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LINEAR TRANSFORMATIONS OF WIENER PROCESS THAT BORN WIENER PROCESS, BROWNIAN BRIDGE OR ORNSTEIN–UHLENBECK PROCESS[1]

PETR LACHOUT

The paper presents a discussion on linear transformations of a Wiener process. The considered processes are collections of stochastic integrals of non-random functions w.r.t. Wiener process. We are interested in conditions under which the transformed process is a Wiener process, a Brownian bridge or an Ornstein–Uhlenbeck process.

1. INTRODUCTION

Investigation of asymptotic properties of a statistical estimator or of a testing statistic often leads to a linear transformation of a Wiener process that born a Wiener process or a Brownian bridge. Let us recall some typical examples of such transformations. Provided a Wiener process $(W(t), t \geq 0)$ the processes $\left(\frac{1}{\alpha}W\left(\alpha^2 \cdot t\right), t \geq 0\right)$, $\alpha \neq 0$ and $\left(tW\left(\frac{1}{t}\right), t > 0\right)$ are Wiener processes and $(W(t) - tW(1), t \in [0,1])$ and $\left(tW\left(\frac{1-t}{t}\right), t \in (0,1)\right)$ are Brownian bridges.

In the paper a more general schema is treated. We consider a collection of stochastic integrals of non-random real functions w.r.t. Wiener process, i. e.

$$\left(\int_0^{+\infty} a_t(\eta)\,\mathrm{d}W(\eta), t \in T\right), \text{ where } a_t \in \mathsf{L}_2(\mathbb{R}_+) \quad \forall\, t \in T.$$

We look for conditions under which the transformed process fulfills

$$\int_0^{+\infty} a_t(\eta)\,\mathrm{d}W(\eta) = V(\xi(t)) \text{ a.s.} \quad \forall\, t \in T,$$

where V is either a Wiener process or a Brownian bridge or an Ornstein–Uhlenbeck process and ξ is a convenient function.

The investigation starts with necessary and sufficient conditions under which the transformed process is a Wiener process. The other two sections provide a study

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on necessary and sufficient conditions under which the transformed process is a Brownian process or an Ornstein–Uhlenbeck process. In each of these cases, we present sufficient conditions based on Radon–Nikodym derivative. Each section is closed with necessary and sufficient conditions for the rescaling of a Wiener process.

To avoid any misunderstanding let us summarize notation used in the paper:

$$\begin{aligned} \mathsf{L}_2 \quad &- \quad \text{the set of all random variables having finite second moment;} \\ \mathsf{L}(\mathbb{R}_+) \quad &- \quad \text{the set of all Lebesgue measurable functions } a : \mathbb{R}_+ \to \mathbb{R}; \\ \mathsf{L}_2(\mathbb{R}_+) \quad &= \quad \left\{ a \in \mathsf{L}(\mathbb{R}_+) \, : \, \int_0^{+\infty} a^2(\eta) \, \mathrm{d}\eta < +\infty \right\}. \end{aligned}$$

2. WIENER PROCESS

This section treats necessary and sufficient conditions under which the linear transformation preserves Wiener processes. For completeness let us recall the definition of Wiener process.

Definition 2.1. A random process $(W(t)\,, t \geq 0)$ is called Wiener process if the following conditions are fulfilled:

i. it is a Gaussian process, i. e. for each $D \subset \mathbb{R}_+$, $\#D < +\infty$ we have $\mathcal{L}\Big((W(t)\,, t \in D) \Big) = \mathsf{N}(\mu_D, \Sigma_D)$ for some vector $\mu_D \in \mathbb{R}^D$ and some positive semidefinite matrix $\Sigma_D \in \mathbb{R}^{D \times D}$;

ii. $\forall\, t \in \mathbb{R}_+ : \ \mathsf{E}[W(t)] = 0$;

iii. $\forall\, t, s \in \mathbb{R}_+ : \ \mathsf{cov}(W(t)\,, W(s)) = \min\{t, s\}$;

iv. the process possesses continuous sample paths.

Definition 2.2. Let T be a non-empty set. We call a process $(\mathcal{W}(t)\,, t \in T)$ to be a pre-Wiener process if

i. the process is a Gaussian process, i. e. for each $D \subset T$, $\#D < +\infty$ we have $\mathcal{L}\Big((\mathcal{W}(t)\,, t \in D) \Big) = \mathsf{N}(\mu_D, \Sigma_D)$ for some vector $\mu_D \in \mathbb{R}^D$ and some positive semidefinite matrix $\Sigma_D \in \mathbb{R}^{D \times D}$;

ii. $\forall\, t \in T : \ \mathsf{E}[\mathcal{W}(t)] = 0$;

iii. there is a function $\xi : T \to \mathbb{R}_+$ such that $\mathsf{cov}(\mathcal{W}(t)\,, \mathcal{W}(s)) = \min\{\xi(t), \xi(s)\}$ $\forall\, t, s \in T$.

Here of course, $\xi(t) = \mathsf{var}(\mathcal{W}(t))$ but we decided to keep the redundant function ξ to engrave a similar scheme to Definitions 2.2, 3. 2 and 4.2.

Let us recall the well-known relation between a Wiener and a pre-Wiener process.

Proposition 2.1. If the process $(\mathcal{W}(t), t \geq 0)$ is a pre-Wiener process with $\mathsf{var}(\mathcal{W}(t)) = t$ for all $t \in \mathbb{R}_+$ then there is a Wiener process $(W(t), t \geq 0)$ which is a modification of the pre-Wiener process, i. e. $\forall t \geq 0 : W(t) = \mathcal{W}(t)$ a. s.

Proof. A proof can be found in [1], Chapter 2, § 9. □

The pre-Wiener process has limits in L_2 and can be extended to the closure of indexes.

Proposition 2.2. Let $T \subset \mathbb{R}_+$ be a non-empty set and $(\mathcal{W}(t), t \in T)$ be a pre-Wiener process with $\mathsf{var}(\mathcal{W}(t)) = t$ for all $t \in T$. Then for each $t \in \mathsf{clo}(T)$, there is $X(t) \in \mathsf{L}_2$ such that $\mathcal{W}(s) \xrightarrow[s \to t,\, s \in T]{\mathsf{L}_2} X(t)$. The process $(X(t), t \in \mathsf{clo}(T))$ is a pre-Wiener process with $\mathsf{var}(X(t)) = t$ for all $t \in \mathsf{clo}(T)$ extending the former one, i. e. $\forall t \in T : X(t) = \mathcal{W}(t)$ a. s.

Proof. The existence of limits in L_2 is evident since

$$\mathsf{E}\Big[(\mathcal{W}(s) - \mathcal{W}(v))^2\Big] = s + v - 2\min\{s, v\} \xrightarrow[\substack{s,v \to t \\ s,v \in T}]{} 0 \quad \text{for each } t \in \mathsf{clo}(T)\,.$$

Let $(X(t), t \in \mathsf{clo}(T))$ denote the limit in L_2.

For each $t \in T$, we immediately have that

$$\mathcal{W}(s) \xrightarrow[\substack{s \to t \\ s \in T}]{\mathsf{L}_2} \mathcal{W}(t)\,.$$

Therefore, $X(t) = \mathcal{W}(t)$ a. s.

The process $(X(t), t \in \mathsf{clo}(T))$ is a pre-Wiener process with $\mathsf{var}(X(t)) = t$ for all $t \in \mathsf{clo}(T)$, since the convergence in L_2 preserves Gaussian processes and implies the convergence of the mean and of the covariance. □

A pre-Wiener processes can be equivalently characterized as a re-indexed Wiener processes.

Theorem 2.1. Let $T \neq \emptyset$ and $(V(t), t \in T)$ be a L_2-random process. Then there exists a Wiener process $(W(t), t \in \mathbb{R}_+)$ such that $V(t) = W(\mathsf{var}(V(t)))$ a. s. for each $t \in T$ if and only if $(V(t), t \in T)$ is a pre-Wiener process.

Proof.

1. Let a Wiener process $(W(t), t \in \mathbb{R}_+)$ be such that $V(t) = W(\mathsf{var}(V(t)))$ a. s. for each $t \in T$. Then $(V(t), t \in T)$ is a zero mean Gaussian process with $\mathsf{cov}(V(t), V(s)) = \min\{\mathsf{var}(V(t)), \mathsf{var}(V(s))\}$ for each $t, s \in T$; i. e. it is a pre-Wiener process.

2. Assume $(V(t), t \in T)$ is a pre-Wiener process.

   Let us denote $D = \{\mathsf{var}(V(t)) \,:\, t \in T\}$.

   For each $d \in D$ we choose $t(d) \in T$ such that $d = \mathsf{var}(V(t(d)))$ and denote $X(d) = V(t(d))$.

   Then the process $(X(d), d \in D)$ is a pre-Wiener process being Gaussian zero-mean process according to the definition and fulfilling

$$\mathsf{cov}(X(d_1), X(d_2)) = \mathsf{cov}(V(t(d_1)), V(t(d_2)))$$

$$= \min\{\mathsf{var}(V(t(d_1))), \mathsf{var}(V(t(d_2)))\} = \min\{d_1, d_2\}.$$

   Further, $X(d) = V(s)$ a.s. whenever $\mathsf{var}(V(s)) = d$ since

$$\mathsf{E}\Big[(X(d) - V(s))^2\Big] = \mathsf{E}\Big[(V(t(d)) - V(s))^2\Big]$$

$$= \mathsf{var}(V(t(d))) + \mathsf{var}(V(s)) - 2\min\{\mathsf{var}(V(t(d))), \mathsf{var}(V(s))\} = 0.$$

   According to Proposition 2.2, there is a pre-Wiener process $(Y(d), d \in \mathsf{clo}(D))$ extending the process $(X(d), d \in D)$. If $0 \notin \mathsf{clo}(D)$, we add $Y(0) \equiv 0$ to have a pre-Wiener process $(Y(d), d \in \mathsf{clo}(D) \cup \{0\})$ extending the process $(X(d), d \in D)$.

   Let us set $D_1 = \mathsf{clo}(D) \cup \{0\}$ and $d_{\max} = \sup_{d \in D_1} d$. Now we start a construction giving a pre-Wiener process with $\mathsf{var}(\mathcal{W}(t)) = t$ defined for all $t \in \mathbb{R}_+$.

   If $d_{\max} = +\infty$ then we set $D_2 = D_1$ else $D_2 = D_1 \cup (d_{\max}, +\infty)$ and one can find a Wiener process $(G(t), t \in \mathbb{R}_+)$ independent with the process $(Y(d), d \in D_1)$ and define $Y(d) = Y(d_{\max}) + G(d - d_{\max})$ for all $d > d_{\max}$. Then the process $(Y(d), d \in D_2)$ is a pre-Wiener process with $\mathsf{var}(Y(d)) = d$.

   The set D_2 is a closed unbounded subset of $\mathbb{R}_+$ and, hence, its complement is an open set and can be expressed as an at most countable union of disjoint bounded intervals, say $\mathbb{R}_+ \setminus D_2 = \bigcup_{i \in I}(a_i, b_i)$, where $I \subset \mathbb{N}$ and $0 \le a_i < b_i < +\infty \ \forall\, i \in I$.

   For all $i \in I$ can be found a Brownian bridge $(B_i(t), t \in [0,1])$ such that the processes $(Y(d), d \in D_2)$, $(B_i(t), t \in [0,1])$, $i \in I$ are independent.

   Now, we define a process $(\mathcal{W}(t), t \in \mathbb{R}_+)$ by

   (i) $\mathcal{W}(t) = Y(t)$ for all $t \in D_2$;

   (ii) $\mathcal{W}(t) = Y(a_i) + \frac{t-a_i}{b_i-a_i}(Y(b_i) - Y(a_i)) + \sqrt{b_i - a_i}\,B_i\left(\frac{t-a_i}{b_i-a_i}\right)$ for all $t \in (a_i, b_i)$, $i \in I$.

   The process $(\mathcal{W}(t), t \in \mathbb{R}_+)$ is a Gaussian process with zero mean. Hence we just need to verify covariances to show that it is a pre-Wiener process.

   Let $0 \le t \le s < +\infty$.

   (a) If $t, s \in D_2$ then $\mathsf{cov}(\mathcal{W}(t), \mathcal{W}(s)) = \mathsf{cov}(Y(t), Y(s)) = t$.

(b) If $t \in D_2$ and $s \notin D_2$ then there is $i \in I$ such that $s \in (a_i, b_i)$ and $t \leq a_i$. Then we conclude

$$\mathsf{cov}(\mathcal{W}(t)\,,\mathcal{W}(s))$$

$$\mathsf{cov}\left(Y(t), Y(a_i) + \frac{s-a_i}{b_i-a_i}(Y(b_i) - Y(a_i)) + \sqrt{b_i - a_i}B_i\left(\frac{s-a_i}{b_i-a_i}\right)\right)$$

$$= \mathsf{cov}(Y(t), Y(a_i)) = t.$$

(c) If $t \notin D_2$ and $s \in D_2$ then there is $i \in I$ such that $t \in (a_i, b_i)$ and $b_i \leq s$. Then we have

$$\mathsf{cov}(\mathcal{W}(t)\,,\mathcal{W}(s))$$

$$\mathsf{cov}\left(Y(a_i) + \frac{t-a_i}{b_i-a_i}(Y(b_i) - Y(a_i)) + \sqrt{b_i - a_i}B_i\left(\frac{t-a_i}{b_i-a_i}\right), Y(s)\right)$$

$$= a_i + \frac{t-a_i}{b_i-a_i}(b_i - a_i) = t.$$

(d) Let $t \in (a_i, b_i)$ and $s \in (a_j, b_j)$ for some $i, j \in I$, $b_i \leq a_j$. Then we have

$$\mathsf{cov}(\mathcal{W}(t)\,,\mathcal{W}(s))$$

$$\mathsf{cov}\left(Y(a_i) + \frac{t-a_i}{b_i-a_i}(Y(b_i) - Y(a_i)) + \sqrt{b_i - a_i}B_i\left(\frac{t-a_i}{b_i-a_i}\right),\right.$$

$$\left. Y(a_j) + \frac{s-a_j}{b_j-a_j}(Y(b_j) - Y(a_j)) + \sqrt{b_j - a_j}B_j\left(\frac{s-a_j}{b_j-a_j}\right)\right)$$

$$a_i + \frac{t-a_i}{b_i-a_i}(b_i - a_i) = t.$$

(e) Let $t, s \in (a_i, b_i)$ for some $i \in I$. Then we have

$$\mathsf{cov}(\mathcal{W}(t)\,,\mathcal{W}(s))$$

$$\mathsf{cov}\left(Y(a_i) + \frac{t-a_i}{b_i-a_i}(Y(b_i) - Y(a_i)) + \sqrt{b_i - a_i}B_i\left(\frac{t-a_i}{b_i-a_i}\right),\right.$$

$$\left. Y(a_i) + \frac{s-a_i}{b_i-a_i}(Y(b_i) - Y(a_i)) + \sqrt{b_i - a_i}B_i\left(\frac{s-a_i}{b_i-a_i}\right)\right)$$

$$= a_i + \frac{(t-a_i)(s-a_i)}{b_i-a_i} + (t - a_i)\left(1 - \frac{s-a_i}{b_i-a_i}\right) = t.$$

According to Proposition 2.1, there is a Wiener process $(W(t)\,, t \in \mathbb{R}_+)$ such that $W(t) = \mathcal{W}(t)$ a.s. for each $t \geq 0$. Because of the construction, we have the desired property: $V(t) = W(\mathsf{var}(V(t)))$ a.s. for each $t \in T$. □

Now, we can start to investigate linear transformations of a Wiener process.

Proposition 2.3. Let $T \neq \emptyset$, $(W(t)\,, t \in \mathbb{R}_+)$ be a Wiener process and for each $t \in T$ a function $a_t \in \mathsf{L}_2(\mathbb{R}_+)$ be given.

Then $\left(\int_0^{+\infty} a_t(\eta)\,\mathrm{d}W(\eta)\,, t \in T\right)$ is a Gaussian process with zero mean and the covariance function $R(t,s) = \int_0^{+\infty} a_t(u)a_s(u)\,\mathrm{d}u$ for each $t, s \in T$.

Proof. The set of all Gaussian random vectors is closed for linear transformations and as well is a closed subset of the space L_2. Therefore, the considered process is always Gaussian.

By the definition of the stochastic integral, the mean must be zero and covariances of the announced form. □

Theorem 2.2. Let $T \neq \emptyset$, $(W(t)\,, t \in \mathbb{R}_+)$ be a Wiener process and for each $t \in T$ a function $a_t \in \mathsf{L}_2(\mathbb{R}_+)$ be given.

Then there exists a Wiener process $(V(t), t \in \mathbb{R}_+)$ such that

$$\int_0^{+\infty} a_t(\eta)\,\mathrm{d}W(\eta) = V\left(\int_0^{+\infty} a_t^2(\eta)\,\mathrm{d}\eta\right) \quad \text{a.s. for each } t \in T \tag{1}$$

if and only if

$$\int_0^{+\infty} a_t(\eta)a_s(\eta)\,\mathrm{d}\eta = \min\left\{\int_0^{+\infty} a_t^2(\eta)\,\mathrm{d}\eta, \int_0^{+\infty} a_s^2(\eta)\,\mathrm{d}\eta\right\} \quad \text{for each } t, s \in T. \tag{2}$$

Proof. According to Theorem 2.1 the expression (1) is fulfilled if and only if $\left(\int_0^{+\infty} a_t(\eta)\,\mathrm{d}W(\eta)\,, t \in T\right)$ is a pre-Wiener process.

The process is zero-mean Gaussian process, according to Proposition 2.3. Hence, (1) and (2) are equivalent. □

Theorem 2.2 gives a necessary and sufficient condition for the expression (1). In the rest of this section, we present sufficient conditions giving (1). At first, let us mention that reversing time procedure preserves the expression (1).

Proposition 2.4. Let $T \neq \emptyset$ and $(W(t)\,, t \in \mathbb{R}_+)$ be a Wiener process. For each $t \in T$ a function $a_t \in \mathsf{L}_2(\mathbb{R}_+)$ is given and

$$\int_0^{+\infty} a_t(\eta)a_s(\eta)\,\mathrm{d}\eta = \min\left\{\int_0^{+\infty} a_t^2(\eta)\,\mathrm{d}\eta, \int_0^{+\infty} a_s^2(\eta)\,\mathrm{d}\eta\right\} \quad \text{for each } t, s \in T. \tag{3}$$

Then there exists a Wiener process $(V(t), t \in \mathbb{R}_+)$ such that

$$\left(\int_0^{+\infty} a_t(\eta)^2\,\mathrm{d}\eta\right)^{-1} \int_0^{+\infty} a_t(\eta)\,\mathrm{d}W(\eta) = V\left(\left(\int_0^{+\infty} a_t(\eta)^2\,\mathrm{d}\eta\right)^{-1}\right) \quad \text{a.s.} \tag{4}$$

for each $t \in T$.

Proof. One can verify the condition from Theorem 2.2 and give the proof. □

The condition of Theorem 2.2 is closed to Radon–Nikodym derivative.

Proposition 2.5. Let $T \neq \emptyset$, $(W(t), t \in \mathbb{R}_+)$ be a Wiener process and for each $t \in T$ a function $a_t \in \mathsf{L}_2(\mathbb{R}_+)$ be given. Let for each $t, s \in T$ we have either $a_t = \frac{\mathrm{d}A_s}{\mathrm{dm}_{\sigma(a_t)}} \cdot \mathbb{I}_{\{\eta \in \mathbb{R}_+ : a_t(\eta) \neq 0\}}$ or $a_s = \frac{\mathrm{d}A_t}{\mathrm{dm}_{\sigma(a_s)}} \cdot \mathbb{I}_{\{\eta \in \mathbb{R}_+ : a_s(\eta) \neq 0\}}$, where A_t denotes the measure on $\mathbb{B}(\mathbb{R}_+)$ with density a_t and $\mathsf{m}_{\mathcal{B}}$ denotes the restriction of Lebesgue measure to a σ-field $\mathcal{B}$.

Then there is a Wiener process $(V(t), t \in \mathbb{R}_+)$ such that

$$\int_0^{+\infty} a_t(\eta)\, \mathrm{d}W(\eta) = V\left(\int_0^{+\infty} a_t(\eta)^2\, \mathrm{d}\eta\right) \text{ a.s. for each } t \in T. \tag{5}$$

P r o o f. Let $t, s \in T$. Properties of Radon–Nikodym derivative are giving

$$\begin{aligned} \int_0^{+\infty} a_t(\eta) a_s(\eta)\, \mathrm{d}\eta &= \int_0^{+\infty} a_t(\eta)^2\, \mathrm{d}\eta \le \int_0^{+\infty} a_s(\eta)^2\, \mathrm{d}\eta \\ &\qquad \text{if } a_t = \frac{\mathrm{d}A_s}{\mathrm{dm}_{\sigma(a_t)}} \cdot \mathbb{I}_{\{\eta \in \mathbb{R}_+ : a_t(\eta) \neq 0\}}, \\ \int_0^{+\infty} a_t(\eta) a_s(\eta)\, \mathrm{d}\eta &= \int_0^{+\infty} a_s(\eta)^2\, \mathrm{d}\eta \le \int_0^{+\infty} a_t(\eta)^2\, \mathrm{d}\eta \\ &\qquad \text{if } a_s = \frac{\mathrm{d}A_t}{\mathrm{dm}_{\sigma(a_s)}} \mathbb{I}_{\{\eta \in \mathbb{R}_+ : a_s(\eta) \neq 0\}}. \end{aligned}$$

Consequently, Theorem 2.2 concludes the proof. □

Proposition 2.6. Let $T \neq \emptyset$, $(W(t), t \in \mathbb{R}_+)$ be a Wiener process and for each $t \in T$ a function $a_t \in \mathsf{L}_2(\mathbb{R}_+)$ be given. Let for each $t, s \in T$ we have either $a_t = \frac{\mathrm{d}A_s}{\mathrm{dm}_{\sigma(a_t)}} \cdot \mathbb{I}_{\{\eta \in \mathbb{R}_+ : a_t(\eta) \neq 0\}}$ or $a_s = \frac{\mathrm{d}A_t}{\mathrm{dm}_{\sigma(a_s)}} \cdot \mathbb{I}_{\{\eta \in \mathbb{R}_+ : a_s(\eta) \neq 0\}}$, where A_t denotes the measure on $\mathbb{B}(\mathbb{R}_+)$ with density a_t and $\mathsf{m}_{\mathcal{B}}$ denotes the restriction of Lebesgue measure to a σ-field $\mathcal{B}$.

Then there is a Wiener process $(V(t), t \in \mathbb{R}_+)$ such that

$$\left(\int_0^{+\infty} a_t(\eta)^2\, \mathrm{d}\eta\right)^{-1} \int_0^{+\infty} a_t(\eta)\, \mathrm{d}W(\eta) = V\left(\left(\int_0^{+\infty} a_t(\eta)^2\, \mathrm{d}\eta\right)^{-1}\right) \text{ a.s.} \tag{6}$$

for each $t \in T$.

P r o o f. The statement is a direct consequence of Propositions 2.5 and 2.4. □

Unfortunately, the condition (2) is weaker than Radon–Nikodym derivative.

Example 2.1. Let $T = \{1, 2\}$ and

$$a_1(\eta) = \begin{cases} 1 & \text{for } 0 \le \eta < 2, \\ 2 & \text{for } 2 \le \eta < 4, \\ 0 & \text{for } 4 \le \eta < +\infty, \end{cases} \qquad a_2(\eta) = \begin{cases} 0 & \text{for } 0 \le \eta < 1, \\ 1 & \text{for } 1 \le \eta < 2, \\ 2 & \text{for } 2 \le \eta < 3, \\ 2.5 & \text{for } 3 \le \eta < 4. \\ 0 & \text{for } 4 \le \eta < +\infty. \end{cases}$$

One can easily check that

$$\int_0^{+\infty} a_1(\eta) a_2(\eta)\,\mathrm{d}\eta = 10 = \int_0^{+\infty} a_1^2(\eta)\,\mathrm{d}\eta \le \int_0^{+\infty} a_2^2(\eta)\,\mathrm{d}\eta = 11.25,$$

$$\frac{\mathrm{d}A_2}{\mathrm{dm}_{\sigma(a_1)}}(\eta) = \frac{\mathrm{d}A_2}{\mathrm{dm}_{\sigma(a_1)}}(\eta) \cdot \mathbb{I}_{\{\eta \in \mathbb{R}_+ : a_1(\eta) \ne 0\}}(\eta) = \begin{cases} 0.5 & \text{for } 0 \le \eta < 2, \\ 2.25 & \text{for } 2 \le \eta < 4, \\ 0 & \text{for } 4 \le \eta < +\infty. \end{cases}$$

Assuming restricted collections of functions, we can show reverse statements.

Proposition 2.7. Let $T \ne \emptyset$, $(W(t), t \in \mathbb{R}_+)$ be a Wiener process, $a \in \mathsf{L}(\mathbb{R}_+)$ and $A_t \subset \mathbb{R}_+$ for each $t \in T$ be such that $\int_{A_t} a(\eta)^2\,\mathrm{d}\eta < +\infty$ for each $t \in T$.

Then there is a Wiener process $(V(t), t \in \mathbb{R}_+)$ such that

$$\int_0^{+\infty} a(\eta)\mathbb{I}_{A_t}(\eta)\,\mathrm{d}W(\eta) = V\left(\int_{A_t} a(\eta)^2\,\mathrm{d}\eta\right) \text{ a.s. for each } t \in T \tag{7}$$

if and only if for each $t, s \in T$

$$\begin{aligned} &\text{either } \mathsf{m}((A_t - A_s) \cap \{\eta \in \mathbb{R}_+ : a(\eta) \ne 0\}) = 0 \\ &\text{or } \mathsf{m}((A_s - A_t) \cap \{\eta \in \mathbb{R}_+ : a(\eta) \ne 0\}) = 0. \end{aligned} \tag{8}$$

Proof. According to Theorem 2.2, the expression by a Wiener process is equivalent to

$$\int_{A_t \cap A_s} a(u)^2\,\mathrm{d}u = \min\left\{\int_{A_t} a(u)^2\,\mathrm{d}u, \int_{A_s} a(u)^2\,\mathrm{d}u\right\} \quad \text{for each } t, s \in T.$$

That is equivalent to

$$\min\left\{\int_{A_t - A_t \cap A_s} a(u)^2\,\mathrm{d}u, \int_{A_s - A_t \cap A_s} a(u)^2\,\mathrm{d}u\right\} = 0 \quad \text{for each } t, s \in T.$$

This condition can be equivalently rewritten as (8). □

Proposition 2.8. Let $T \neq \emptyset$, $(W(t), t \in \mathbb{R}_+)$ be a Wiener process, $a \in \mathsf{L}(\mathbb{R}_+)$ and $\alpha_t \in \mathbb{R}$, $\alpha_t \neq 0$, $A_t \subset \mathbb{R}_+$ for each $t \in T$. Let

$$\text{either } \mathsf{m}((A_t - A_s) \cap \{\eta \in \mathbb{R}_+ : a(\eta) \neq 0\}) = 0$$
$$\text{or } \mathsf{m}((A_s - A_t) \cap \{\eta \in \mathbb{R}_+ : a(\eta) \neq 0\}) = 0 \text{ whenever } t, s \in T$$

and $0 < \int_{A_t} a(\eta)^2 \, d\eta < +\infty$ for each $t \in T$. Let there be $t_0, s_0 \in T$ such that $\alpha_{t_0} \neq \alpha_{s_0}$.

Then there is a Wiener process $(V(t), t \in \mathbb{R}_+)$ such that

$$\int_0^{+\infty} \alpha_t a(\eta) \mathbb{I}_{A_t}(\eta) \, dW(\eta) = V\left(\alpha_t^2 \int_{A_t} a(\eta)^2 \, d\eta\right) \text{ a.s. for each } t \in T \tag{9}$$

if and only if

$$\alpha_t = \frac{\int_{A_{t_0}} a(\eta)^2 \, d\eta}{\int_{A_t} a(\eta)^2 \, d\eta} \cdot \alpha_{t_0} \text{ for each } t \in T. \tag{10}$$

Proof. According to Theorem 2.2, the expression by a Wiener process is equivalent to

$$\alpha_t \alpha_s \int_{A_t \cap A_s} a(u)^2 \, du = \min\left\{\alpha_t^2 \int_{A_t} a(u)^2 \, du, \alpha_s^2 \int_{A_s} a(u)^2 \, du\right\} \text{ for each } t, s \in T.$$

Assume $t, s \in T$ with $\alpha_t \neq \alpha_s$ and $\mathsf{m}((A_t - A_s) \cap \{\eta \in \mathbb{R}_+ : a(\eta) \neq 0\}) = 0$, then

$$\alpha_s \int_{A_s} a(u)^2 \, du = \alpha_t \int_{A_t} a(u)^2 \, du \text{ because}$$

$$0 < \int_{A_t} a(\eta)^2 \, d\eta < +\infty, \; 0 < \int_{A_s} a(\eta)^2 \, d\eta < +\infty.$$

Since $\alpha_{t_0} \neq \alpha_{s_0}$, we have either $\alpha_t \neq \alpha_{s_0}$ or $\alpha_t \neq \alpha_{t_0}$ for each $t \in T$. Therefore, we can conclude

$$\alpha_t \int_{A_t} a(u)^2 \, du = \alpha_{t_0} \int_{A_{t_0}} a(u)^2 \, du \text{ for each } t \in T.$$

Evidently, such constants fulfill the condition of Theorem 2.2. Therefore, we have proved the statement of the proposition. □

These two propositions solve the problem of the rescaling of a Wiener process.

Theorem 2.3. Let $T \neq \emptyset$ and $(W(t), t \in \mathbb{R}_+)$ be a Wiener process. Let for each $t \in T$ constants $\alpha_t, \beta_t \in \mathbb{R}$, $\alpha_t \neq 0$, $\beta_t > 0$ be given.

Then there is a Wiener process $(V(t), t \in \mathbb{R}_+)$ such that

$$\alpha_t W(\beta_t) = V\left(\alpha_t^2 \beta_t\right) \text{ a.s. for each } t \in T$$

if and only if either $\alpha_t = \alpha_s$ for each $s, t \in T$ or $\alpha_t \beta_t = \alpha_s \beta_s$ for each $s, t \in T$.

Proof.

1. Let $\alpha_t = \alpha$ for all $t \in T$.

   Setting $a \equiv \alpha$, $A_t = [0, \beta_t]$ the assumptions of Proposition 2.7 are fulfilled and the statement becomes to be obvious.

2. Let $t, s \in T$ exist such that $\alpha_t < \alpha_s$.

   Setting $a \equiv 1$, $A_t = [0, \beta_t]$ the assumptions of Proposition 2.8 are fulfilled and the statement follows. □

3. BROWNIAN BRIDGE

This section treats necessary and sufficient conditions under which the linear transformation gives Brownian bridges. For completeness let us recall the definition of Brownian bridge.

Definition 3.1. A process $(B(t)\,, t \in [0,1])$ is called Brownian bridge if

1. it is a Gaussian process;
2. $\forall\, t \in [0,1]: \ \mathsf{E}[B(t)] = 0$;
3. $\forall\, t, s \in [0,1]: \ \mathsf{cov}(B(t)\,, B(s)) = \min\{t,s\}\,(1 - \max\{t,s\})$;
4. its sample paths are continuous.

Definition 3.2. Let T be a non-empty set. We call the process $(\mathcal{B}(t)\,, t \in T)$ to be a pre-Brownian bridge if

1. it is a Gaussian process;
2. $\forall\, t \in T: \ \mathsf{E}[\mathcal{B}(t)] = 0$;
3. there is a function $\xi : T \to [0,1]$ such that $\forall\, t, s \in T:$

$$\mathsf{cov}(\mathcal{B}(t)\,, \mathcal{B}(s)) = \min\{\xi(t), \xi(s)\}\,(1 - \max\{\xi(t), \xi(s)\}).$$

A Brownian bridge exists.

Proposition 3.1. Let $(W(t)\,, t \in \mathbb{R}_+)$ be a Wiener process. Then the process $(W(t) - t \cdot W(1)\,, t \in [0,1])$ is a Brownian bridge.

Proof. The transformed process is a zero-mean Gaussian process with continuous sample paths. Evaluating covariances one can easily check that the process is a Brownian bridge. □

Any pre-Brownian bridge can be described as a re-indexed Brownian bridge.

Lemma 3.1. Let $T \neq \emptyset$, G be a standard Gaussian r.v. independent with a random process $(\mathcal{B}(t), t \in T)$ and $\xi : T \to [0,1]$. Then

$$\begin{gathered}(\mathcal{B}(t), t \in T) \text{ is a pre-Brownian bridge with} \\ \mathsf{cov}(\mathcal{B}(t), \mathcal{B}(s)) = \min\{\xi(t), \xi(s)\}\,(1 - \max\{\xi(t), \xi(s)\})\ t, s \in T\end{gathered} \tag{11}$$

if and only if

$$\begin{gathered}(\mathcal{B}(t) + \xi(t) \cdot G, t \in T) \text{ is a pre-Wiener process with} \\ \mathsf{var}(\mathcal{B}(t) + \xi(t) \cdot G) = \xi(t)\ \forall\ t \in T.\end{gathered} \tag{12}$$

Proof. For each $t, s \in T$ we immediately receive

$$\begin{aligned}\mathsf{cov}(\mathcal{B}(t) + \xi(t) \cdot G, \mathcal{B}(s) + \xi(s) \cdot G) &= \mathsf{cov}(\mathcal{B}(t), \mathcal{B}(s)) + \xi(t) \cdot \xi(s) \cdot \mathsf{var}(G) \\ &= \mathsf{cov}(\mathcal{B}(t), \mathcal{B}(s)) + \xi(t) \cdot \xi(s).\end{aligned}$$

That is straightforwardly giving the desired statement. □

Theorem 3.1. Let $T \neq \emptyset$ and $(\mathcal{B}(t), t \in [0,1])$ be a L_2-random process. Then $(\mathcal{B}(t), t \in [0,1])$ is a pre-Brownian bridge if and only if there exists a Brownian bridge $(B(t), t \in [0,1])$ and a function $\xi : T \to [0,1]$ such that

$$\mathcal{B}(t) = B(\xi(t)) \ \text{a.s. for each } t \in T.$$

Proof. Let G be standard Gaussian r.v. independent with the process $(\mathcal{B}(t), t \in T)$ and $\xi : T \to [0,1]$.

According to Lemma 3.1, $(\mathcal{B}(t), t \in T)$ is a pre-Brownian bridge with

$$\mathsf{cov}(\mathcal{B}(t), \mathcal{B}(s)) = \min\{\xi(t), \xi(s)\}\,(1 - \max\{\xi(t), \xi(s)\})$$

if and only if $(\mathcal{B}(t) + \xi(t) \cdot G, t \in T)$ is a pre-Wiener process with

$$\mathsf{var}(\mathcal{B}(t) + \xi(t) \cdot G) = \xi(t).$$

According to Theorem 2.1, the process $(\mathcal{B}(t) + \xi(t) \cdot G, t \in T)$ is a pre-Wiener process with $\mathsf{var}(\mathcal{B}(t) + \xi(t) \cdot G) = \xi(t)$ if and only if there is a Wiener process $(W(t), t \in \mathbb{R}_+)$ such that $\mathcal{B}(t) + \xi(t) \cdot G = W(\xi(t))$ a.s. for each $t \in T$.

Consequently, $\mathcal{B}(t) = W(\xi(t)) - \xi(t) \cdot W(1)$ a.s. for each $t \in T$.

This concludes the proof since the process $(W(t) - t \cdot W(1), t \in [0,1])$ is a Brownian bridge, accordingly to Proposition 3.1. □

Brownian bridges and pre-Brownian bridges are preserved if the time is reversed.

Proposition 3.2. If $(B(t), t \in T)$ is a pre-Brownian bridge then $(B(1-t), t \in T)$ is a pre-Brownian bridge, as well.

Proof. The covariance structure must be verified, only, and that is evident. □

Proposition 3.3. If $(B(t)\,, t \in [0,1])$ is a Brownian bridge then $(B(1-t)\,, t \in [0,1])$ is a Brownian bridge, as well.

Proof. The process $(B(1-t)\,, t \in [0,1])$ is a pre-Brownian bridge, according to Propositions 3.2 and its sample path are evidently continuous. □

Theorem 3.2. Let $T \neq \emptyset$, $(W(t)\,, t \in \mathbb{R}_+)$ be a Wiener process, for each $t \in T$ a function $a_t \in \mathsf{L}_2(\mathbb{R}_+)$ be given and $\xi : T \to [0,1]$.

Then there exists a Brownian bridge $(B(t)\,, t \in [0,1])$ such that

$$\int_0^{+\infty} a_t(\eta)\,\mathrm{d}W(\eta) = B(\xi(t)) \text{ a.s. for each } t \in T \tag{13}$$

if and only if

$$\int_0^{+\infty} a_t(\eta)a_s(\eta)\,\mathrm{d}\eta = \min\{\xi(t), \xi(s)\}\,(1 - \max\{\xi(t), \xi(s)\}) \quad \text{for each } t, s \in T. \tag{14}$$

(Particularly $\int_0^{+\infty} a_t^2(\eta)\,\mathrm{d}\eta \leq \xi(t)(1-\xi(t)) \leq \frac{1}{4}$.)

Proof. The assertion is a direct consequence of Theorem 3.1. □

Now, we give two systems of functions fulfilling the condition of the previous theorem.

Proposition 3.4. Let $T \neq \emptyset$ and $(W(t)\,, t \in \mathbb{R}_+)$ be a Wiener process. For each $t \in T$ a function $a_t \in \mathsf{L}_2(\mathbb{R}_+)$ is given such that

$$\int_0^{+\infty} a_t(\eta)a_s(\eta)\,\mathrm{d}\eta = \min\left\{\int_0^{+\infty} a_t^2(\eta)\,\mathrm{d}\eta, \int_0^{+\infty} a_s^2(\eta)\,\mathrm{d}\eta\right\} \quad \text{for each } t, s \in T.$$

Then there exists a Brownian bridge $(B(t)\,, t \in \mathbb{R}_+)$ such that

$$\left(\int_0^{+\infty} a_t(\eta)^2\,\mathrm{d}\eta + 1\right)^{-1} \int_0^{+\infty} a_t(\eta)\,\mathrm{d}W(\eta) = B\left(\left(\int_0^{+\infty} a_t(\eta)^2\,\mathrm{d}\eta + 1\right)^{-1}\right) \text{ a.s.}$$

for each $t \in T$.

Proof. One can verify the condition of Theorem 3.2 for

$$\xi(t) = \left(\int_0^{+\infty} a_t(\eta)^2\,\mathrm{d}\eta + 1\right)^{-1}. \qquad \square$$

Proposition 3.5. Let $T \neq \emptyset$ and $(W(t), t \in \mathbb{R}_+)$ be a Wiener process. For each $t \in T$ a function $a_t \in \mathsf{L}_2(\mathbb{R}_+)$ is given such that

$$\int_0^{+\infty} a_t(\eta) a_s(\eta)\,\mathrm{d}\eta = \min\left\{\int_0^{+\infty} a_t^2(\eta)\,\mathrm{d}\eta, \int_0^{+\infty} a_s^2(\eta)\,\mathrm{d}\eta\right\} \quad \text{for each } t, s \in T.$$

Let $t_0 \in T$ be such that $\int_0^{+\infty} a_t(\eta)^2\,\mathrm{d}\eta \leq \int_0^{+\infty} a_{t_0}(\eta)^2\,\mathrm{d}\eta$ for each $t \in T$ and $\int_0^{+\infty} a_{t_0}^2(\eta)\,\mathrm{d}\eta > 0$.

Then there exists a Brownian bridge $(B(t), t \in \mathbb{R}_+)$ such that

$$\left(\int_0^{+\infty} a_{t_0}(\eta)^2\,\mathrm{d}\eta\right)^{-\frac{1}{2}} \int_0^{+\infty} \left(a_t(\eta) - \frac{\int_0^{+\infty} a_t(\nu)^2\,\mathrm{d}\nu}{\int_0^{+\infty} a_{t_0}(\nu)^2\,\mathrm{d}\nu} \cdot a_{t_0}(\eta)\right) \mathrm{d}W(\eta)$$

$$= B\left(\frac{\int_0^{+\infty} a_t(\eta)^2\,\mathrm{d}\eta}{\int_0^{+\infty} a_{t_0}(\eta)^2\,\mathrm{d}\eta}\right) \quad \text{a. s.}$$

for each $t \in T$.

Proof. One can verify the condition of Theorem 3.2 for $\xi(t) = \frac{\int_0^{+\infty} a_t(\eta)^2\,\mathrm{d}\eta}{\int_0^{+\infty} a_{t_0}(\eta)^2\,\mathrm{d}\eta}$. □

Again, we can employ Radon–Nikodym derivative.

Proposition 3.6. Let $T \neq \emptyset$ and $(W(t), t \in \mathbb{R}_+)$ be a Wiener process. For each $t \in T$ a function $a_t \in \mathsf{L}_2(\mathbb{R}_+)$ is given such that for each $t, s \in T$ we have either $a_t = \frac{\mathrm{d}A_s}{\mathrm{dm}_{\sigma(a_t)}} \cdot \mathbb{I}_{\{\eta \in \mathbb{R}_+ : a_t(\eta) \neq 0\}}$ or $a_s = \frac{\mathrm{d}A_t}{\mathrm{dm}_{\sigma(a_s)}} \cdot \mathbb{I}_{\{\eta \in \mathbb{R}_+ : a_s(\eta) \neq 0\}}$, where A_t denotes the measure on $\mathbb{B}(\mathbb{R}_+)$ with density a_t and $\mathsf{m}_{\mathcal{B}}$ denotes the restriction of Lebesgue measure to a σ-field $\mathcal{B}$.

Then there exists a Brownian bridge $(B(t), t \in \mathbb{R}_+)$ such that

$$\left(\int_0^{+\infty} a_t(\eta)^2\,\mathrm{d}\eta + 1\right)^{-1} \int_0^{+\infty} a_t(\eta)\,\mathrm{d}W(\eta) = B\left(\left(\int_0^{+\infty} a_t(\eta)^2\,\mathrm{d}\eta + 1\right)^{-1}\right) \quad \text{a. s.}$$

for each $t \in T$.

Proof. According to Proposition 2.5, the considered functions fulfill the condition required in Proposition 3.4. Hence, the proof is done. □

Proposition 3.7. Let $T \neq \emptyset$ and $(W(t), t \in \mathbb{R}_+)$ be a Wiener process. For each $t \in T$ a function $a_t \in \mathsf{L}_2(\mathbb{R}_+)$ is given such that for each $t, s \in T$ we have either $a_t = \frac{\mathrm{d}A_s}{\mathrm{dm}_{\sigma(a_t)}} \cdot \mathbb{I}_{\{\eta \in \mathbb{R}_+ : a_t(\eta) \neq 0\}}$ or $a_s = \frac{\mathrm{d}A_t}{\mathrm{dm}_{\sigma(a_s)}} \cdot \mathbb{I}_{\{\eta \in \mathbb{R}_+ : a_s(\eta) \neq 0\}}$, where A_t denotes the measure on $\mathbb{B}(\mathbb{R}_+)$ with density a_t and $\mathsf{m}_{\mathcal{B}}$ denotes the restriction of Lebesgue measure to a σ-field $\mathcal{B}$. Let $t_0 \in T$ be such that $\int_0^{+\infty} a_t(\eta)^2\,\mathrm{d}\eta \leq \int_0^{+\infty} a_{t_0}(\eta)^2\,\mathrm{d}\eta$ for each $t \in T$ and $\int_0^{+\infty} a_{t_0}^2(\eta)\,\mathrm{d}\eta > 0$.

Then there exists a Brownian bridge $(B(t)\,, t \in \mathbb{R}_+)$ such that

$$\left(\int_0^{+\infty} a_{t_0}(\eta)^2\,\mathrm{d}\eta\right)^{-\frac{1}{2}} \int_0^{+\infty} \left(a_t(\eta) - \frac{\int_0^{+\infty} a_t(\nu)^2\,\mathrm{d}\nu}{\int_0^{+\infty} a_{t_0}(\nu)^2\,\mathrm{d}\nu} \cdot a_{t_0}(\eta)\right) \mathrm{d}W(\eta)$$

$$= B\left(\frac{\int_0^{+\infty} a_t(\eta)^2\,\mathrm{d}\eta}{\int_0^{+\infty} a_{t_0}(\eta)^2\,\mathrm{d}\eta}\right) \text{ a. s.}$$

for each $t \in T$.

P r o o f. According to Proposition 2.5, the considered functions fulfill the condition required in Proposition 3.5. Hence, the proof is done. □

Assuming restricted collections of functions, we can show reverse statements.

Proposition 3.8. Let $T \neq \emptyset$, $\xi : T \to [0,1]$, $(W(t)\,, t \in \mathbb{R}_+)$ be a Wiener process, $a \in \mathsf{L}(\mathbb{R}_+)$ and $\alpha_t \in \mathbb{R}$, $\alpha_t \neq 0$, $A_t \subset \mathbb{R}_+$ for each $t \in T$. Let

$$\text{either } \mathsf{m}((A_t - A_s) \cap \{\eta \in \mathbb{R}_+ : a(\eta) \neq 0\}) = 0$$
$$\text{or } \mathsf{m}((A_s - A_t) \cap \{\eta \in \mathbb{R}_+ : a(\eta) \neq 0\}) = 0 \text{ whenever } t, s \in T$$

and $0 < \int_{A_t} a(\eta)^2\,\mathrm{d}\eta < +\infty$ for each $t \in T$.

Then there is a Brownian bridge $(B(t)\,, t \in \mathbb{R}_+)$ such that

$$\int_0^{+\infty} \alpha_t a(\eta) \mathbb{I}_{A_t}(\eta)\,\mathrm{d}W(\eta) = B(\xi(t)) \text{ a. s. for each } t \in T \tag{15}$$

if and only if

$$\xi(t)(1 - \xi(t)) = \alpha_t^2 \cdot \int_{A_t} a(\eta)^2\,\mathrm{d}\eta \text{ for all } t \in T \tag{16}$$

and there is $c \in \mathbb{R}$, $c \neq 0$ such that either

$$\Big(\mathsf{m}((A_t - A_s) \cap \{\eta \in \mathbb{R}_+ : a(\eta) \neq 0\}) = 0 \iff \xi(t) \leq \xi(s)\Big) \text{ for all } t, s \in T,$$
$$\xi(t) = 1 - c \cdot \alpha_t \text{ for all } t \in T \tag{17}$$

or

$$\Big(\mathsf{m}((A_t - A_s) \cap \{\eta \in \mathbb{R}_+ : a(\eta) \neq 0\}) = 0 \iff \xi(t) \geq \xi(s)\Big) \text{ for all } t, s \in T,$$
$$\xi(t) = c \cdot \alpha_t \text{ for all } t \in T. \tag{18}$$

P r o o f. We need just to verify that under the theorem assumptions (16), (17), (18) are equivalent to the condition (14) of Theorem 3.2.

1. Let the conditions (16) and (17) be fulfilled.
   Thus for $t, s \in T$ with $\xi(t) \leq \xi(s)$ we have

$$\alpha_t \alpha_s \cdot \int_{A_t} a(\eta)^2 \, d\eta = \frac{\alpha_s}{\alpha_t} \xi(t)(1-\xi(t)) = \frac{1-\xi(s)}{1-\xi(t)} \xi(t)(1-\xi(t)) = \xi(t)(1-\xi(s)).$$

2. Let the conditions (16) and (18) be fulfilled.
   Thus for $t, s \in T$ with $\xi(t) \geq \xi(s)$ we have

$$\alpha_t \alpha_s \cdot \int_{A_t} a(\eta)^2 \, d\eta = \frac{\alpha_s}{\alpha_t} \xi(t)(1-\xi(t)) = \frac{\xi(s)}{\xi(t)} \xi(t)(1-\xi(t)) = \xi(s)(1-\xi(t)).$$

3. Let the statement (14) be fulfilled.
   Immediately, we have the following observations.
   (a) $\alpha_t^2 \cdot \int_{A_t} a(\eta)^2 \, d\eta = \xi(t)(1-\xi(t))$ for each $t \in T$.
   (b) Let $t, s \in T$, $\mathsf{m}((A_t - A_s) \cap \{\eta \in \mathbb{R}_+ : a(\eta) \neq 0\}) = 0$ and $\xi(t) \leq \xi(s)$. Then $\alpha_t \alpha_s \cdot \int_{A_t} a(\eta)^2 \, d\eta = \xi(t)(1-\xi(s))$. Consequently,

$$\frac{\alpha_s}{1-\xi(s)} = \frac{\xi(t)}{\alpha_t \int_{A_t} a(\eta)^2 \, d\eta} = \frac{\alpha_t \xi(t)}{\xi(t)(1-\xi(t))} = \frac{\alpha_t}{1-\xi(t)}.$$

   (c) Let $t, s \in T$, $\mathsf{m}((A_t - A_s) \cap \{\eta \in \mathbb{R}_+ : a(\eta) \neq 0\}) = 0$ and $\xi(t) \geq \xi(s)$. Then $\alpha_t \alpha_s \cdot \int_{A_t} a(\eta)^2 \, d\eta = \xi(s)(1-\xi(t))$. Consequently,

$$\frac{\alpha_s}{\xi(s)} = \frac{1-\xi(t)}{\alpha_t \int_{A_t} a(\eta)^2 \, d\eta} = \frac{\alpha_t(1-\xi(t))}{\xi(t)(1-\xi(t))} = \frac{\alpha_t}{\xi(t)}.$$

   It gives $\xi(t) = \xi(s)$ and $\int_{A_t} a(\eta)^2 \, d\eta = \int_{A_s} a(\eta)^2 \, d\eta$ whenever $\alpha_t = \alpha_s$.
   Assume $t_0, s_0 \in T$ such that $\alpha_{t_0} \neq \alpha_{s_0}$ and

$$\mathsf{m}((A_{t_0} - A_{s_0}) \cap \{\eta \in \mathbb{R}_+ : a(\eta) \neq 0\}) = 0.$$

   (a) If $\xi(t_0) \leq \xi(s_0)$ then $\frac{\alpha_t}{1-\xi(t)} = \frac{\alpha_{t_0}}{1-\xi(t_0)}$ for all $t \in T$.
   It is because any $t \in T$ not fulfilling this equality must fulfill $\frac{\alpha_t}{\xi(t)} = \frac{\alpha_{t_0}}{\xi(t_0)}$ and $\frac{\alpha_t}{\xi(t)} = \frac{\alpha_{s_0}}{\xi(s_0)}$. That is impossible since $\alpha_{t_0} \neq \alpha_{s_0}$.
   (b) If $\xi(t_0) \geq \xi(s_0)$ then $\frac{\alpha_t}{\xi(t)} = \frac{\alpha_{t_0}}{\xi(t_0)}$ for all $t \in T$.
   It is because any $t \in T$ not fulfilling this equality must fulfill $\frac{\alpha_t}{1-\xi(t)} = \frac{\alpha_{t_0}}{1-\xi(t_0)}$ and $\frac{\alpha_t}{1-\xi(t)} = \frac{\alpha_{s_0}}{1-\xi(s_0)}$. That is impossible since $\alpha_{t_0} \neq \alpha_{s_0}$. □

The proposition solves the problem of the rescaling of a Wiener process.

Theorem 3.3. Let $T \neq \emptyset$, $\xi : T \to [0,1]$ and $(W(t), t \in \mathbb{R}_+)$ be a Wiener process. Let for each $t \in T$ constants $\alpha_t, \beta_t \in \mathbb{R}$, $\alpha_t \neq 0$, $\beta_t > 0$ be given.

Then there is a Brownian bridge $(B(t), t \in \mathbb{R}_+)$ such that

$$\alpha_t W(\beta_t) = B(\xi(t)) \text{ a.s. for each } t \in T \tag{19}$$

if and only if

$$\xi(t)(1-\xi(t)) = \alpha_t^2 \beta_t \text{ for all } t \in T \tag{20}$$

and there is $c \in \mathbb{R}$, $c \neq 0$ such that either

$$\Big(\beta_t \leq \beta_s \iff \xi(t) \leq \xi(s)\Big) \text{ for all } t,s \in T,\ \xi(t) = 1 - c \cdot \alpha_t \text{ for all } t \in T \tag{21}$$

or

$$\Big(\beta_t \leq \beta_s \iff \xi(t) \geq \xi(s)\Big) \text{ for all } t,s \in T,\ \xi(t) = c \cdot \alpha_t \text{ for all } t \in T. \tag{22}$$

Proof. One can easily show the statement setting $a \equiv 1$, $A_t = [0, \beta_t]$ and applying Proposition 3.8. □

4. ORNSTEIN–UHLENBECK PROCESS

This section treats necessary and sufficient conditions under which the linear transformation gives Ornstein–Uhlenbeck processes. To help to the reader we usher the definition of such a process.

Definition 4.1. A process $(\mathcal{OU}(t), t \in \mathbb{R})$ is called an Ornstein–Uhlenbeck process if

1. it is a Gaussian process;
2. $\forall\, t \in \mathbb{R}:\ \mathsf{E}[\mathcal{OU}(t)] = 0$;
3. $\forall\, t,s \in \mathbb{R}:\ \mathsf{cov}(\mathcal{OU}(t), \mathcal{OU}(s)) = \mathrm{e}^{-|t-s|}$.
4. its sample paths are continuous.

Definition 4.2. Let $T \neq \emptyset$. We call a process $(\mathcal{OU}(t), t \in T)$ a pre–Ornstein–Uhlenbeck process if

1. it is a Gaussian process;
2. $\forall\, t \in T:\ \mathsf{E}[\mathcal{OU}(t)] = 0$;
3. there is a function $\xi : T \to \mathbb{R}$ such that $\mathsf{cov}(\mathcal{OU}(t), \mathcal{OU}(s)) = \mathrm{e}^{-|\xi(t)-\xi(s)|}$ $\forall\, t,s \in T$.

Linear transformation of time preserves Ornstein–Uhlenbeck processes.

Proposition 4.1. If $(\mathcal{OU}(t)\,, t \in T)$ is a pre-Ornstein–Uhlenbeck process and $\alpha, \beta \in \mathbb{R}$, $|\alpha| = 1$ then $(\mathcal{OU}(\alpha t + \beta)\,, t \in T)$ is a pre-Ornstein–Uhlenbeck process, as well.

Proof. One can easily show the statement. □

Proposition 4.2. If $(\mathcal{OU}(t)\,, t \in \mathbb{R})$ is an Ornstein–Uhlenbeck process and $\alpha, \beta \in \mathbb{R}$, $|\alpha| = 1$ then $(\mathcal{OU}(\alpha t + \beta)\,, t \in \mathbb{R})$ is an Ornstein–Uhlenbeck process, as well.

Proof. The process $(\mathcal{OU}(\alpha t + \beta)\,, t \in \mathbb{R})$ is a pre-Ornstein–Uhlenbeck process, according to Propositions 4.1 and its sample path are evidently continuous. □

An Ornstein–Uhlenbeck process can be easily constructed from a Wiener process and vice versa.

Proposition 4.3. Let $(W(t)\,, t \in \mathbb{R}_+)$ be a Wiener process. Then the process $\big(\mathrm{e}^{-t} \cdot W(\mathrm{e}^{2t})\,, t \in \mathbb{R}\big)$ is an Ornstein–Uhlenbeck process.

Proof. The considered transformation of the Wiener process is a Gaussian process with zero mean and continuous sample paths. We have to check its covariances, only. Let $t, s \in [0,1]$ then

$$\begin{aligned}\mathrm{cov}\big(\mathrm{e}^{-t} \cdot W(\mathrm{e}^{2t})\,, \mathrm{e}^{-s} \cdot W(\mathrm{e}^{2s})\big) &= \mathrm{e}^{-t-s} \cdot \mathrm{cov}\big(W(\mathrm{e}^{2t})\,, W(\mathrm{e}^{2s})\big)\\ &= \mathrm{e}^{-t-s} \cdot \min\{\mathrm{e}^{2t}, \mathrm{e}^{2s}\} = \mathrm{e}^{-t-s+2\min\{t,s\}} = \mathrm{e}^{-|t-s|}.\end{aligned}$$

□

From an Ornstein–Uhlenbeck process we can construct a Wiener process, too.

Proposition 4.4. Let $(\mathcal{OU}(t)\,, t \in \mathbb{R})$ be an Ornstein–Uhlenbeck process. Then there is a Wiener process $(W(t)\,, t \in \mathbb{R}_+)$ such that $\sqrt{t} \cdot \mathcal{OU}\big(\tfrac{1}{2}\log t\big) = W(t)$ a.s. for each $t > 0$.

Proof. The transformed process is a zero-mean Gaussian process. We need to check the covariances, only. Let $0 < t \le s < +\infty$ then

$$\begin{aligned}\mathrm{cov}\big(\sqrt{t}\mathcal{OU}\big(\tfrac{1}{2}\log t\big)\,, \sqrt{s}\mathcal{OU}\big(\tfrac{1}{2}\log s\big)\big) &= \sqrt{ts} \cdot \mathrm{cov}\big(\mathcal{OU}\big(\tfrac{1}{2}\log t\big)\,, \mathcal{OU}\big(\tfrac{1}{2}\log s\big)\big)\\ &= \sqrt{ts} \cdot \mathrm{e}^{-\left|\frac{1}{2}\log t - \frac{1}{2}\log s\right|} = \sqrt{ts} \cdot \mathrm{e}^{\frac{1}{2}\log t - \frac{1}{2}\log s} = t.\end{aligned}$$

Thus, the transformed process is a pre-Wiener process and the statement follows from Theorem 2.1. □

Theorem 4.1. Let $T \neq \emptyset$, $(W(t)\,, t \in \mathbb{R}_+)$ be a Wiener process, for each $t \in T$ a function $a_t \in \mathsf{L}_2(\mathbb{R}_+)$ be given and $\xi : T \to \mathbb{R}$.

Then there exists an Ornstein–Uhlenbeck process $(\mathcal{OU}(t)\,, t \in \mathbb{R})$ such that

$$\int_0^{+\infty} a_t(\eta)\,\mathrm{d}W(\eta) = \mathcal{OU}(\xi(t)) \ \text{ a.s. for each } t \in T \tag{23}$$

if and only if

$$\int_0^{+\infty} a_t(\eta)a_s(\eta)\,\mathrm{d}\eta = \mathrm{e}^{-|\xi(t)-\xi(s)|} \quad \text{for each } t, s \in T. \tag{24}$$

(Particularly $\int_0^{+\infty} a_t^2(\eta)\,\mathrm{d}\eta = 1$.)

Proof.

1. Assuming the representation (23) the condition (24) is fulfilled.

2. Let the condition (24) be fulfilled.

   We define $\mathcal{W}(t) = \mathrm{e}^{\xi(t)} \int_0^{+\infty} a_t(\eta)\,\mathrm{d}W(\eta)$ for all $t \in T$. One can check that the process $(\mathcal{W}(t)\,, t \in T)$ is a pre-Wiener process. According to Theorem 2.1, there is a Wiener process $(W(t)\,, t \in \mathbb{R}_+)$ such that $\mathcal{W}(t) = W(\mathrm{e}^{2t})$ a. s. for each $t \in T$. According to Proposition 4.3, we have

$$\int_0^{+\infty} a_t(\eta)\,\mathrm{d}W(\eta) = \mathrm{e}^{-\xi(t)} \cdot \mathcal{W}(t) = \mathrm{e}^{-\xi(t)} \cdot W(\mathrm{e}^{2t}) = \mathcal{OU}(t) \quad \text{a. s. for all } t \in T.$$

□

Now, we give two systems of functions fulfilling the condition of the previous theorem.

Proposition 4.5. Let $T \neq \emptyset$ and $(W(t)\,, t \in \mathbb{R}_+)$ be a Wiener process. For each $t \in T$ a function $a_t \in \mathsf{L}_2(\mathbb{R}_+)$ is given such that

$$\int_0^{+\infty} a_t(\eta)a_s(\eta)\,\mathrm{d}\eta = \min\left\{\int_0^{+\infty} a_t^2(\eta)\,\mathrm{d}\eta, \int_0^{+\infty} a_s^2(\eta)\,\mathrm{d}\eta\right\} \quad \text{for each } t, s \in T.$$

Then there exists an Ornstein–Uhlenbeck process $(\mathcal{OU}(t)\,, t \in \mathbb{R}_+)$ such that

$$\left(\int_0^{+\infty} a_t(\eta)^2\,\mathrm{d}\eta\right)^{-\frac{1}{2}} \int_0^{+\infty} a_t(\eta)\,\mathrm{d}W(\eta) = \mathcal{OU}\left(\log \int_0^{+\infty} a_t(\eta)^2\,\mathrm{d}\eta\right) \quad \text{a. s. for each } t \in T.$$

Proof. One can verify the condition of Theorem 4.1 for $\xi(t) = \log \int_0^{+\infty} a_t(\eta)^2\,\mathrm{d}\eta$. □

Again, we can employ Radon–Nikodym derivative.

Proposition 4.6. Let $T \neq \emptyset$ and $(W(t), t \in \mathbb{R}_+)$ be a Wiener process. For each $t \in T$ a function $a_t \in \mathsf{L}_2(\mathbb{R}_+)$ is given such that for each $t, s \in T$ we have either $a_t = \frac{\mathrm{d}A_s}{\mathrm{dm}_{\sigma(a_t)}} \cdot \mathbb{I}_{\{\eta \in \mathbb{R}_+ : a_t(\eta) \neq 0\}}$ or $a_s = \frac{\mathrm{d}A_t}{\mathrm{dm}_{\sigma(a_s)}} \cdot \mathbb{I}_{\{\eta \in \mathbb{R}_+ : a_s(\eta) \neq 0\}}$, where A_t denotes the measure on $\mathbb{B}(\mathbb{R}_+)$ with density a_t and $\mathsf{m}_{\mathcal{B}}$ denotes the restriction of Lebesgue measure to a σ-field $\mathcal{B}$.

Then there exists an Ornstein–Uhlenbeck process $(\mathcal{OU}(t), t \in \mathbb{R}_+)$ such that

$$\left(\int_0^{+\infty} a_t(\eta)^2 \, \mathrm{d}\eta\right)^{-\frac{1}{2}} \int_0^{+\infty} a_t(\eta) \, \mathrm{d}W(\eta) = \mathcal{OU}\left(\log \int_0^{+\infty} a_t(\eta)^2 \, \mathrm{d}\eta\right) \text{ a.s. for each } t \in T.$$

Proof. According to Proposition 2.5, the considered functions fulfill the condition required in Proposition 4.5. Hence, the proof is done. □

Assuming restricted collections of functions, we can show reverse statements.

Proposition 4.7. Let $T \neq \emptyset$, $(W(t), t \in \mathbb{R}_+)$ be a Wiener process, $a \in \mathsf{L}(\mathbb{R}_+)$ and $\alpha_t \in \mathbb{R}$, $\alpha_t \neq 0$, $A_t \subset \mathbb{R}_+$ for each $t \in T$. Let

$$\text{either } \mathsf{m}((A_t - A_s) \cap \{\eta \in \mathbb{R}_+ : a(\eta) \neq 0\}) = 0$$
$$\text{or } \mathsf{m}((A_s - A_t) \cap \{\eta \in \mathbb{R}_+ : a(\eta) \neq 0\}) = 0 \text{ whenever } t, s \in T$$

and $0 < \int_{A_t} a(\eta)^2 \, \mathrm{d}\eta < +\infty$ for each $t \in T$.

Then there is an Ornstein–Uhlenbeck process $(\mathcal{OU}(t), t \in \mathbb{R}_+)$ and a function $\xi : T \to \mathbb{R}$ such that

$$\alpha_t \cdot \int_0^{+\infty} a(\eta) \mathbb{I}_{A_t}(\eta) \, \mathrm{d}W(\eta) = \mathcal{OU}(\xi(t)) \text{ a.s. for each } t \in T \tag{25}$$

if and only if

$$\alpha_t^2 \cdot \int_{A_t} a(\eta)^2 \, \mathrm{d}\eta = 1 \tag{26}$$

and there is a constant $c \in \mathbb{R}$ such that either

$$\left(\mathsf{m}((A_t - A_s) \cap \{\eta \in \mathbb{R}_+ : a(\eta) \neq 0\}) = 0 \iff \xi(t) \leq \xi(s)\right) \text{ for all } t, s \in T,$$
$$\xi(t) = \log \int_{A_t} a(\eta)^2 \, \mathrm{d}\eta + c \text{ for all } t \in T, \tag{27}$$

or

$$\left(\mathsf{m}((A_t - A_s) \cap \{\eta \in \mathbb{R}_+ : a(\eta) \neq 0\}) = 0 \iff \xi(t) \geq \xi(s)\right) \text{ for all } t, s \in T,$$
$$\xi(t) = -\log \int_{A_t} a(\eta)^2 \, \mathrm{d}\eta + c \text{ for all } t \in T, \tag{28}$$

Proof. According to Theorem 4.1, the expression (25) is equivalent to the property

$$\alpha_t \alpha_s \cdot \int_{A_t} a(\eta)^2 \, \mathrm{d}\eta = \mathrm{e}^{-|\xi(t) - \xi(s)|}$$

whenever $t, s \in T$ and $\mathsf{m}((A_t - A_s) \cap \{\eta \in \mathbb{R}_+ : a(\eta) \neq 0\}) = 0$.

Hence, we have the following observations:

1. $\alpha_t^2 \cdot \int_{A_t} a(\eta)^2 \, d\eta = 1$ for each $t \in T$.

2. For $t, s \in T$, $\mathsf{m}((A_t - A_s) \cap \{\eta \in \mathbb{R}_+ : a(\eta) \neq 0\}) = 0$, $\xi(t) \leq \xi(s)$ we receive

$$\xi(t) = \frac{1}{2} \cdot \log \int_{A_t} a(\eta)^2 \, d\eta + \xi(s) - \frac{1}{2} \cdot \log \int_{A_s} a(\eta)^2 \, d\eta.$$

3. For $t, s \in T$, $\mathsf{m}((A_t - A_s) \cap \{\eta \in \mathbb{R}_+ : a(\eta) \neq 0\}) = 0$, $\xi(t) \geq \xi(s)$ we receive

$$\xi(t) = -\frac{1}{2} \cdot \log \int_{A_t} a(\eta)^2 \, d\eta + \xi(s) + \frac{1}{2} \cdot \log \int_{A_s} a(\eta)^2 \, d\eta.$$

Therefore, $\xi(t) = \xi(s)$ and $\int_{A_t} a(\eta)^2 \, d\eta = \int_{A_s} a(\eta)^2 \, d\eta$ whenever $\alpha_t = \alpha_s$. Let us assume $t_0, s_0 \in T$ such that $\alpha_{t_0} \neq \alpha_{s_0}$.

1. If $\xi(t_0) - \frac{1}{2} \cdot \log \int_{A_{t_0}} a(\eta)^2 \, d\eta = \xi(s_0) - \frac{1}{2} \cdot \log \int_{A_{s_0}} a(\eta)^2 \, d\eta$ then $\xi(t) = \frac{1}{2} \cdot \log \int_{A_t} a(\eta)^2 \, d\eta + c$ for all $t \in T$ and some $c \in \mathbb{R}$.
   It is because assuming $t \in T$ without this property, we have
   $$\xi(t) + \frac{1}{2} \cdot \log \int_{A_t} a(\eta)^2 \, d\eta = \xi(t_0) + \frac{1}{2} \cdot \log \int_{A_{t_0}} a(\eta)^2 \, d\eta = \xi(s_0) + \frac{1}{2} \cdot \log \int_{A_{s_0}} a(\eta)^2 \, d\eta,$$
   which is a contradiction with $\alpha_{t_0} \neq \alpha_{s_0}$.

2. If $\xi(t_0) + \frac{1}{2} \cdot \log \int_{A_{t_0}} a(\eta)^2 \, d\eta = \xi(s_0) + \frac{1}{2} \cdot \log \int_{A_{s_0}} a(\eta)^2 \, d\eta$ then $\xi(t) = -\frac{1}{2} \cdot \log \int_{A_t} a(\eta)^2 \, d\eta + c$ for all $t \in T$ and some $c \in \mathbb{R}$.
   It is because assuming $t \in T$ without this property, we have
   $$\xi(t) - \frac{1}{2} \cdot \log \int_{A_t} a(\eta)^2 \, d\eta = \xi(t_0) - \frac{1}{2} \cdot \log \int_{A_{t_0}} a(\eta)^2 \, d\eta = \xi(s_0) - \frac{1}{2} \cdot \log \int_{A_{s_0}} a(\eta)^2 \, d\eta,$$
   which is a contradiction with $\alpha_{t_0} \neq \alpha_{s_0}$. □

The following proposition solves the problem of the rescaling of a Wiener process.

Theorem 4.2. Let $T \neq \emptyset$ and $(W(t), t \in \mathbb{R}_+)$ be a Wiener process. Let for each $t \in T$ constants $\alpha_t, \beta_t \in \mathbb{R}$, $\alpha_t \neq 0$, $\beta_t > 0$ be given.

Then there is an Ornstein–Uhlenbeck process $(\mathcal{OU}(t), t \in \mathbb{R}_+)$ and a function $\xi : T \to [0, 1]$ such that

$$\alpha_t W(\beta_t) = \mathcal{OU}(\xi(t)) \text{ a.s. for each } t \in T \tag{29}$$

if and only if

$$\alpha_t^2 \cdot \beta_t = 1$$

and there is a constant $c \in \mathbb{R}$ such that either

$$\Big(\beta_t \le \beta_s \iff \xi(t) \le \xi(s)\Big) \text{ for all } t, s \in T,\ \xi(t) = \log \beta_t + c \text{ for all } t \in T \qquad (30)$$

or

$$\Big(\beta_t \le \beta_s \iff \xi(t) \ge \xi(s)\Big) \text{ for all } t, s \in T,\ \xi(t) = -\log \beta_t + c \text{ for all } t \in T. \qquad (31)$$

Proof. One can easily show the statement setting $a \equiv 1$, $A_t = [0, \beta_t]$ and applying Proposition 4.7. □

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RNDr. Petr Lachout, CSc., Department of Probability and Statistics, Charles University, Sokolovská 83, 186 00 Praha 8 and Institute of Information Theory and Automation – Academy of Sciences of the Czech Republic, Pod vodárenskou věží 4, 182 08 Praha 8. Czech Republic.
e-mail: lachout@karlin.mff.cuni.cz