KYBERNETIKA — VOLUME 37 (2001), NUMBER 5, PAGES 575–583

SOLUTION FOR A CLASSICAL PROBLEM IN THE CALCULUS OF VARIATIONS VIA RATIONALIZED HAAR FUNCTIONS

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A numerical technique for solving the classical brachistochrone problem in the calculus of variations is presented. The brachistochrone problem is first formulated as a nonlinear optimal control problem. Application of this method results in the transformation of differential and integral expressions into some algebraic equations to which Newton-type methods can be applied. The method is general, and yields accurate results.

1. INTRODUCTION

There has been a considerable renewal of interest in the classical problems of the calculus of variations both from the point of view of mathematics and of applications in physics, engineering, and applied mathematics.

Finding the brachistochrone, or path of quickest decent, is a historically interesting problem that is discussed in virtually all textbooks dealing with the calculus of variations. In 1696, the brachistochrone problem was posed as a challenge to mathematicians by John Bernoulli. The solution of the brachistochrone problem is often cited as the origin of the calculus of variations as suggested in [19].

The classical brachistochrone problem deals with a mass moving along a smooth path in a uniform gravitational field. A mechanical analogy is the motion of a bead sliding down a frictionless wire. The solution to this problem was obtained by various methods such as the gradient method [3] and successive sweep algorithm in [1, 4].

Orthogonal functions have received considerable attention in dealing with various problems of dynamic systems. The main characteristic of this technique is that it reduces these problems to those of solving a system of algebraic equations thus greatly simplifying the problem. The approach is based on converting the underlying differential equations into an integral equations through integration, approximating a various signals involved in the equation by truncated orthogonal series $\phi(t) = [\phi_0, \phi_1, \cdots, \phi_{r-1}]^T$ and using the operational matrix of integration P, to eliminate the integral operations. The elements $\phi_0, \phi_1, \cdots, \phi_{r-1}$ are the basis functions, orthogonal on certain interval, and the matrix P can be uniquely determined

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based on the particular orthogonal functions. Special attention has been given to applications of Walsh functions [6], block-pulse functions [10], Laguerre series [9], Legendre polynomials [5], Chebyshev polynomials [8] and Fourier series [17].

The orthogonal set of Haar functions is a group of square waves with magnitude of $+2^{\frac{i}{2}}$, $-2^{\frac{i}{2}}$ and 0, $i = 0, 1, 2, \ldots$ [15]. The use of the Haar functions comes from the rapid convergence feature of Haar series in expansion of function compared with that of Walsh series [2]. Lynch et al [11] have rationalized the Haar transform by deleting the irrational numbers and introducing the integral powers of two. This modification results in what is called the rationalized Haar (RH) transform. The RH transform preserves all the properties of the original Haar transform and can be efficiently implemented using digital pipeline architecture [18]. The corresponding functions are known as RH functions. The RH functions are composed of only three amplitude +1, −1 and 0. Further, Ohkita and Kobayayashi [12, 13] applied RH functions to solve linear ordinary differential equations [12] and linear first and second order partial differential equations [13].

In the present paper we apply RH functions to solve the brachistochrone problem. The brachistochrone problem is first formulated as an optimal control problem. The method consist of reducing the optimal control problem into a set of algebraic equations by expanding the state rate $\dot{x}(t)$ as RH functions with unknown coefficients. The operational matrix of integration is then used to evaluate the coefficients of RH functions in such a way that the necessary conditions for extremization is imposed. The paper is organized as follows: In Section 2 we describe the properties of the RH functions required for our subsequent development. Section 3 is devoted to the formulation of the brachistochrone problem as an optimal control problem. In Sections 4 and 5 the proposed method is used to approximation the brachistochrone problem. In Section 6, we report our numerical finding and demonstrate the accuracy of the proposed numerical scheme by considering the illustrative example discussed in [1, 3, 4].

2. PROPERTIES OF RATIONALIZED HAAR FUNCTIONS

2.1. Rationalized Haar functions

The RH functions $\mathrm{RH}(r,t)$, $r = 1, 2, 3, \cdots$ are composed of three values +1, −1 and 0 and can be defined on the interval $[0, 1)$ as [12]

$$\mathrm{RH}(r,t) = \begin{cases} 1, & J_1 \le t < J_{\frac{1}{2}} \\ -1, & J_{\frac{1}{2}} \le t < J_0 \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where

$$J_u = \frac{j-u}{2^i}, \quad u = 0, \frac{1}{2}, 1.$$

The value of r is defined by two parameters i and j as

$$r = 2^i + j - 1, \quad i = 0, 1, 2, 3, \ldots \quad j = 1, 2, 3, \ldots, 2^i.$$

RH$(0,t)$ is defined for $i = j = 0$ and is given by

$$\mathrm{RH}(0,t) = 1, \qquad 0 \le t < 1. \tag{2}$$

A set of the first eight RH functions is shown in Figures 1 – 8, where, $r = 0, 1, 2, \ldots, 7$. The orthogonality property is given by

$$\int_0^1 \mathrm{RH}(r,t)\,\mathrm{RH}(v,t)\,\mathrm{d}t = \begin{cases} 2^{-i}, & \text{for} \quad r = v \\ 0 & \text{for} \quad r \neq v \end{cases}$$

where

$$v = 2^n + m - 1, \quad n = 0, 1, 2, \ldots, \quad m = 1, 2, \ldots, 2^n.$$

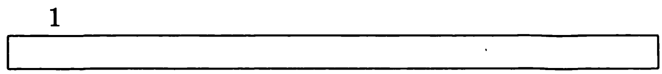

Fig. 1. RH$(0,t)$ obtained for $i = 0$ and $j = 0$.

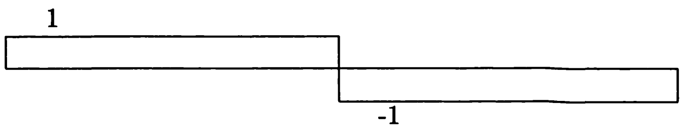

Fig. 2. RH$(1,t)$ obtained for $i = 0$ and $j = 1$.

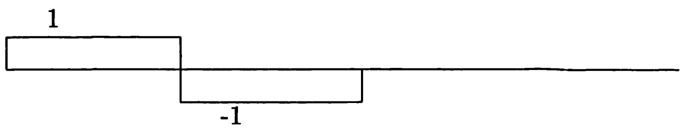

Fig. 3 RH$(2,t)$ obtained for $i = 1$ and $j = 1$.

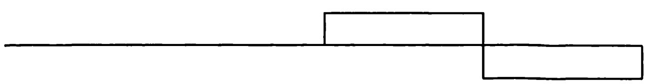

Fig. 4. RH$(3,t)$ obtained for $i = 1$ and $j = 2$.

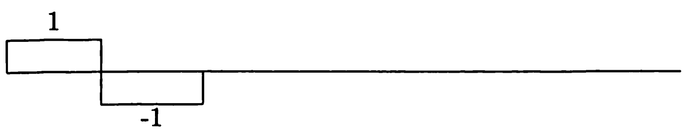

Fig. 5. RH$(4,t)$ obtained for $i = 2$ and $j = 1$.

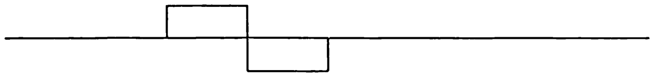

Fig. 6. RH$(5,t)$ obtained for $i=2$ and $j=2$.

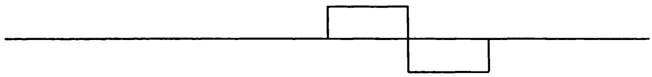

Fig. 7. RH$(6,t)$ obtained for $i=2$ and $j=3$.

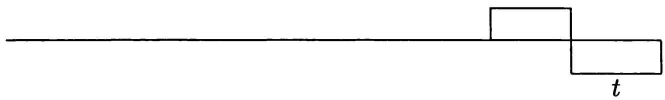

Fig. 8. RH$(7,t)$ obtained for $i=2$ and $j=4$.

2.2. Function approximation

A function $f(t)$ defined over the interval $[0,1)$ may be expanded in RH functions as

$$f(t)=\sum_{r=0}^{+\infty} a_r\,\mathrm{RH}(r,t), \tag{3}$$

where a_r, $r=0,1,2,\cdots$ is given by

$$a_r=2^i\int_0^1 f(t)\,\mathrm{RH}(r,t)\,\mathrm{d}t,$$

with

$$r=2^i+j-1,\quad i=0,1,2,3,\ldots\quad j=1,2,3,\ldots,2^i \text{ and } \quad r=0 \text{ for } i=j=0. \tag{4}$$

The series in Eq. (3) contains an infinite number of terms. If we let $i=0,1,2,\ldots,\alpha$ then the infinite series in Eq. (3) is truncated up to its first k terms as

$$f(t)=\sum_{r=0}^{k-1} a_r\,\mathrm{RH}(r,t)=A^T\phi(t), \tag{5}$$

where

$$k=2^{\alpha+1},\quad \alpha=0,1,2,\ldots.$$

The RH functions coefficient vector A and RH functions vector $\phi(t)$ are defined as

$$A = [a_0,a_1,\ldots,a_{k-1}]^T \tag{6}$$

$$\phi(t) = [\phi_0(t),\phi_1(t),\ldots,\phi_{k-1}(t)]^T \tag{7}$$

where

$$\phi_r(t) = \mathrm{RH}(r,t), \quad r = 0,1,2,\cdots,k-1.$$

If each waveform is divided into eight intervals, the magnitude of the waveform can be represented as

$$\hat{\Phi}_{8\times 8} = \begin{bmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \phi_7 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}. \tag{8}$$

In Eq. (8) the row denotes the order of the Haar function. The matrix $\hat{\Phi}_{k\times k}$ can be expressed as

$$\hat{\Phi}_{k\times k} = [\phi(1/2k), \phi(3/2k), \ldots, \phi((2k-1)/2k)]. \tag{9}$$

2.3. Operational matrix of integration

The integration of the function $\phi(t)$ defined in Eq. (7) is given by

$$\int_0^t \phi(t')\,\mathrm{d}t' = P\phi(t) \tag{10}$$

where $P = P_{k\times k}$ is the $k \times k$ operational matrix for integration and is given in [12] as

$$P_{k\times k} = \frac{1}{2k}\begin{bmatrix} 2kP_{\frac{k}{2}\times\frac{k}{2}} & -\hat{\Phi}_{\frac{k}{2}\times\frac{k}{2}} \\ \hat{\Phi}^{-1}_{\frac{k}{2}\times\frac{k}{2}} & 0 \end{bmatrix}$$

where $\hat{\Phi}_{1\times 1} = [1]$, $P_{1\times 1} = [\frac{1}{2}]$, and

$$\hat{\Phi}^{-1}_{k\times k} = \left(\frac{1}{k}\right)\hat{\Phi}^T_{k\times k}\,\mathrm{diag}\left(1,1,2,2,\underbrace{2^2,\ldots,2^2}_{2^2},\underbrace{2^3,\ldots,2^3}_{2^3},\ldots,\underbrace{\frac{k}{2},\ldots,\frac{k}{2}}_{\frac{k}{2}}\right).$$

3. THE BRACHISTOCHRONE PROBLEM AS AN OPTIMAL CONTROL PROBLEM

The brachistochrone problem may be formulated as an optimal control problem as in [7]. Minimize the performance index J,

$$J = \int_0^1 \left[\frac{1+u^2(t)}{1-x(t)}\right]^{\frac{1}{2}} \mathrm{d}t \tag{11}$$

subject to

$$\dot{x}(t) = u(t) \tag{12}$$

with

$$x(0) = 0, \quad x(1) = -0.5 \tag{13}$$

Equations (11) – (13) describe the motion of a bead sliding down a frictionless wire between two points (0, 0) and (1, −0.5) in a constant gravitational field. It is desired to find the shape of the wire that will produce the minimum time pass between the two points.

The minimal time transfer expression in Eq. (11) is obtained from the law of conservation of energy. Here $x(t)$ and $u(t)$ are dimensionless and they represent respectively the vertical and horizontal coordinates of the sliding bead.

As is well known the exact solution to the brachistochrone problem is the cycloid defined by the parametric equations.

$$x = 1 - \frac{c}{2}(1 + \cos 2\alpha), \quad t = t_0 + c(2\alpha + \sin 2\alpha)$$

where

$$\operatorname{tag} \alpha = \dot{x}$$

with the given boundary conditions, the integration constants are found to be

$$c = 1.6184891, \quad t_0 = 2.7500631.$$

4. THE NUMERICAL METHOD

Suppose, the rate variable $\dot{x}(t)$ can be expressed approximately as

$$\dot{x}(t) = A^T \phi(t) \tag{14}$$

using Eqs. (10) and (14), $x(t)$ can be represented as

$$\begin{aligned} x(t) &= \int_0^t \dot{x}(t')\mathrm{d}t' + x(0) \\ &= A^T P \phi(t), \end{aligned} \tag{15}$$

also by using Eqs. (12) and (14) we have

$$u(t) = A^T \phi(t). \tag{16}$$

To apply Eqs. (15) and (16), we first collocate these equations in k points. Suitable collocation points are Newton–Cotes nodes given in [14] as

$$t_p = \frac{2p-1}{2k}, \quad p = 1, 2, 3, \ldots, k. \tag{17}$$

By using Eqs. (7) and (17) we get

$$\phi(t_p) = \hat{\Phi}_{k\times k} e_p \quad , \quad p = 1, 2, \ldots, k, \tag{18}$$

where $\hat{\Phi}_{k\times k}$ can be obtained similarly to $\hat{\Phi}_{8\times 8}$ in Eq. (8) and

$$e_p = [\underbrace{0, 0, \ldots, 0}_{p-1}, 1, \underbrace{0, \ldots, 0}_{k-p}]^T.$$

Next we express Eqs. (15) and (16) at collocation points as

$$x_p = x(t_p) = A^T P \phi(t_p) = A^T P \hat{\Phi}_{k\times k} e_p \quad p = 1, 2, \ldots, k \tag{19}$$

$$u_p = u(t_p) = A^T \hat{\Phi}_{k\times k} e_p, \quad p = 1, 2, \ldots, k. \tag{20}$$

5. THE PERFORMANCE INDEX APPROXIMATION

Let

$$g\big(x(t), u(t), t\big) = \left(\frac{1 + u^2(t)}{1 - x(t)}\right)^{\frac{1}{2}}. \tag{21}$$

Substituting Eqs. (15) and (16) in Eq. (21) we get

$$g\big(A^T P \phi(t), A^T \phi(t), t\big) = \left(\frac{1 + (A^T \phi(t))^2}{1 - A^T P \phi(t)}\right)^{\frac{1}{2}}. \tag{22}$$

At the collocation points t_p, $p = 1, 2, 3, \ldots, k$, Eq. (22) reduces to

$$g\big(x_p, u_p, t_p\big) = \left(\frac{1 + u_p^2}{1 - x_p}\right)^{\frac{1}{2}}.$$

Finally, the performance index in Eq. (11) can be written as:

$$\begin{aligned} J &= \int_0^1 g\big(x(t), u(t), t\big)\, \mathrm{d}t = \int_0^1 g\left(A^T P \phi(t), A^T \phi(t), t\right)\, \mathrm{d}t \\ &= \sum_{p=1}^{k} \omega_p \left(\frac{1 + u_p^2}{1 - x_p}\right)^{\frac{1}{2}} \end{aligned} \tag{23}$$

where ω_p are the corresponding weights, given by [14]

$$\omega_j = \frac{1}{k} \quad , \quad p = 1, 2, \ldots, k.$$

6. EVALUATING THE VECTOR A

The optimal control problem has been reduced to a parameter optimization problem which can be stated as follows:

Find a_r, $r = 0, 1, 2, \ldots, k-1$, which minimizes Eq. (23) subject to

$$x(0) = A^T P \hat{\Phi}_{k\times k} e_1 = 0, \quad x(1) = A^T P \hat{\Phi}_{k\times k} e_k = -0.5. \tag{24}$$

We now minimize Eq. (23) subject Eq. (24) using the Lagrange multiplier technique. Suppose

$$J^* = J + \lambda_1 x(0) + \lambda_2 [x(1) + 0.5].$$

The necessary conditions for minimum are

$$\frac{\partial J^*}{\partial a_r} = 0, \quad \frac{\partial J^*}{\partial \lambda_1} = 0, \quad \frac{\partial J^*}{\partial \lambda_2} = 0, \quad n = 0, 1, 2, \cdots, k-1,$$

which gives $(k+2)$ non-linear equations which can be solved for a_r, λ_1 and λ_2 using Newton's iterative method. The initial values required to start Newton's iterative method have been chosen by taking $x(t)$ as linear function between the initial value $x(0) = 0$ and final value $x(1) = -0.5$. In Table 1 the results for RH functions approximation with $k = 4$, 8 and 16 together with gradient [3], successive sweep [1, 4] methods and exact are listed.

Table 1. The RH functions and other solutions.

Methods	$x(1)$	$u(-1)$	J
Gradient method [3]	-0.5	-0.7832273	0.9984988
Successive sweep method [1, 4]	-0.5	-0.7834292	0.9984989
RH functions			
$k = 4$	-0.51	-0.7864563	0.9984684
$k = 8$	-0.5	-0.7864415	0.9984973
$k = 16$	-0.5	-0.7864407	0.9984982
Exact Solution [4]	-0.5	-0.7864408	0.9984981

7. CONCLUSION

The operational matrix of integration of the RH functions together with the Newton–Cotes nodes are applied to solve the brachistochrone problem. The matrices $\hat{\Phi}_{k\times k}$ and P introduced in Eqs. (9) and (10) contain many zeros, and thus make the rationalized Haar functions computationally very attractive.

(Received October 6, 2000.)

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