediately from the Assertion 1.

Corollary 1. The transfer function matrix of the system (1), (2) with $n = r\nu$ can be always factored into the product $G(s) = C(s)\,F(s)^{-1}$ where the polynomial $l \times r$ and $r \times r$ matrices $C(s)$ and $F(s)$ are the following matrix polynomials

$$C(s) = C_1 + C_2 s + \dots + C_\nu s^{\nu-1} \tag{12}$$

$$F(s) =. F_1 + F_2 s + \dots + F_\nu s^{\nu-1} + I_r s^\nu. \tag{13}$$

Corollary 2. The transfer function vector of system (1), (2) with scalar input ($r = 1$, $l_1 = l_2 = \dots = l_n = 1$) can be always factored into the product $G(s) = C(s) f(s)^{-1}$ where the vector polynomial $C(s)$ has structure (12) for $\nu = n$ and $f(s) = \det(sI - A)$.

Remark 1. The similar approach may be applied to an observable pair (H, A) to find the left coprime fraction $G(s) = N(s)^{-1} Q(s)$ with $l \times l$ and $l \times r$ matrices $N(s)$ and $Q(s)$ respectively. The factorization obtained will be the left coprime for the controllable pair (A, B).

In the next section we will scrutinize the block coefficients C_i and F_i, $i = 1, 2, \dots, \nu$ of the generalized matrix (matrix) polynomials $C(s)$ and $F(s)$. First we analyze the relationship between the block coefficients C_i and Markov parameters (matrices) $HB, HAB, HA^2B, \dots$.

4. CALCULATION OF C_i

Let us denote by R_1, R_2,..., R_ν the $n \times l_i$ blocks of the matrix in the partition

$$N^{-1} = [R_1, R_2, \dots, R_\nu] \tag{14}$$

and express matrix C (11) in the following form

$$C = HN^{-1} = [HR_1, HR_2, \dots, HR_\nu] = [C_1, C_2, \dots, C_\nu]. \tag{15}$$

Then generalized matrix polynomial (8) can be represented as

$$\begin{aligned} C(s) &= H[0,R_1] + H[0,R_2]\mathrm{diag}(I_{r-l_1}, I_{l_1}s) \\ &+ H[0,R_3]\mathrm{diag}(I_{r-l_2}, I_{l_2-l_1}s, I_{l_1}s^2) + \dots \\ &\dots + HR_\nu \mathrm{diag}(I_{r-l_{\nu-1}}, I_{l_{\nu-1}-l_{\nu-2}}s, \cdots, I_{l_1}s^{\nu-1}) \end{aligned} \tag{16}$$

where $[0, R_i]$ are $n\times r$ blocks. We are going to express the blocks R_i via the matrices A, B, F_i and Q. At first, we use the relation $\bar{B} = NBM$: $BM = N^{-1}\bar{B} = [R_1, R_2, \dots, R_\nu]\bar{B} = R_\nu Q$ that allows to evaluate the last block of the matrix N^{-1} (14)

$$R_\nu = BMQ^{-1}. \tag{17}$$

To find the $n\times r$ blocks $[0, R_i]$, $i = \nu-1, \nu-2, \dots, 1$ we apply the formula $AN^{-1} = N^{-1}\bar{A}$ that can be written as $A[R_1, R_2, \dots, R_\nu] = [R_1, R_2, \dots, R_\nu]\bar{A}$. Taking into account the special structure of the matrix $\bar{A}$ we can write

$$AR_i = R_{i-1}[0, I_{l_{i-1}}] - R_\nu F_i, \quad i = \nu,\ \nu-1, \dots, 2 \tag{18}$$

and then express the blocks $R_{i-1}[0, I_{l_{i-1}}]$ as

$$R_{i-1}[0, I_{l_{i-1}}] \;=\; AR_i + BMQ^{-1}F_i, \quad i = \nu, \nu-1, \dots, 2. \tag{19}$$

Recurrent formula (19) can be applied to find the $n\times r$ blocks $[0, R_{\nu-1}]$, $[0, R_{\nu-2}]$,..., $[0, R_1]$. For $i = \nu$ relation (19) takes the following form

$$R_{\nu-1}[0, I_{l_{\nu-1}}] \;=\; AR_\nu + BMQ^{-1}F_\nu.$$

Using the formula $R_{\nu-1}[0, I_{l_{\nu-1}}] = [0, R_{\nu-1}]$ we can present $[0, R_{\nu-1}]$ as

$$[0, R_{\nu-1}] = ABMQ^{-1} + BMQ^{-1}F_\nu. \tag{20}$$

For $i = \nu - 1$ relation (19) can be written as

$$R_{\nu-2}[0, I_{l_{\nu-2}}] \;=\; AR_{\nu-1} + BMQ^{-1}F_{\nu-1}. \tag{21}$$

Postmultiply both sides of (21) by the $l_{\nu-1}\times r$ matrix $[\tilde{0}, I_{l_{\nu-1}}]$ where $\tilde{0}$ is $l_{\nu-1}\times(r - l_{\nu-1})$ zero block. Then using the formula $[0, I_{l_{\nu-2}}][\tilde{0}, I_{l_{\nu-1}}] = [\tilde{0}, I_{l_{\nu-2}}]$ where $[\tilde{0}, I_{l_{\nu-2}}]$ is $l_{\nu-2}\times r$ matrix we can present (21) as

$$[0, R_{\nu-2}] = A[0, R_{\nu-1}] + BMQ^{-1}[0, F_{\nu-1}] \tag{22}$$

where $[0, R_{\nu-2}]$, $[0, R_{\nu-1}]$, $[0, F_{\nu-1}]$ are $n\times r$, $n\times r$, $r\times r$ matrices respectively. Substitution the matrix $[0, R_{\nu-1}]$ from (20) into (22) results in

$$[0, R_{\nu-2}] = A^2BMQ^{-1} + ABMQ^{-1}F_\nu + BMQ^{-1}[0, F_{\nu-1}]. \tag{23}$$

Continue the same procedure we can obtain the $n\times r$ matrices $[0, R_{\nu-3}], \dots, [0, R_1]$

$$\begin{aligned} [0, R_{\nu-i}] &= A^iBMQ^{-1} + A^{i-1}BMQ^{-1}F_\nu + \dots \\ &\dots + ABMQ^{-1}[0, F_{\nu-i+2}] + BMQ^{-1}[0, F_{\nu-i+1}]. \end{aligned} \tag{24}$$

Substituting R_ν, $[0, R_{\nu-1}]$,..., $[0, R_1]$ into (16) we can find the structure of $C(s)$

$$\begin{aligned} C(s) = & \ (HA^{\nu-1}BMQ^{-1} + HA^{\nu-2}BMQ^{-1}F_\nu + \cdots \\ & \cdots + HABMQ^{-1}[0, F_3] + HBMQ^{-1}[0, F_2]) \\ & +(HA^{\nu-2}BMQ^{-1} + HA^{\nu-3}BMQ^{-1}F_\nu + \cdots \\ & \cdots + HBMQ^{-1}[0, F_3]) \operatorname{diag}(I_{r-l_1}, I_{l_1}s) + \cdots \\ & \cdots + (HABMQ^{-1} + HBMQ^{-1}F_\nu) \operatorname{diag}(I_{r-l_{\nu-2}}, I_{l_{\nu-2}-l_{\nu-3}}s, \cdots, I_{l_1}s^{\nu-2}) \\ & +HBMQ^{-1}\operatorname{diag}(I_{r-l_{\nu-1}}, I_{l_{\nu-1}-l_{\nu-2}}s, \cdots, I_{l_1}s^{\nu-1}). \end{aligned} \tag{25}$$

Remark 2. If the system (1), (2) is reduced to Asseo's canonical form with $l_1 = l_2 = \cdots = l_\nu = r$, $n = r\nu$, $Q = I_r$, $M = I_r$, $[0,\ F_i] = F_i$ then the matrix polynomial $C(s)$ takes the following form

$$\begin{aligned} C(s) \ = \ & (HA^{\nu-1}B + HA^{\nu-2}BF_\nu + \cdots + HABF_3 + HBF_2) \\ & +(HA^{\nu-2}B + HA^{\nu-3}BF_\nu + \cdots \\ & \cdots + HBF_3)s + \cdots + (HA^2B + HABF_\nu + HBF_{\nu-1})s^{\nu-3} \\ & \cdots + (HAB + HBF_\nu)s^{\nu-2} + HBs^{\nu-1}. \end{aligned} \tag{26}$$

Remark 3. For system (1), (2) with scalar input the vector polynomial $C(s)$ has the simple structure

$$\begin{aligned} C(s) \ = \ & (HA^{n-1}b + HA^{n-2}b\alpha_n + \cdots + HAb\alpha_3 + Hb\alpha_2) \\ & +(HA^{n-2}b + HA^{n-3}b\alpha_n + \cdots \\ & \cdots + Hb\alpha_3)s + \cdots + (HA^2b + HAb\alpha_n)s^{n-2} + Hbs^{n-1} \end{aligned} \tag{27}$$

where $\alpha_2, \ldots, \alpha_n$ are the coefficients of the characteristic polynomial of $A : \det(sI - A) = s^n + \alpha_n s^{n-1} + \cdots + \alpha_2 s + \alpha_1$.

5. CALCULATION OF F_i

In this section we discuss a new recurrent method for calculating the matrix Q and block coefficients $F_1, F_2, \ldots, F_\nu$ of the polynomial matrix $F(s)$ in right coprime polynomial fraction (4). Contrary to the previous approach [19] the method does not use the calculation of the full transformation matrix N. Moreover, the original formula for the characteristic polynomial coefficients can be easy obtained from the proposed method.

As it has been shown in [11] the transformation $n \times n$ matrix N has the following structure

$$N^T = [\ N_\nu^T, \quad N_{\nu-1}^T, \quad \cdots \quad N_1^T\ ] \tag{28}$$

where $(l_{\nu-i+1} \times n)$ blocks N_i, $i = 1, 2, \ldots, \nu$ are calculated from the formulas

$$N_\nu = P_\nu,\; N_{\nu-1} = \begin{bmatrix} P_{\nu-1} \\ P_\nu A \end{bmatrix},\; N_{\nu-2} = \begin{bmatrix} P_{\nu-2} \\ P_{\nu-1}A \\ P_\nu A^2 \end{bmatrix}, \ldots, N_1 = \begin{bmatrix} P_1 \\ P_2 A \\ \vdots \\ P_\nu A^{\nu-1} \end{bmatrix} \tag{29}$$

and $(l_\nu - l_{\nu-1}) \times n$, $(l_{\nu-1} - l_{\nu-2}) \times n, \ldots$, $(l_2 - l_1) \times n$, $l_1 \times n$ blocks P_1, $P_2, \ldots, P_\nu$ are computed from the following linear algebraic equation

$$\begin{bmatrix} P_\nu \\ P_{\nu-1} \\ \vdots \\ P_1 \end{bmatrix} \left(BM, ABM \begin{bmatrix} 0 \\ I_{l_{\nu-1}} \end{bmatrix}, \ldots, A^{\nu-1}BM \begin{bmatrix} 0 \\ I_{l_1} \end{bmatrix} \right)$$

$$= \begin{bmatrix} 0 & 0 & \vdots & \cdots & \vdots & 0 & 0 & \vdots & I^\nu \\ 0 & 0 & \vdots & \cdots & \vdots & I^{\nu-1} & 0 & \vdots & X \\ 0 & 0 & \vdots & \cdots & \vdots & X & X & \vdots & X \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ I^1 & 0 & \vdots & \cdots & \vdots & X & X & \vdots & X \end{bmatrix} \tag{30}$$

where I^i are the identity matrices of the order $(l_{\nu-i+1} - l_{\nu-i})$, X are some matrices.

Assertion 2. In Yokoyama's canonical form (10) the blocks Q, $F_\nu, F_{\nu-1}, \ldots, F_1$ are expressed via the $r \times n$ matrix N_1 in the recurrent formulas

$$\begin{aligned} Q &= N_1 BM \\ F_\nu &= -N_1 ABMQ^{-1} \\ F_{\nu-1} &= (-N_1A^2 - F_\nu N_1 A)\, BMQ^{-1} \begin{bmatrix} 0 \\ I_{l_{\nu-1}} \end{bmatrix} \\ &\vdots \\ F_1 &= (-N_1A^\nu - F_\nu N_1 A^{\nu-1} - [0, F_{\nu-1}]\, N_1 A^{\nu-2} - \cdots \\ &\qquad \cdots - [0, F_2]\, N_1 A) BMQ^{-1} \begin{bmatrix} 0 \\ I_{l_1} \end{bmatrix} \end{aligned} \tag{31}$$

(see the Appendix for proof).

Corollary 3. Expressions (31) take a simple shape for the pair (A, B) reduced in Asseo's canonical form

$$\begin{aligned} Q &= N_1 B \\ F_\nu &= -N_1 AB \\ F_{\nu-1} &= -N_1 A^2 B - F_\nu N_1 AB \\ &\vdots \\ F_1 &= -N_1 A^\nu B - F_\nu N_1 A^{\nu-1} B - F_{\nu-1} N_1 A^{\nu-2} B - \cdots - F_2 N_1 AB \end{aligned} \tag{32}$$

where the $r \times n$ matrix N_1 is calculated from the formula [11]

$$N_1 = [0, \ldots, 0, I_r]\,[B, AB, \ldots, A^{\nu-1}B]^{-1}A^{\nu-1}. \tag{33}$$

Corollary 4. If $r = 1$ ($B = b$ is a column vector) then the recurrent formula for coefficients of the matrix A characteristic polynomial can be derived from

$$\begin{aligned} \alpha_n &= -qAb, \quad \alpha_{n-1} = -qA^2b - \alpha_n qAb, \quad \ldots, \\ \alpha_1 &= -qA^n b - \alpha_n qA^{n-1}b - \cdots - \alpha_2 qAb \end{aligned} \tag{34}$$

where n row vector q satisfies the relation:

$$q = [0, 0, \ldots, 0, 1]\,[b, Ab, \ldots, A^{n-1}b]^{-1}A^{n-1}.$$

In conclusion, we present an algorithm of calculating the right coprime MFD that can be easy implemented on the computer.

STEP 1. Calculate integers ν and $l_1, l_2, \ldots, l_\nu$ for the controllable pair (A, B).

STEP 2. Calculate the matrix M and blocks Q, F_1, $F_2, \ldots, F_\nu$ (formulas (31) – (34)).

STEP 3. Using formula (9) or (13) construct the polynomial matrix $F(s)$.

STEP 4. Calculate HB, $HAB, \ldots, HA^{\nu-1}B$. Using formulas (25) – (27) find the polynomial matrix $C(s)$.

6. EXAMPLE

Let us find factorization (7) for controllable and observable system (1), (2) with the matrices

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad H = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}. \tag{35}$$

As it has been shown in ([11], p.31) this system has $\nu = 3$, $l_1 = l_2 = 1$, $l_3 = 2$. The pair (A, B) is transformed into Yokoyama's canonical form with $M = I_2$. Using (29) – (31) we can calculate

$$N_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$F_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad F_3 = \begin{bmatrix} -2 & 0 \\ -1 & -1 \end{bmatrix}$$

and then construct the 2×2 generalized matrix polynomial using formula (9)

$$\begin{aligned} F(s) &= [0, F_1] + [0, F_2]\,\mathrm{diag}(I_{r-l_1}, I_{l_1} s) + F_3 \mathrm{diag}(I_{r-l_2}, I_{l_2-l_1} s, I_{l_1} s^2) \\ &\quad + I_r \mathrm{diag}(I_{r-l_2} s, I_{l_2-l_1} s^2, I_{l_1} s^3) \\ &= [0, F_1] + [0, F_2]\,\mathrm{diag}(1, s) + F_3 \mathrm{diag}(1, s^2) + \mathrm{diag}(s, s^3) \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -s \\ 0 & -s \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ -1 & -s^2 \end{bmatrix} + \begin{bmatrix} s & 0 \\ 0 & s^3 \end{bmatrix} = \begin{bmatrix} s-2 & -s \\ -1 & s^3 - s^2 - s \end{bmatrix}. \end{aligned}$$

To evaluate the the 2×2 generalized matrix polynomial $C(s)$ we will apply formula (25) that takes the following form for $Q = I_2$, $M = I_2$

$$C(s) = (HA^2B + HABF_3 + HB[0, F_2]) + (HAB + HBF_3)\,\mathrm{diag}(1, s) + HB\,\mathrm{diag}(1, s^2).$$

Substituting the matrices

$$HB = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad HAB = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}, \qquad HA^2B = \begin{bmatrix} 3 & 2 \\ 7 & 3 \end{bmatrix}$$

we find

$$C(s) = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -s \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & s^2 \end{bmatrix} = \begin{bmatrix} 1 & 2-s \\ 1 & s^2 \end{bmatrix}$$

and write the right polynomial fraction of $G(s)$ as follows

$$G(s) = \begin{bmatrix} 1 & 2-s \\ 1 & s^2 \end{bmatrix} \begin{bmatrix} s-2 & -s \\ -1 & s^3 - s^2 - s \end{bmatrix}^{-1}.$$

This factorization is right coprime as the pair (H, A) is observable.

Examples for the cases when $n = r\nu$, $l_1 = l_2 = \cdots = l_\nu = r$ and $r = 1$, $l_1 = l_2 = \cdots = l_{n-1} = 1$ can be found in [13].

7. CONCLUSION

In the paper a new polynomial fraction representation for a transfer function matrix of a multivariable system in state-space has been discussed. The 'numerator' and 'denominator' of the fraction have the structure of matrix or generalized matrix polynomial. It has been shown that the block coefficients of the 'numerator' are defined via Markov matrices HB, HAB, HA^2B, ... The results presented in the paper can be considered as a generalization of the classic TF notions.

APPENDIX

Proof of the Assertion 2. Since $\bar{A} = NAN^{-1}$, $\bar{B} = NBM$ then the following equality takes place

$$[\bar{B}, \bar{A}\bar{B}, \ldots, \bar{A}^\nu \bar{B}] = N[BM, ABM, \ldots, A^\nu BM]. \tag{A1}$$

Using (10) we can calculate

$$\bar{B} = \begin{bmatrix} 0 \\ Q \end{bmatrix}, \quad \bar{A}\bar{B} = \begin{bmatrix} 0 \\ [0, I_{l_{\nu-1}}]\, Q \\ -F_\nu Q \end{bmatrix},$$

$$\bar{A}^2\bar{B} = \begin{bmatrix} 0 \\ [0, I_{l_{\nu-2}}]\, Q \\ -[0, I_{l_{\nu-1}}]\, FQ \\ -[0, F_{\nu-1}]\, Q - F_\nu(-F_\nu)Q \end{bmatrix}, \ldots$$

Then using the following notations

$$[W_1, W_2, \ldots, , W_{\nu+1}] = [0, I_r]\,[\bar{B}, \bar{A}\bar{B}, \ldots, \bar{A}^\nu \bar{B}] \tag{A2}$$

we can obtain the recurrent formulas for W_i : $W_1 = Q$, $W_i = -[0, F_{\nu-i+2}]W_1 - [0, F_{\nu-i+3}]W_2 - \ldots - [0, F_{\nu-1}]W_{i-2} - F_\nu W_{i-1}$, $i = 2, \ldots, \nu+1$. These relations allow to express Q, F_ν, $[0, F_{\nu-1}]$,..., $[0, F_1]$ $(\det Q \neq 0)$ as

$$\begin{aligned} Q &= W_1, \\ F_\nu &= -W_2 Q^{-1} \\ [0, F_{\nu-1}] &= (-W_3 - F_\nu W_2)Q^{-1} \\ [0, F_{\nu-2}] &= (-W_4 - F_\nu W_3 - [0, F_{\nu-1}]\, W_2)Q^{-1} \\ &\vdots \\ [0, F_1] &= (-W_{\nu+1} - F_\nu W_\nu - [0, F_{\nu-1}]\, W_{\nu-1} - \ldots - [0, F_2]\, W_2)Q^{-1}. \end{aligned} \tag{A3}$$

On the other hand using (A2) and (A1) we can represent the blocks W_i in the form

$$[W_1, W_2, \ldots, W_{\nu+1}] = [0, I_r]N[BM, ABM, \ldots, A^\nu BM] = N_1[BM, ABM, \ldots, A^\nu BM].$$

Substitution these W_i in (A3) results in formulas (31). □

ACKNOWLEDGEMENTS

Ye. Smagina was supported by the Center for Absorption in Science, Ministry of Immigrant Absorption, State of Israel.

(Received August 29, 2000.)

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Dr. Yelena M. Smagina, Department of Chemical Engineering, Technion – Israel Institute of Technology, Haifa, 32000. Israel.
e-mail: cermssy@tx.technion.ac.il