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ON NONLINEAR EQUIVALENCE AND BACKSTEPPING OBSERVER*

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An observer design based on backstepping approach for a class of state affine systems is proposed. This class of nonlinear systems is determined via a constructive algorithm applied to a general nonlinear Multi Input–Multi Output systems. Some examples are given in order to illustrate the proposed methodology.

1. INTRODUCTION

It is well-known that when a state control law is designed its application is limited if the components of the state vector are not all measurable. This problem can be overcome by using observers. For linear systems, it is traditionally solved by using either a Luenberger observer or Kalman-filter. Moreover, the observability property for linear systems does not depend on the input. However, the observability property of nonlinear systems does depend on the input. There are some inputs for which the system could become unobservable (for more details see [1, 8, 10]). Hence, the inputs which render the system unobservable should be considered when observer is constructed. For these reasons, the observer problem for nonlinear systems remains an interesting field of research. Although the problem of observer synthesis for linear systems is solved, no general methodology exists for the observer design for nonlinear systems. However, some results have been obtained in this direction ([8, 10, 12, 13, 16, 18, 20]), where the observer design has been investigated for a class of nonlinear system which can be transformed into another observable form.

Several authors (see for instances [13, 14]) have considered the case when a nonlinear system can be transformed into a linear system up to input-output injection. On the other hand, a straightforward approach verifying and computing the linearization condition for those systems have been given in ([15, 17]).

The design of an observer for a class of nonlinear systems can be solved via a change of coordinates which transforms the system into another nonlinear system for which an observer can be constructed (see [10, 14, 20]). Some results related to

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the coordinate transformation of a nonlinear system into a state affine systems have been obtained (see for instances [1, 8, 10, 14, 18]). The design of an observer for these state affine systems has been studied in [3].

Furthermore, necessary and sufficient conditions transforming a nonlinear system into a state affine system has been proposed in [2, 10]. However, no construction procedure characterizing such systems exits so far for multi-input-multi-output case. On the other hand, a constructive methodology for the single output case, computing the change of coordinates, is presented in [14].

This paper deals with the observer synthesis of nonlinear systems via their equivalence to state affine systems. Necessary and sufficient conditions are given to characterize a class of nonlinear systems, which can be transformed into a class of multivariable state affine systems up to input-output injection. Furthermore, for the class of state affine systems an observer is designed using a backstepping observer approach.

The paper is organized as follows. In Section 3, a computation algorithm is described which allows the transformation of a nonlinear system into a multi-output affine system. In Section 4, the unmeasurable components of the vector state are estimated using a backstepping observer. For this observer, conditions are given to characterize the inputs which render the system observable. In Section 5, some examples illustrating the proposed methodology are given. Finaily, some conclusions are given.

2. PRELIMINARIES

Now, consider the following nonlinear system

$$\Sigma : \begin{cases} \dot{x} = f(x,u) \\ y = h(x) \end{cases} \tag{1}$$

where $x \in I\!R^n$ is the state, $u \in I\!R^m$ is the input, $y \in I\!R^p$ is the controlled output, f and h are meromorphic functions of their arguments. Assume that there exists a change of coordinates transforming Σ into the state affine system of the form

$$\Sigma_{\text{affine}} : \begin{cases} \dot{z}_i &= A_i(u,y)z_i + \phi_i(u,y) \\ y_i &= C_i z_i, \quad i = 1, \dots, p, \end{cases} \tag{2}$$

where $z_i = \text{col}\,(z_{i,1}, \dots, z_{i,k_i})$, $A_i \in I\!R^{k_i \times k_i}$ are matrices of the form

$$A_i = \begin{pmatrix} 0 & a_{i,1}(u) & 0 & \dots & 0 \\ 0 & 0 & a_{i,2}(u,y) & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & a_{i,k_i-1}(u,y) \\ 0 & 0 & \dots & \dots & 0 \end{pmatrix} \tag{3}$$

$$\phi_i = \begin{pmatrix} \varphi_{i,1} \\ \vdots \\ \varphi_{i,k_i} \end{pmatrix}; \text{ and } C_i = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}_{1\times k_i}; \; i = 1,\dots,p,$$

where the k_i denote observability index related with the output y_i,which are ordered as $k_1 \geq k_2 \geq \ldots \geq k_p$ and $\sum_{i=1}^{p} k_i = n$.

Remark 1. In order to simplify the notation and without loss of generality, the outputs are reordered in function of the observability indices; i. e. the output y_i is associated to the index observability k_i, for $i = 1,\dots,p$.

All definitions and results given in the paper can be written locally around a regular point x_0 of M, an open subset of $I\!R^n$. If this property is generically satisfied, it means that this property is satisfied locally around a regular point x_0 of M. Let $\mathcal{O}$ denote the generic observability space defined by (see [16]).

$$\mathcal{O} = \mathcal{X} \cap (\mathcal{Y} + \mathcal{U}) \tag{4}$$

where $\mathcal{X} = \text{Span}_{\mathcal{K}}\{\mathrm{d}x\}$, $\mathcal{Y} = \text{Span}_{\mathcal{K}}\{\mathrm{d}y^{(w)}, w \geq 0\}$, $\mathcal{U} = \text{Span}_{\mathcal{K}}\{\mathrm{d}u^{(w)}, w \geq 0\}$, ($\text{Span}_{\mathcal{K}}$ is a space spanned over the field $\mathcal{X}$ of meromorphic functions of x and a finite number of time derivatives of u).

Definition 1. The system Σ is generically observable if

$$\dim \mathcal{O} = n.$$

The first goal of this paper is to find a state coordinate transformation $z = \Phi(x)$, such that system Σ is locally equivalent to system Σ_{affine} in order to design an observer. The approach consists in checking that the Input-Output (I/O) differential equation associated to the observable system Σ, which is given by

$$y_i^{(k_i)} = P_0^i(y_1, \dot{y}_1, \dots y_1^{(k_1-1)}, \dots, y_p, \dots, y_p^{(k_p-1)}, u, \dot{u}, \ddot{u}, \dots, u^{(k_1-1)}), \tag{5}$$

has the same I/O differential equation as Σ_{affine}, which verifies

$$\begin{aligned} y_i^{(k_i)} = P_{a0}^i &= F_{k_i}^i(a_{i,1},\dots,a_{i,n-1}) \\ &+ \sum_{r=1}^{k_i-1} K_{k_i-r-1}^i F_r^i(a_{i,k_i-r},\dots,a_{i,k_i-1},\varphi_{i,k_i-r}) + K_{k_i-1}^i F_0^i(\varphi_{i,k_i}) \\ &= F_{k_i}^i(a_{i,1},\dots,a_{i,n-1}) + \Gamma_0^{k_i-1}(a_{i,1},\dots,a_{i,k_i-1},\varphi_{i,1},\dots,\varphi_{i,k_i}) \end{aligned} \tag{6}$$

where $K_r^i = a_{i,0}\dots a_{i,r} = \prod_{j=0}^{r} a_{i,j}$, and $a_{i,0} = 1$. The functions F_r^i, $r = 0,\dots,k_i$; are given as a sum of monomials depending on

$$\left(y_i^{(n_i)}\right)^{q_i} \text{ and } \left(u_i^{(m_i)}\right)^{s_i}, \text{ for } i = 1,\dots,p;$$

where n_i, $m_i = 0, \ldots, k_i$; represent the order of derivation of the outputs and the inputs respectively; and $q_i, s_i = 0, 1, \ldots$; are the exponents of the outputs and the inputs and their derivatives, respectively. These parameters satisfy the following relation

$$\sum_i n_i q_i + \sum_i m_i s_i = r; \text{ for } i = 1, \ldots, p.$$

Remark 2. The functions F_r^i involves monomials depending on functions $\left(y_i^{(n_i)}\right)^{q_i}$ and $\left(u_i^{(m_i)}\right)^{s_i}$ of degree $\sum_i n_i q_i + \sum_i m_i s_i = (k_i - r)$.

On the other hand, the proposed results are obtained from the analysis of I/O differential equations. The observable nonlinear system Σ in the state space representation will be transformed into a set of higher-order differential equations depending on the inputs and outputs. These equations are obtained by using state elimination techniques (see [5]). Moreover, considering the assumption of generic observability of the system, the elimination problem has a solution (see [15, 19]). Hence, the state affine transformation problem is solved as a realization problem.

The classification problem of nonlinear systems which can be steered by a change of coordinates to some observable form has received significant attention during the last years. In [7] and [8], locally uniformly observable systems are studied. Necessary and sufficient conditions have been stated to guarantee the transformation of nonlinear systems into state affine systems (see [1, 10, 11]). These conditions guarantee the existence of a vector field transforming the system into another observable one. However, this vector field cannot be computed directly and hence, the application of this methodology is limited (see [1]). On the other hand, a constructive methodology for the single output case, computing the change of coordinates, is presented in [14]. In this paper, using the results given in [14], an extension for the class of multivariable systems will be considered.

3. STATE AFFINE TRANSFORMATION ALGORITHM

The problem of verifying the equivalence between a nonlinear system and state affine system is considered in this section. Necessary and sufficient conditions allowing to characterize a class of nonlinear systems, which are diffeomorphic to state affine systems, are given. These conditions are obtained using the exterior differential system theory (for more details see [4, 9, 14, 16]).

Now, the algorithm allowing us to know if a diffeomorphism exists between (1) and (2) is given. Let $S_j^i = \{k_1, k_2, \ldots, k_j\}$ be the set of observability indices such that k_j satisfies the following inequality

$$k_j > k_i - k$$

for a given k. Denote d_i^k the number of outputs whose observability index is greater than $k_i - k$, as

$$d_i^k = \text{Card}\{k_1, k_2, \ldots, k_j\}. \tag{7}$$

Algorithm.

Step 1. Computation of the functions $a_{i,j}$.

Let $P_0^i = y_i^{(k_i)}$, $i = 1, \dots, p$; be the I/O differential equation obtained from the nonlinear system Σ. Let ω_k^i be the one-form defined by

$$\omega_k^i = c_k^i \sum_{j=1}^{d_i^k} \frac{\partial^2 P_0^i}{\partial y_j^{(k)} \partial y_j^{(k_i-k)}} \mathrm{d}y_j + \sum_{j=1}^{d_i^k} \sum_{l=1}^{m} \frac{\partial^2 P_0^i}{\partial u_l^{(k)} \partial y_j^{(k_i-k)}} \mathrm{d}u_l \tag{8}$$

for $k = 1, \dots, k_i - 1$; with $c_1^i = \dots = c_{k_i-2}^i = 1$ and $c_{k_i-1}^i = 0$. Now, in order to verify if it is possible to find an equivalence between Σ and Σ_{affine}, it is necessary to check the following conditions:

— Case $d_i^k < p$.

If $\mathrm{d}\omega_k^i \wedge \mathrm{d}u \neq 0$ or $\mathrm{d}\omega_k^i \wedge \mathrm{d}y_{d_i^k+1} \wedge \cdots \wedge \mathrm{d}y_p \neq 0$; then, there is no solution.

— Case $d_i^k = p$:

If $\mathrm{d}\omega_k^i \neq 0$, then the problem has no solution.

Otherwise, let the $a_{i,k}$ functions be any solution of

$$\omega_k^i = c_k^i \sum_{j=1}^{d_i^k} \frac{\partial^2 P_{a0}^i}{\partial y_j^{(k)} \partial y_j^{(k_i-k)}} \mathrm{d}y_j + \sum_{j=1}^{d_i^k} \sum_{l=1}^{m} \frac{\partial^2 P_{a0}^i}{\partial u_l^{(k)} \partial y_j^{(k_i-k)}} \mathrm{d}u_l \tag{9}$$

where the right-hand side of this equation is deduced from the I/O differential equation P_{a0}^i, which is computed from system Σ_{affine}.

This ends the Step 1.

On the other hand, the previous one-forms do not allow to know the functions $\varphi_{i,k}$. Then, in order to identify the functions $\varphi_{i,j}$, all $a_{i,j}$ obtained from Step 1 will be used to determine the $\varphi_{i,j}$, as it is presented in the next step.

Step 2. Determination of φ_{i,k_i}.

Consider P_0^i as in Step 1, and let

$$P_r^i = P_{r-1}^i - F_{k_i-r+1}^i, \tag{10}$$

for $r := 1, \dots, k_i - 1$; where the $F_{k_i-r+1}^i$ are functions as in (6). Let ϖ_r^i the one-form given by

$$\varpi_r^i = \frac{1}{K_r^i} \left\{ \sum_{j=1}^{d_i^r} \frac{\partial P_r^i}{\partial y_j^{(k_i-r)}} \mathrm{d}y_j + \sum_{l=1}^{m} \frac{\partial P_r^i}{\partial u_l^{(k_i-r)}} \mathrm{d}u_l \right\} \tag{11}$$

where

$$K_r^i = a_{i,1} \dots a_{i,r} = \prod_{j=0}^{r} a_{i,j},$$

and $a_{i,0} = 1$. Now, in order to compute the functions $\varphi_{i,r}$, we check the following conditions:

— Case $d_i^r < p$.

If $\mathrm{d}\overline{\omega}_r^i \wedge \mathrm{d}u \neq 0$ or $\mathrm{d}\overline{\omega}_r^i \wedge \mathrm{d}y_{d_i^r+1} \wedge \cdots \wedge \mathrm{d}y_p \neq 0$, then, the problem has no solution.

— Case $d_i^r = p$.

If $\mathrm{d}\overline{\omega}_r^i \neq 0$, then the problem has no solution.
Otherwise, if $\mathrm{d}\overline{\omega}_r^i = 0$, for $\forall\, r = 1, \dots, k_i - 1$; then $\varphi_{i,r}$ is a solution of

$$\overline{\omega}_r^i = \frac{1}{a_{i,r}} \left\{ \sum_{j=1}^{d_i^r} \frac{\partial \varphi_{i,r}}{\partial y_j} \mathrm{d}y_j + \sum_{j=1}^{m} \frac{\partial \varphi_{i,r}}{\partial u_j} \mathrm{d}u_j - \frac{\varphi_{i,r}}{a_{i,r}} \left(\sum_{j=1}^{d_i^r} \frac{\partial a_{i,r}}{\partial y_j} \mathrm{d}y_j + \sum_{j=1}^{m} \frac{\partial a_{i,r}}{\partial u_j} \mathrm{d}u_j \right) \right\}. \tag{12}$$

And for $r = k_i$,

$$P_{k_i}^i = a_{i,1} \dots a_{i,k_i-1} \varphi_{i,k_i} = K_{k_i}^i \varphi_{i,k_i}. \tag{13}$$

End of the Algorithm.

This Algorithm allows to establish the following theorem.

Theorem 1. The system Σ is locally equivalent by state coordinates transformation to the system Σ_{affine} if and only if the following conditions are verified:

1. For $d_i^k < p$,

$$\begin{aligned} &\mathrm{d}\omega_k^i \wedge \mathrm{d}u = 0, \text{ and } \mathrm{d}\omega_k^i \wedge \mathrm{d}y_{d_i^k+1} \wedge \cdots \wedge \mathrm{d}y_p = 0, \\ &\mathrm{d}\overline{\omega}_k^i \wedge \mathrm{d}u = 0, \text{ and } \mathrm{d}\overline{\omega}_k^i \wedge \mathrm{d}y_{d_i^k+1} \wedge \cdots \wedge \mathrm{d}y_p = 0. \end{aligned} \tag{14}$$

2. For $d_i^k = p$,

$$\mathrm{d}\omega_k^i = 0, \qquad \text{and} \qquad \mathrm{d}\overline{\omega}_k^i = 0;$$

where ω_k^i and $\overline{\omega}_k^i$are one-forms defined in (8) and (11).

If the conditions of Theorem 1 are satisfied, system Σ is locally equivalent to system Σ_{affine}, and the state coordinates transformation $z = \Phi(x)$ is given by

$$\begin{aligned} z_{i,1} &= y_i \\ z_{i,2} &= \frac{1}{a_{i,1}} \{\dot{y}_i(x) - \varphi_{i,1}(u,y)\} \\ z_{i,j} &= \frac{y_i^{(j-1)} - P_{j-1}^i}{K_{j-1}^i}, \text{ for } j = 3, \dots, k_i \end{aligned} \tag{15}$$

where $z_i = \mathrm{col}(z_{i,1} \dots z_{i,k_i})$ and

$$P_k^i = K_{k-1}^i \varphi_{i,k} + \frac{\mathrm{d}P_{k-1}^i}{\mathrm{d}t} + z_{i,k} \frac{\mathrm{d}K_{k-1}^i}{\mathrm{d}t} \tag{16}$$

for $k = 1, \dots, k_i$, $a_{i,k_i} = 0$ and $P_1^i = \varphi_{i,1}$.

Proof of Theorem 1 (see Appendix B).

This result gives the conditions to transform system Σ into system Σ_{affine} (2). The next section introduces a procedure to design a backstepping observer for this class of systems.

4. BACKSTEPPING OBSERVER

The propose of this section is to design an observer for the class of state affine systems (2) based on the backstepping approach. From the structure of the state affine system, which is represented by state affine subsystems, an observer will be designed for each subsystem independently. For this reason, consider the following class of single output state affine systems which are in the observable form

$$\begin{aligned}
\dot{x}_1 &= a_1(u,y)x_2 + g_1(u,x_1) \\
\dot{x}_i &= a_i(u,y)\, x_{i+1} + g_i(u,x_1,\dots,x_i), \quad i = 2,\dots,n-1; \\
\dot{x}_n &= f_n(x) + g_n(u,x), \\
y &= Cx = x_1.
\end{aligned} \tag{17}$$

It is clear that system (17) is uniformly observable if the applied inputs are persistently exciting. For instance, there are some inputs which render the unmeasured states unobservable. Then, in order to design an observer for the unmeasured states the inputs must be satisfy some observability conditions (see [11]).

The observer for the class of systems considered is described by

$$\begin{aligned}
\dot{z}_1 &= a_1(u,y)\, z_2 + g_1(u,z_1) + \psi_1(z)(x_1 - z_1) \\
\dot{z}_i &= a_i(u,y)\, z_{i+1} + g_i(u,z_1,z_2,\dots,z_i) + \psi_i(z)(x_1 - z_1), \\
&\qquad \text{for } i = 2,\dots,n-1 \\
\dot{z}_n &= f_n(z) + g_n(u,z) + \psi_n(z)(x_1 - z_1)
\end{aligned} \tag{18}$$

where $z = \mathrm{col}(z_1, z_2, \dots, z_n)$ is the estimated state and $\psi_i(z)$, $i = 2, \dots, n-1$; are the observer gains which must be determined in order to guarantee the convergence of the observer. Defining the estimation error $e_i = x_i - z_i$, for $i = 1, \dots, n$; whose dynamics is given by

$$\begin{aligned}
\dot{e}_1 &= a_1(u,y)e_2 - \psi_1(z)\, e_1 \\
\dot{e}_i &= a_i(u,y)\, e_{i+1} + g_i(u,x_1,\dots,x_i) - g_i(u,z_1,z_2,\dots,z_i) - \psi_i(z)\, e_1, \\
&\qquad \text{for } i = 2,\dots,n-1 \\
\dot{e}_n &= f_n(x) - f_n(z) + g_n(u,x) - g_n(u,z) - \psi_n(z)\, e_1.
\end{aligned} \tag{19}$$

Using similar arguments given in [12], we will find the observer gains $\psi_i(z), i = 1, \dots, n$, such that the estimation error tends to zero as $t \to \infty$. Now, in order to design the observer the following assumptions are introduced.

A1) *There exist positive constants c_1 and c_2, where $0 < c_1 < c_2 < \infty$, such that for all $x \in I\!R^n$;*

$$0 < c_1 \leq |a_i(u, y)| \leq c_2 < \infty, \qquad i = 1, \dots, n-1$$

A2) *The functions $g_i(u, y, \dots, x_i)$, $i = 2, \dots, n$, are globally Lipschitz with respect to $(x_1, \dots, x_i)$, and uniformly with respect to u and y.*

Remark 3. The condition (20) corresponds to a characterization of "good" inputs, which are required to recover state observability.

Let be $O(e)^k$ a function of z and e for $k > 0$ such that for $z \in \Xi \subset I\!R^n$, there exist constants $N > 0$, $\epsilon > 0$ such that

$$\left|O(e)^k\right| \leq N \left\|e\right\|^k, \quad \forall \left\|e\right\| < \epsilon, \quad \forall z \in \Xi.$$

Now, consider the following variables s_i for $i = 1, \dots, n+1$;

$$\begin{aligned} s_1 &= e_1 \\ s_2 &= c_1 s_1 + \dot{s}_1 + O(e)^2 \\ s_i &= s_{i-2} + c_{i-1} s_{i-1} + \dot{s}_{i-1} + O(e)^2, \text{ for } i = 3, \dots, n+1, \end{aligned} \tag{20}$$

where the parameters c_i are positive constants and the error terms are chosen so that s is a linear function of the error e. Next, writing the above equations in terms of the error e, we obtain

$$s_{l+1} = \sum_{i=1}^{l} \left(b_{l+1,i} - K_{l-i} K_{i-1} \psi_{l-i+1}\right) e_i + K_l e_{l+1}, \text{ for } l = 1, \dots, n-1 \tag{21}$$

and for $l = n$,

$$s_{n+1} = \sum_{i=1}^{n} \left(b_{n+1,i} - K_{n-i} K_{i-1} \psi_{n-i+1} + K_{n-1} \left(\frac{\partial f_n}{\partial z_i}\right)\right) e_i \tag{22}$$

where $b_{l+1,i}$ and K_{i-1} for $i = 1, \dots, l$; and $l = 1, \dots, n$; are given in Appendix C. Furthermore, let U_ρ be the ρ-neighborhood of $\mathcal{C}$ an open subset of $I\!R^n$, there exists constants $\lambda_1 > 0$ and $\lambda_2 > 0$ such that for all $z \in \bar{U}_\rho$, a compact subset,with e and $s \in \mathcal{C}$, the following inequality is satisfied

$$\lambda_1 \left\|e\right\| \leq \left\|s\right\| \leq \lambda_2 \left\|e\right\|, \tag{23}$$

where $s = \text{col}\,(s_1, s_2, \dots, s_n)$ and $e = \text{col}\,(e_1, e_2, \dots, e_n)$. Then we can establish the following result.

Theorem 2. Consider the system (17), and assume that assumptions A1 and A2 are satisfied. For any subset $\mathcal{C} \subset I\!R^n$ of the dynamical system (17) there exist constants $\lambda_1, \lambda_2 > 0$; $\epsilon > 0$; $\gamma > 0$ such that if $x(0) \in \mathcal{C}$ and $\|e(0)\| < \epsilon$ then the system (18) is a locally exponential observer for system (17). Thus, the estimation error

$$\|e(t)\| \leq \frac{\lambda_2}{\lambda_1} \|e(0)\| \exp^{-2\gamma t}$$

converges exponentially to zero as t tends to ∞.

Proof. Defining the following Lyapunov function

$$V = \sum_{i=1}^{n} V_i = \frac{1}{2} \sum_{i=1}^{n} s_i^2.$$

Taking the time derivative of V along (20), we obtain

$$\dot{V} = -\sum_{i=1}^{n} c_i s_i^2 + s_n s_{n+1} + O(e)^3.$$

Next, the observer gains ψ_i, $i = 1, \dots, n$; are chosen as follows

$$\psi_i = \frac{b_{n+1,n-i+1}}{K_{n-i} K_{i-1}} + \frac{K_{n-1}}{K_{i-1} K_{n-i}} \left(\frac{\partial f_n}{\partial z_{n-i+1}} \right), \text{ for } i = 1, \dots, n,$$

where $b_{n+1,i}$ and K_{n-1} are given in Appendix C. Then, from (38) the term s_{n+1} is equal to 0 (see Appendix C). Hence, we obtain

$$\dot{V} = -\sum_{i=1}^{n} c_i s_i^2 + O(e)^3. \tag{24}$$

Now, let U_ρ be the ρ-neighborhood of $\mathcal{C}$ an open subset of $I\!R^n$, then its closure $\bar{U}_\rho$ is a compact subset. Hence there exist constants $N > 0$, $\epsilon > 0$ such that the error term (24) satisfies

$$\left|O(e)^3\right| \leq N \|e\|^3$$

for all $z \in \bar{U}_\rho$, and $\|e\| < \epsilon$. Next, let be $\bar{\epsilon} = \min(\rho, \epsilon)$.

From $s = M(b_{i,j}, \psi_i)\, e$ where s is a linear function of e (see equation (20) and Appendix C), we know that there exists constants $\lambda_1 > 0, \lambda_2 > 0$ such that for all $z \in \bar{U}_\rho$, and $e, s \in \mathcal{C}$, the following inequality is satisfied

$$\lambda_1 \|e\| \leq \|s\| \leq \lambda_2 \|e\|. \tag{25}$$

Since $c_i > 0$, there exists a constant $\gamma > 0$ such that

$$4\gamma \|s\|^2 \leq \sum_{i=1}^{n} c_i s_i^2.$$

Hence, there exist an $\bar{\epsilon} > 0$ sufficiently small such that the error term in (24) satisfies

$$|O(e)^3| \leq \frac{1}{2} \sum_{i=1}^{n} c_i s_i^2$$

for all $z \in \bar{U}_\rho$, and $||e|| < \bar{\epsilon}$. For these z and e, we have

$$\dot{V} = -\frac{1}{2} \sum_{i=1}^{n} c_i s_i^2 \leq -2\gamma V. \tag{26}$$

And using Gronwall's inequality

$$V(t) \leq V(0) \exp^{-2\gamma t}.$$

Using the inequality (25), we have

$$||e(t)|| \leq \frac{\lambda_2}{\lambda_1} ||e(0)|| \exp^{-2\gamma t}.$$

Then, the estimation error converges exponentially to zero as $t \to \infty$. This ends the proof. □

5. EXAMPLES

Example 1. Single Output Case.

Consider the dynamics of a rigid body

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} \gamma_1 x_2 x_3 \\ \gamma_2 x_1 x_3 \\ \gamma_3 x_1 x_2 \end{pmatrix}$$
$$y = x_1$$

in which x_1, x_2 and x_3 are the components of the angular velocity with respect to the principal axes of inertia, J_1, J_2 and J_3 the moments of inertia with respect to the principal axes of inertia $\gamma_1 = \frac{J_3 - J_2}{J_1}$, $\gamma_2 = \frac{J_1 - J_3}{J_2}$ and $\gamma_3 = \frac{J_2 - J_1}{J_3}$. Assume that the angular velocity x_1 is measured. The observation problem is the estimation of the angular velocities x_2 and x_3.

Now, we apply the Algorithm presented in Section 3, to check if there exists a transformation for the above system.

Step 1. Determination of a_i.

Applying the proposed algorithm, the I/O differential equation (5), for $i = 1$ and $k_1 = 3$ is given by

$$\begin{aligned} y^{(3)} = P_0^i(y, \dot{y}, y^{(2)}) &= \frac{y^{(2)}\dot{y}}{y} + 4\gamma_2\gamma_3 y^2 \dot{y} \\ &= F_3 + F_2 + K_1 F_1 + K_2 F_0 \end{aligned}$$

where $F_2 = F_0 = 0$. On the other hand, the I/O differential equation of the affine system is given by

$$\begin{aligned} y_a^{(3)} &= y_a^{(1)} \left(\ddot{\overline{\ln a_1}} - \dot{\overline{\ln a_1 a_2}}\ \dot{\overline{\ln a_1}}\right) + y_a^{(2)} \left(\dot{\overline{\ln a_1}} + \dot{\overline{\ln a_1 a_2}}\right) - (\ddot{\overline{\ln a_1}} - \dot{\overline{\ln a_1 a_2}}\ \dot{\overline{\ln a_1}})\varphi_1 \\ &\quad - \dot{\overline{\ln a_1}}\dot{\varphi}_1 + \ddot{\varphi}_1 - \left(\dot{\overline{\ln a_1 a_2}}\right) \dot{\varphi}_1 - a_1(\dot{\overline{\ln a_1 a_2}} + \dot{\overline{\ln a_1}})\varphi_2 + \dot{\overline{a_1 \varphi_2}} + a_1 a_2 \varphi_3 \\ &= F_{3a} + F_{2a} + K_1 F_{1a} + K_2 F_{0a} \end{aligned}$$

where

$$\begin{aligned} F_{0a} &= \varphi_3, \\ F_{1a} &= -(\dot{\overline{\ln a_1 a_2}} + \dot{\overline{\ln a_1}})\varphi_2 + \dot{\overline{\varphi_2}} + \dot{\overline{\ln a_1}}\varphi_2, \\ F_{2a} &= -(\ddot{\overline{\ln a_1}} - \dot{\overline{\ln a_1 a_2}}\ \dot{\overline{\ln a_1}})\varphi_1 - \dot{\overline{\ln a_1}}\dot{\varphi}_1 + \ddot{\varphi}_1 - \left(\dot{\overline{\ln a_1 a_2}}\right) \dot{\varphi}_1, \\ F_{3a} &= y_a^{(1)} \left(\ddot{\overline{\ln a_1}} - \dot{\overline{\ln a_1 a_2}}\ \dot{\overline{\ln a_1}}\right) + y_a^{(2)} \left(\dot{\overline{\ln a_1}} + \dot{\overline{\ln a_1 a_2}}\right). \end{aligned}$$

From equation (8), the one-form ω_1 is given by

$$\omega_1 = \frac{1}{y}\mathrm{d}y.$$

Now, for $k = 2$, the one-form ω_2 is given by

$$\omega_2 = \frac{1}{y}\mathrm{d}y.$$

It is easy to see that the one-form ω_1 verify the conditions (14).

Now, computing one-form ω_{1a}, we have

$$\omega_{1a} = \frac{\partial^2 y_a^{(3)}}{\partial y_a^{(1)} \partial y_a^{(2)}}\mathrm{d}y = \left\{ 2\frac{\partial \log a_1}{\partial y_a} + \frac{\partial \log a_1 a_2}{\partial y_a} \right\} \mathrm{d}y.$$

In the same way, $\omega_{2a} = \omega_{1a}$. Then, in order to determine the a_i's, it is necessary to solve the following equation

$$\left\{ 2\frac{\partial \log a_1}{\partial y} + \frac{\partial \log a_1 a_2}{\partial y} \right\} = \frac{1}{y}.$$

Notice that the function a_1 depends on y, then the proposed algorithm can be extended to a large class of nonlinear systems where $a_{i,1}$ depends on u and y. However, for this class of systems the algorithm gives several solutions for a given system. For example, setting the arbitrary choice

$$a_1 = \frac{1}{a_2^2}.$$

It follows that a solution is of the form

$$a_1 = y, \qquad a_2 = \frac{1}{y^2}.$$

Step 2. Determination of φ_i.

Consider I/O differential equation P_0 and F_3, then

$$\begin{aligned} P_1 = P_0 - F_3 &= P_0 - \frac{y^{(2)}\dot{y}}{y} \\ &= 4\gamma_2\gamma_3 y^2 \dot{y}. \end{aligned}$$

Computing the one-form $\overline{\omega}_1$ from equation (12), we obtain $\overline{\omega}_1 = 0$.

$$\begin{aligned} \overline{\omega}_1 &= \frac{1}{a_1}\left\{\frac{\partial \varphi_1}{\partial y}\mathrm{d}y - \frac{\varphi_1}{a_1}\left(\frac{\partial a_1}{\partial y}\right)\mathrm{d}y\right\} \\ &= \mathrm{d}\left(\frac{\varphi_1}{a_1}\right) = 0. \end{aligned}$$

Since, $a_1 \neq 0$, then, this implies that $\varphi_1 = 0$.

Next, to determine $\overline{\omega}_2$, using equation for $r = 2$, we have

$$P_2 = P_1 - F_2 = P_1$$

since $F_2 = 0$, then

$$\overline{\omega}_2 = \frac{1}{a_1 a_2}\frac{\partial P_2}{\partial \dot{y}}\mathrm{d}y = 4\gamma_2\gamma_3 y^2 \mathrm{d}y$$

then, we have

$$\begin{aligned} \overline{\omega}_2 &= \frac{1}{a_2}\left\{\frac{\partial \varphi_2}{\partial y}\mathrm{d}y - \frac{\varphi_2}{a_2}\left(\frac{\partial a_2}{\partial y}\right)\mathrm{d}y\right\} \\ &= \mathrm{d}\left(\frac{\varphi_2}{a_2}\right) = 4\gamma_2\gamma_3 y^2 \mathrm{d}y. \end{aligned}$$

Solving the above equation, we obtain

$$\varphi_2 = \gamma_2\gamma_3 y^2.$$

Now, for $r = 3$, and from (13)

$$P_3 = a_1 a_2 \varphi_3.$$

Since $P_3 = 0$, it follows that $\varphi_3 = 0$.

After computation, the change of coordinates obtained is

$$\begin{aligned} z_1 &= x_1, \qquad z_2 = \frac{\gamma_1 x_2 x_3}{x_1} \\ z_3 &= \gamma_1\gamma_2 x_1^2 x_3^2 + \gamma_1\gamma_3 x_1^3 x_2 + \gamma_1^2 x_2^2 x_3^2 + \gamma_2\gamma_3 x_1^4. \end{aligned}$$

Then, the transformed system Σ_{affine} in the new coordinates is given by

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix} = \begin{pmatrix} 0 & y & 0 \\ 0 & 0 & \frac{1}{y^2} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + \begin{pmatrix} 0 \\ \gamma_2\gamma_3 y^2 \\ 0 \end{pmatrix}. \tag{27}$$

An observer backstepping for the above system can be design as follows.

$$\begin{pmatrix} \dot{\hat{z}}_1 \\ \dot{\hat{z}}_2 \\ \dot{\hat{z}}_3 \end{pmatrix} = \begin{pmatrix} 0 & y & 0 \\ 0 & 0 & \frac{1}{y^2} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{z}_1 \\ \hat{z}_2 \\ \hat{z}_3 \end{pmatrix} + \begin{pmatrix} 0 \\ \gamma_2\gamma_3 y^2 \\ 0 \end{pmatrix} + \begin{pmatrix} \psi_1(\hat{z}) \\ \psi_2(\hat{z}) \\ \psi_3(\hat{z}) \end{pmatrix} \left(z_1 - \hat{z}_1 \right) \tag{28}$$

where the observer gains are given by

$$\begin{aligned} \psi_1(\hat{z}) &= y b_{4,3} \\ \psi_2(\hat{z}) &= \frac{b_{4,2}}{y^2} \\ \psi_3(\hat{z}) &= y b_{4,1} \end{aligned}$$

where $K_1 = y$, $K_2 = \frac{1}{y}$, $g_1 = 0$, $g_2 = 0$, $g_3 = 0$, and

$$\begin{aligned} b_{2,1} &= c_1 \\ b_{3,1} &= 1 + c_2(c_1 - \psi_1) - (c_1 - \psi_1)\psi_1 - \frac{\mathrm{d}}{\mathrm{d}t}(\psi_1) \\ b_{3,2} &= y\,(c_2 + c_1) + \frac{\mathrm{d}y}{\mathrm{d}t} \\ b_{4,1} &= c_1 - \psi_1 + c_3(b_{3,1} - y\psi_2) - (b_{3,1} - y\psi_2)\psi_1 + \frac{\mathrm{d}}{\mathrm{d}t}(b_{3,1} - y\psi_2) \\ &\quad -(b_{3,2} - y\psi_1)\psi_2 + \frac{\mathrm{d}}{\mathrm{d}t}(b_{3,2} - y\psi_1) \\ b_{4,2} &= y + c_3(b_{3,2} - y\psi_1) + y b_{3,1} \\ b_{4,3} &= c_3\frac{1}{y} + \frac{1}{y^2}b_{3,2} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{y}\right). \end{aligned}$$

Example 2. Multi-Input Multi-Output.

Consider the following multivariable system:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{pmatrix} = \begin{pmatrix} u e^{x_2} \\ x_1 x_3 e^{-x_2} - u^2 e^{-x_2} \\ u x_1 \\ u^2 x_5 + u x_1 \\ x_1^2 x_4 \end{pmatrix}$$

$$y_1 = x_1, \quad y_2 = x_4.$$

It is easy to verify that the system is observable with indices of observability given by $k_1 = 3$ and $k_2 = 2$. Moreover, the I/O differential equations (5) of this system are

$$y_1^{(3)} = \frac{\dot{u}}{u}y_1^{(2)} + \overline{\ln(uy_1)}^{\cdot}y_1^{(2)} + \overline{\ln u}^{\cdot\cdot}\dot{y}_1 - \overline{\ln(uy_1)}^{\cdot}\overline{\ln u}^{\cdot}\dot{y}_1 - \overline{\ln(uy_1)}^{\cdot}u^3 + \dot{\overline{u^3}} + u^2y_1^2$$

$$y_2^{(2)} = 2\frac{\dot{u}}{u}(\dot{y}_2 - uy_1) + u^2y_1^2y_2 + \dot{u}y_1 + u\dot{y}_1.$$

Next, the I/O differential equations associated to the equivalent state affine system are

$$\begin{aligned} y_{1,a}^{(3)} &= y_{1,a}^{(1)}\left(\ddot{\overline{\ln a_{1,1}}} - \dot{\overline{\ln a_{1,1}a_{1,2}}}\ \dot{\overline{\ln a_{1,1}}}\right) + y_{1,a}^{(2)}\left(\dot{\overline{\ln a_{1,1}}} + \dot{\overline{\ln a_{1,1}a_{1,2}}}\right) \\ &\quad - (\ddot{\overline{\ln a_{1,1}}} - \dot{\overline{\ln a_{11}a_{12}}}\ \dot{\overline{\ln a_{11}}})\varphi_{1,1} - \dot{\overline{\ln a_{11}}}\dot{\varphi}_{1,1} + \ddot{\varphi}_{1,1} - \left(\dot{\overline{\ln a_{1,1}a_{1,2}}}\right)\dot{\varphi}_{1,1} \\ &\quad - a_{1,1}(\dot{\overline{\ln a_{1,1}a_{1,2}}} + \dot{\overline{\ln a_{1,1}}})\varphi_{1,2} + \dot{\overline{a_{1,1}\varphi_{1,2}}} + a_{1,1}a_{1,2}\varphi_{1,3} \end{aligned}$$

and

$$y_{2,a}^{(2)} = \ln \dot{a}_{2,1}(\dot{y}_2 - \varphi_{2,1}) + a_{2,1}\varphi_{2,2} + \dot{\varphi}_{2,1}.$$

Now, we apply the algorithm

Step 1. Computation of $a_{i,j}$.

For $i = 1$, the I/O differential equation P_0^1 is given by

$$\begin{aligned} P_0^1 &= y_1^{(3)} \\ &= \frac{\dot{u}}{u}y_1^{(2)} + \dot{\overline{\ln(uy_1)}}y_1^{(2)} + \ddot{\overline{\ln u}}\dot{y}_1 - \dot{\overline{\ln(uy_1)}}\ \dot{\overline{\ln u}}\dot{y}_1 - \dot{\overline{\ln(uy_1)}}u^3 + \dot{\overline{u^3}} + u^2y_1^2. \end{aligned}$$

For $k = 1$, it follows that the number of output that verify condition (7) is given $d_1^1 = 1$.

Now, computing the one-form ω_1^1, which is derived from (8), we obtain

$$\omega_1^1 = \frac{1}{y_1}\mathrm{d}y_1 + \frac{2}{u}\,\mathrm{d}u.$$

It is clear that $\mathrm{d}\omega_1^1 = 0$. Then, this implies that $\mathrm{d}\omega_1^1 \wedge \mathrm{d}u = 0$ and $\mathrm{d}\omega_1^1 \wedge dy_2 = 0$.

Next, for $k = 2$, and following the same procedure as above, we compute the one-form ω_2^1, which is given by

$$\omega_2^1 = \frac{1}{y_1}\mathrm{d}y_1 + \frac{1}{u}\mathrm{d}u.$$

Then, checking the condition of the theorem, it follows that

$$\mathrm{d}\omega_1^1 \wedge \mathrm{d}u = 0, \quad \mathrm{d}\omega \wedge \mathrm{d}y_2 = 0 \text{ and } \mathrm{d}\omega_2^1 = 0.$$

Given that the conditions of the theorem are verified, now we identify the unknown functions $a_{i,j}$ from the I/O differential equation $P_{a0}^1 := y_{1,a}^{(3)}$.

Now, computing the one-form from the I/O differential equation P^1_{a0}, we obtain

$$\omega^1_1 = \frac{\partial}{\partial \dot{y}_1}\left(\frac{\dot{a}_{1,2}(u,y)}{a_{1,2}(u,y)}\right)\mathrm{d}y_1 + \frac{\partial}{\partial \dot{u}}\left(\frac{2\dot{a}_{1,1}}{a_{1,1}} + \frac{\dot{a}_{1,2}}{a_{1,2}}\right)\mathrm{d}u.$$

The above equation allows to compute the functions $a_{1,1}$ and $a_{1,2}$.

Finally, after straightforward computation, we obtain

$$a_{1,1} = u \text{ and } a_{1,2} = y_1.$$

Now, for $i = 2$, the corresponding one-form obtained from $P^2_0 = y^{(2)}_2$ is given by

$$\omega^2_1 = \omega^2_{k_2-1} = \frac{2}{u}\mathrm{d}u.$$

Similarly, the one-form obtained from the I/O differential equation $P^2_{a0} := y^{(2)}_{2,a}$, is given by

$$\omega^2_1 = \frac{\partial}{\partial \dot{u}}\left(\frac{\dot{a}_{2,1}}{a_{2,1}}\right)\mathrm{d}u.$$

Comparing both one-forms, we can deduce that a solution is

$$a_{2,1} = u^2.$$

Step 2. Computation of $\varphi_{i,j}$.

Now, the components of the vector $\phi_i = \mathrm{col}(\ \varphi_{i,1} \quad \dots \quad \varphi_{i,k_i}\ )$ for each subsystem are determined.

For $i = 1$ and $r = 1$, we have that

$$\begin{aligned} P^1_1 &= P^1_0 - F^1_3 \\ &= -\left(\overline{\ln(uy_1)}^{\cdot}\right)\left(\overline{\ln u\dot{y}_1}^{\cdot}\right) - \left(\overline{\ln(uy_1)}^{\cdot}\right)u^3 + \overline{u^3}^{\cdot} + u^2y_1^2. \end{aligned}$$

Computing the one-form $\overline{\omega}^1_1$, it is easy to verify that $\overline{\omega}^1_1 = 0$, and this implies the function $\varphi_{1,1} = 0$.

Now, for $i = 1$ and $r = 2$,it follows that

$$P^1_2 = P^1_1 - F^1_2 = P^1_1$$

since $F^1_2 = 0$. Hence, the one-form $\overline{\omega}^1_2$ is given by

$$\overline{\omega}^1_2 = \frac{1}{a_{1,1}a_{1,2}}\left(\frac{u^3}{y_1}\right)\mathrm{d}y_1 + u^2\mathrm{d}u.$$

Comparing with following the I/O differential equation

$$\overline{\omega}^1_2 = \frac{1}{a_{1,2}}\left\{\sum_{j=1}^{d^2_1}\frac{\partial\varphi_{1,2}}{\partial y_j}\mathrm{d}y_j + \frac{\partial\varphi_{1,2}}{\partial u}\mathrm{d}u - \frac{\varphi_{1,2}}{a_{1,2}}\left(\sum_{j=1}^{d^2_1}\frac{\partial a_{1,2}}{\partial y_j}\mathrm{d}y_j + \frac{\partial a_{1,2}}{\partial u}\mathrm{d}u\right)\right\}.$$

This implies that $\varphi_{1,2} = u^2$.

The last iteration for this output leads to

$$uy_1\varphi_{1,3} = P_3^1 = P_2^1 - F_1^1 = uy_1^2.$$

Repeating the same procedure for $i = 2$, it follows that

$$P_1^2 = P_0^2 - F_2^2 = 2\frac{\dot{u}}{u}(-uy_1) + u^2y_1^2y_2 + \dot{u}y_1 + u\dot{y}_1$$

and the one-form $\overline{\omega}_1^2$ is given by

$$\overline{\omega}_1^2 = \frac{1}{u}\mathrm{d}y_1 + \frac{1}{y_1}\mathrm{d}u.$$

By comparison with the I/O differential equation, we obtain that

$$\varphi_{2,1} = uy_1.$$

Second iteration yields

$$a_{2,1}\varphi_{2,2} = P_2^2 = u^2y_1^2y_2.$$

Finally, we obtain $\varphi_{2,2} = y_1^2y_2$.

Then the transformed system is of the form

$$\begin{pmatrix} \dot{z}_{1,1} \\ \dot{z}_{1,2} \\ \dot{z}_{1,3} \end{pmatrix} = \begin{pmatrix} 0 & u & 0 \\ 0 & 0 & y_1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_{1,1} \\ z_{1,2} \\ z_{1,3} \end{pmatrix} + \begin{pmatrix} 0 \\ u^2 \\ uy_1 \end{pmatrix}$$

$$\begin{pmatrix} \dot{z}_{2,1} \\ \dot{z}_{2,2} \end{pmatrix} = \begin{pmatrix} 0 & u^2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z_{2,1} \\ z_{2,2} \end{pmatrix} + \begin{pmatrix} uy_1 \\ y_1^2y_2 \end{pmatrix} \tag{29}$$

$$y_1 = z_{1,1}, \quad y_2 = z_{2,1}.$$

The state coordinate transformation is

$$z_{1,1} = x_1, \quad z_{1,2} = e^{x_2}, \quad z_{1,3} = x_3$$
$$z_{2,1} = x_4, \quad z_{2,2} = x_5.$$

The observer for the system (29) is given by

$$\begin{pmatrix} \dot{\hat{z}}_{1,1} \\ \dot{\hat{z}}_{1,2} \\ \dot{\hat{z}}_{1,3} \end{pmatrix} = \begin{pmatrix} 0 & u & 0 \\ 0 & 0 & y_1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{z}_{1,1} \\ \hat{z}_{1,2} \\ \hat{z}_{1,3} \end{pmatrix} + \begin{pmatrix} 0 \\ u^2 \\ uy_1 \end{pmatrix} + \begin{pmatrix} \psi_{1,1}(\hat{z}_1) \\ \psi_{1,2}(\hat{z}_1) \\ \psi_{1,3}(\hat{z}_1) \end{pmatrix} \left(z_{1,1} - \hat{z}_{1,1} \right)$$

$$\begin{pmatrix} \dot{\hat{z}}_{2,1} \\ \dot{\hat{z}}_{2,2} \end{pmatrix} = \begin{pmatrix} 0 & u^2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{z}_{2,1} \\ \hat{z}_{2,2} \end{pmatrix} + \begin{pmatrix} uy_1 \\ y_1^2y_2 \end{pmatrix} + \begin{pmatrix} \psi_{2,1}(\hat{z}_2) \\ \psi_{2,2}(\hat{z}_2) \end{pmatrix} \left(z_{2,1} - \hat{z}_{2,1} \right)$$

where the observer gains are given by

$$\psi_{1,1}(\hat{z}_1) = \frac{b^1_{4,3}}{uy_1}, \quad \psi_{1,2}(\hat{z}_1) = \frac{b^1_{4,2}}{u^2}, \quad \psi_{1,3}(\hat{z}_1) = \frac{b^1_{4,1}}{uy_1}$$
$$\psi_{2,1}(\hat{z}_2) = \frac{b^2_{3,2}(\hat{z}_2)}{u^2}, \quad \psi_{2,2}(\hat{z}_2) = \frac{b^2_{3,1}(\hat{z}_2)}{u^2}$$

and for the first subsystem, we obtain

$$K^1_1 = u,\ K^1_2 = uy_1,\ g_{1,1} = 0,\ g_{1,2} = u^2,\ g_{1,3} = uy_1;$$

$$\begin{aligned}
b^1_{2,1} &= c_{1,1} \\
b^1_{3,1} &= 1 + c_{1,2}(c_{1,1} - \psi_{1,1}) - (c_{1,1} - \psi_{1,1})\psi_{1,1} - \frac{\mathrm{d}}{\mathrm{d}t}(\psi_{1,1}) \\
b^1_{3,2} &= u\,(c_{1,2} + c_{1,1}) + \frac{\mathrm{d}u}{\mathrm{d}t} \\
b^1_{4,1} &= c_{1,1} - \psi_{1,1} + c_{1,3}(b^1_{3,1} - u\psi_{1,2}) - (b_{3,1} - u\psi_{1,2})\psi_{1,1} + \frac{\mathrm{d}}{\mathrm{d}t}(b_{3,1} - u\psi_{1,2}) \\
&\quad - (b^1_{3,2} - u\psi_{1,1})\psi_{1,2} + uy_1\frac{\partial g_3}{\partial z_1} + \frac{\mathrm{d}}{\mathrm{d}t}(b_{3,2} - u\psi_{1,1}) \\
b^1_{4,2} &= u + c_{1,3}(b^1_{3,2} - u\psi_{1,1}) + ub^1_{3,1} \\
b^1_{4,3} &= c_{1,3}uy_1 + y_1 b^1_{3,2} + \frac{\mathrm{d}}{\mathrm{d}t}\,(uy_1)\,.
\end{aligned}$$

And for the second subsystem, we have

$$K^2_1 = u^2, \quad g_{2,1} = uy_1, \quad g_{2,2} = y_1^2 y_2;$$

$$\begin{aligned}
b^2_{2,1} &= c_{2,1} \\
b^2_{3,1} &= 1 + c_{2,2}(c_{2,1} - \psi_{2,1}) - (c_{2,1} - \psi_{2,1})\psi_{2,1} - \frac{\mathrm{d}}{\mathrm{d}t}(\psi_{2,1}) \\
b^2_{3,2} &= u^2\,(c_{2,2} + c_{2,1}) + \frac{\mathrm{d}u^2}{\mathrm{d}t}.
\end{aligned}$$

6. CONCLUSIONS

The observer synthesis for nonlinear systems has been considered in this paper. Based on their equivalence to state affine systems, necessary and sufficient conditions have been given to characterize a class of nonlinear systems which can be transformed into a multivariable state affine form up to input-output injection. For this class of systems a backstepping observer approach has been presented in order to design an observer. Several examples have been given in order to illustrate the proposed methodology.

APPENDIX A

Let $\mathcal{K}$ the field of meromorphic functions of $a \in I\!R^\lambda$ and $b \in I\!R^\rho$.

$$\omega \in \mathrm{Span}_{\mathcal{K}(a,b)}\{\mathrm{d}a_1, \ldots, \mathrm{d}a_\lambda, \mathrm{d}b_1, \ldots, \mathrm{d}b_\rho\}.$$

Definition A1. A one-form ω is closed if $\mathrm{d}\omega = 0$.

Definition A2. A one-form ω is exact if there exists a function $\psi(a,b)$ such that $\omega = \mathrm{d}\psi$.

Proposition A3. Any exact one-form is closed.

Lemma de Poincaré A4. Let ω be a closed one-form of the form

$$\omega \in \mathrm{Span}_{\mathcal{K}(a,b)}\{\mathrm{d}a_1, \ldots, \mathrm{d}a_\lambda, \mathrm{d}b_1, \ldots, \mathrm{d}b_\rho\}\ .$$

Then ω is locally exact if and only if $\mathrm{d}\omega = 0$.

Theorem A5. Given ω one-form, there exist a function ψ such that $\mathrm{Span}_{\mathcal{K}}\{\omega\} = \mathrm{Span}_{\mathcal{K}}\{\mathrm{d}\psi\}$ if and only if

$$\mathrm{d}\omega \wedge \omega = 0.$$

Theorem A6 (Frobenius Theorem). Let $\mathcal{V}$

$$\mathcal{V} = \mathrm{Span}_{\mathcal{K}}\{\omega_1, \ldots, \omega_n\}$$

be a subspace of $\mathcal{E}$. $\mathcal{V}$ is closed if and only if

$$\mathrm{d}\omega \wedge \omega_1 \wedge \ldots \wedge \omega_n, \text{ for any } i = 1, \ldots, n.$$

APPENDIX B

Proof of Theorem 1.

Necessity.

Assume that there exists a state transformation $z = T(x)$ transforming system Σ into system Σ_{affine}. Thus, the I/O differential equation of the system Σ, $P_0^i = y_i^{(k_i)}$ is equal to $P_{a0}^i := y_{ia}^{(k_i)}$;

$$P_{a0}^i = F_{k_i}^i(a_{i,1}, \ldots, a_{i,n-1}) + \Gamma_0^{k_i-1}(a_{i,1}, \ldots, a_{i,k_i-1}, \varphi_{i,1}, \ldots, \varphi_{i,k_i}).$$

Notice that the first term of the right hand does not depends on $\varphi_{i,1},\cdots,\varphi_{i,k_i}$, and can be written as

$$\begin{aligned} F^i_{k_i}(a_{i,1},\dots,a_{i,n-1}) &= y_j^{(k_i-1)}\frac{\mathrm{d}f^i_{1,1}}{\mathrm{d}t} + y_j^{(1)}\left\{\frac{\mathrm{d}^{k_i-1}f^i_{j,1}}{\mathrm{d}t^{k_i-1}} + \delta^i_{j,1}\right\} \\ &\quad + \sum_{j=2}^{k_i-2} y_j^{(k_i-j)}\left\{\frac{\mathrm{d}^j f^i_{j,1}}{\mathrm{d}t^j} + \delta^i_{j,1}\right\} \end{aligned} \tag{30}$$

where the $\delta^i_{j,1}(\cdot)$ are functions which depend only on functions $y^{(l)}$ and $u^{(l)}$, with $l<j$. The functions $F^i_{k_i-j}$, $j=1,\dots,k_i-1$, have the following form

$$F^i_{k_i-j} = \varphi_j^{(k_i-j)} + \begin{pmatrix} \varphi_j^{(k_i-j-1)} & \varphi_j^{(k_i-j-2)} & \cdots & \varphi_j \end{pmatrix} \begin{pmatrix} \frac{\mathrm{d}f^i_{1,j}}{\mathrm{d}t} \\ \frac{\mathrm{d}^2 f^i_{2,j}}{\mathrm{d}t^2} + \delta^i_{2,j} \\ \vdots \\ \frac{\mathrm{d}^{k_i-j}f^i_{k_i,j}}{\mathrm{d}t^{k_i-j}} + \delta^i_{k_i-j,j} \end{pmatrix} \tag{31}$$

for $j=1,\dots,k_i-1$; and the function $F^i_0=\varphi_{k_i}$. Then, the I/O differential equation can be written as

$$P^i_{a0} = y_i^{(k_i-1)}\frac{\mathrm{d}f^i_{1,1}}{\mathrm{d}t} + y_i^{(1)}\left(\frac{\mathrm{d}^{k_i-1}f^i_{j,1}}{\mathrm{d}t^{k_i-1}}\right) + \Delta(\cdot)$$

where $\Delta(\cdot)=\Gamma_0^{k_i-1}(a_{i,1},\dots,a_{i,k_i-1},\varphi_{i,1},\dots,\varphi_{i,k_i}) + y_j^{(1)}\delta^i_{j,1}$, and Δ represents to all monomials with a degree less than k_i-2.

Notice that

$$\begin{aligned} \frac{\mathrm{d}f^i_{1,1}}{\mathrm{d}t} &= \frac{\partial f^i_{1,1}}{\partial y}\dot y + \sum_{l=1}^m \frac{\partial f^i_{1,1}}{\partial u_l}\dot u_l \\ \frac{\mathrm{d}^{k_i-1}f^i_{j,1}}{\mathrm{d}t^{k_i-1}} &= \frac{\partial \log a_{i,j}}{\partial y}y^{(k_i-1)} + \sum_{l=1}^m \frac{\partial \log a_{i,j}}{\partial u_l}u_l^{(k_i-1)}. \end{aligned}$$

Now, let us apply the first step of the algorithm.

For $k=1$, the one-form is given by

$$\begin{aligned} \omega^i_1 &= \sum_{j=1}^{d^1_i}\frac{\partial^2 P^i_{a0}}{\partial y_j^{(1)}\partial y_j^{(k_i-1)}}\mathrm{d}y_j + \sum_{l=1}^m \frac{\partial^2 P^i_{a0}}{\partial u_l^{(1)}\partial y_j^{(k_i-1)}}\mathrm{d}u_l \\ &= \frac{1}{f^i_{1,1}}\left\{\sum_{j=1}^{d^1_i}\frac{\partial f^i_{1,1}}{\partial y_j}\mathrm{d}y_j + \sum_{l=1}^m \frac{\partial f^i_{1,1}}{\partial u_l}\mathrm{d}u_l\right\} \\ &= \frac{1}{f^i_{1,1}}\mathrm{d}f^i_{1,1}(u,y). \end{aligned}$$

Thus, the one-form ω_1^i is given by

$$\mathrm{d}\omega_1^i = \sum_{q=d_i^1+1}^{p} \left\{ \sum_{j=1}^{d_i^1} \frac{\partial}{\partial y_q} \left(\frac{1}{f_{1,1}^i} \frac{\partial f_{1,1}^i}{\partial y_j} \right) \mathrm{d}y_q \wedge \mathrm{d}y_j + \sum_{l=1}^{m} \frac{\partial}{\partial y_q} \left(\frac{1}{f_{1,1}^i} \frac{\partial f_{1,1}^i}{\partial u_l} \right) \mathrm{d}y_q \wedge \mathrm{d}u_l \right\}.$$

Then, the conditions of Theorem 1, for $d_i^k < p$,

$$\mathrm{d}\omega_1^i \wedge \mathrm{d}u = 0 \text{ and } \mathrm{d}\omega_1^i \wedge \mathrm{d}y_{d_i^1+1} \wedge \cdots \wedge \mathrm{d}y_p = 0$$

are verified directly.

The proof for $2 \leq k \leq k_i - 1$ follows the same lines as for $k = 1$.

Substituting the $a_{i,j}$ functions in $F_{k_i}^i$ in (30), and from equation (31), $F_{k_i-j}^i$ verifies

$$\begin{aligned} F_{k_i-j}^i = {} & \frac{\partial \varphi_j}{\partial y} y_j^{(k_i-j)} + \sum_{l=1}^{m} \frac{\partial \varphi_j}{\partial u_l} u_l^{(k_i-j)} \\ & - \varphi_j \left\{ \frac{\partial \log a_{i,j}}{\partial y} y^{(k_i-j)} + \sum_{l=1}^{m} \frac{\partial \log a_{i,j}}{\partial u_l} u_l^{(k_i-j)} \right\} + \Theta_{k_i-j}(\cdot) \end{aligned}$$

where the functions $\Theta_{k_i-j}(\cdot)$ involves monomials depending on functions $y^{(l)}$ and $u^{(l)}$, with $l < k_i - j$.

Applying Step 2 for $r = 1$, P_1^i is computed as follows

$$\begin{aligned} P_1^i &= P_0^i - F_{k_i}^i, = y_i^{(k_i)} - F_{k_i}^i \\ &= \frac{\partial \varphi_1}{\partial y} y_i^{(k_i-1)} + \sum_{l=1}^{m} \frac{\partial \varphi_1}{\partial u_l} u_l^{(k_i-1)} \\ &\quad - \varphi_1 \left\{ \sum_{j=1}^{d_i^1} \frac{\partial \log a_{i,1}}{\partial y_j} y_j^{(k_i-1)} + \sum_{l=1}^{m} \frac{\partial \log a_{i,1}}{\partial u_l} u_l^{(k_i-1)} \right\} + \Theta_{k_i-1}(\cdot) \end{aligned}$$

and set $K_1^i = a_{i,1}$.

Computing the one-form $\overline{\omega}_1^i$ as follows

$$\begin{aligned} \overline{\omega}_1^i &= \frac{1}{K_1^i} \left\{ \sum_{j=1}^{d_i^1} \frac{\partial P_1^i}{\partial y_j^{(k_i-1)}} \mathrm{d}y_j + \sum_{l=1}^{m} \frac{\partial P_1^i}{\partial u_l^{(k_i-1)}} \mathrm{d}u_l \right\} \\ &= \frac{1}{a_{i,1}} \left\{ \sum_{j=1}^{d_i^1} \frac{\partial \varphi_1}{\partial y_j} \mathrm{d}y_j + \sum_{l=1}^{m} \frac{\partial \varphi_1}{\partial u_l} \mathrm{d}u_l - \frac{\varphi_1}{a_{i,1}} \left\{ \sum_{j=1}^{d_i^1} \frac{\partial \log a_{i,1}}{\partial y_j} \mathrm{d}y_j + \sum_{l=1}^{m} \frac{\partial \log a_{i,1}}{\partial u_l} \mathrm{d}u_l \right\} \right\}. \end{aligned}$$

Thus, $\overline{\omega}_1^i = \mathrm{d}\left(\frac{\varphi_1}{a_{i,1}}\right)$, and it is easy to see that the conditions

$$\mathrm{d}\overline{\omega}_1^i \wedge \mathrm{d}u = 0 \text{ and } \mathrm{d}\overline{\omega}_1^i \wedge \mathrm{d}y_{d_i^k+1} \wedge \cdots \wedge \mathrm{d}y_p = 0$$

are satisfied. The necessary condition of Theorem 1 is proved for the first iteration. For proving the iterations $r = 2, \dots, k_i$, a similar procedure can be followed.

Sufficiency:

Step 1. Determination of $a_{i,j}$.

Consider the nonlinear system Σ and suppose that the conditions

$$\mathrm{d}\omega_k^i \wedge \mathrm{d}u = 0, \text{ and } \mathrm{d}\omega_k^i \wedge \mathrm{d}y_{d_i^k+1} \wedge \cdots \wedge \mathrm{d}y_p = 0$$

are satisfied. The one-form ω_k^i given by

$$\omega_k^i = c_k^i \sum_{j=1}^{d_i^k} \frac{\partial^2 P_{a0}^i}{\partial y_j^{(k)} \partial y_j^{(k_i-k)}} \mathrm{d}y_j + \sum_{j=1}^{d_i^k} \sum_{l=1}^{m} \frac{\partial^2 P_{a0}^i}{\partial u_l^{(k)} \partial y_j^{(k_i-k)}} \mathrm{d}u_l$$

satisfies the above conditions. Then,

$$\omega_k^i \in \mathrm{Span}\{\mathrm{d}y_1, \dots, \mathrm{d}y_{d_i^k}\}.$$

On the other hand, the one-form obtained from the I/O differential equation P_{a0}^i, satisfies the following relation

$$\omega_{ka}^i = c_k^i \sum_{j=1}^{d_i^k} \frac{\partial^2 P_{a0}^i}{\partial y_j^{(k)} \partial y_j^{(k_i-k)}} \mathrm{d}y_j + \sum_{j=1}^{d_i^k} \sum_{l=1}^{m} \frac{\partial^2 P_{a0}^i}{\partial u_l^{(k)} \partial y_j^{(k_i-k)}} \mathrm{d}u_l.$$

Solving the set of $(d_i^k - 1)$ partial differential equations, it is possible to obtain the $a_{i,j}$ functions. This ends the proof of Step 1.

Step 2. Determination of $\varphi_{i,j}$.

In order to obtain the functions $\varphi_{i,j}$, we assume the $a_{i,j}$ are known from Step 1, and for $r = 1$, replacing the function $a_{i,1}$, the one-form ϖ_1^i is given by

$$\varpi_{1a}^i = \frac{1}{a_{i,1}} \left\{ \sum_{j=1}^{d_i^1} \frac{\partial \varphi_1}{\partial y_j} \mathrm{d}y_j + \sum_{l=1}^{m} \frac{\partial \varphi_1}{\partial u_l} \mathrm{d}u_l - \frac{\varphi_1}{a_{i,1}} \left\{ \sum_{j=1}^{d_i^1} \frac{\partial \log a_{i,1}}{\partial y_j} \mathrm{d}y_j + \sum_{l=1}^{m} \frac{\partial \log a_{i,1}}{\partial u_l} \mathrm{d}u_l \right\}\right\}.$$

On the other hand, the one-form ϖ_k^i obtained from the I/O differential equation of the nonlinear system Σ and the conditions

$$\mathrm{d}\varpi_k^i \wedge \mathrm{d}u = 0 \text{ and } \mathrm{d}\varpi_k^i \wedge \mathrm{d}y_{d_i^k+1} \wedge \cdots \wedge \mathrm{d}y_p = 0$$

allows to conclude that

$$\varpi_1^i \in \mathrm{Span}\{\mathrm{d}y_1, \dots, \mathrm{d}y_{d_i^k}\}.$$

Then, the $\varphi_{i,j}$ can be determined as follows. Let $z_i = \mathrm{col}(z_{i,1} \dots z_{i,k_i}) \in I\!R^{k_i}$, for $i = 1, \dots, p$; and $z_{i,1} = y_i = h_i(x)$, where h_i is the ith component of the output equation $y = h(x)$.

Now, for $k = 2, \dots, k_i$, let be

$$z_{i,k} = \frac{\dot{z}_{i,k-1} - \varphi_{i,k-1}}{a_{i,k-1}}.$$

which represent the $k_i - 1$ first dynamics of Σ.

To compute the last dynamic equation $\dot{z}_{i,k_i}$, we note that

$$y_i^{(k)} = z_{i,k+1} K_k^i + P_k^i$$

where

$$P_k^i = \varphi_{i,k} K_k^i + \dot{P}_{k-1}^i + z_{i,k} \frac{\mathrm{d}K_k^i}{\mathrm{d}t},$$

and $a_{i,k_i} = 0$ by construction and $P_1^i = \varphi_{i,1}$.

Thus the last dynamic equation obtained as follows

$$z_{i,k_i} = \frac{\dot{z}_{i,k-1} - \varphi_{i,k_i-1}}{a_{i,k_i-1}} = \frac{y_i^{(k_i-1)} - P_{k_i-1}^i}{K_{k_i-1}^i}.$$

Taking the time derivative of the above equation, it follows that

$$\dot{z}_{i,k_i} = \frac{\left(y_i^{(k_i)} - \dot{P}_{k_i-1}^i\right) K_{k_i-1}^i - \left(y_i^{(k_i-1)} - P_{k_i-1}^i\right) \dot{K}_{k_i-1}^i}{\left(K_{k_i-1}^i\right)^2}.$$

After substitution of the function $P_{k_i-1}^i$, one finally gets

$$\dot{z}_{i,k_i} = \varphi_{i,k_i}.$$

This ends the proof. □

APPENDIX C

Let be

$$s_{l+1} = \sum_{i=1}^{l} \left(b_{l+1,i} - K_{l-i} K_{i-1} \psi_{l-i+1}\right) e_i + K_l e_{l+1} \tag{32}$$

where $s = \operatorname{col}(s_1, s_2, \dots, s_l, s_{l+1})$, $e = \operatorname{col}(e_1, e_2, \dots, e_{l+1})$.

Now, writing in terms of the estimation error, we obtain

$$s = M(b_{i,j}, \psi_i)\, e \tag{33}$$

$$M(b_{i,j}, \psi_i) =$$

$$= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ b_{2,1} - \psi_1 & K_1 & \cdots & 0 \\ b_{3,1} - K_1\psi_2 & b_{3,2} - K_1\psi_1 & \cdots & 0 \\ b_{4,1} - K_2\psi_3 & b_{4,2} - (K_1)^2\psi_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} - K_{l-2}\psi_{n-1} & b_{l,2} - K_{l-3}K_1\psi_{l-2} & \cdots & K_{l-1} \\ b_{n+1,1} - K_{l-1}\psi_n & b_{l+1,2} - K_{l-2}K_1\psi_{l-1} & \cdots & b_{l+1,l} - K_{l-1}\psi_1 \end{pmatrix} \tag{34}$$

where

$$K_r = \prod_{i=0}^{r} a_i$$

and $a_0 = 1$; the $b_{i,j} = b_{i,j}(z)$ are given by,
for $i = 2$

$$b_{2,1} = c_1 + \frac{\partial g_1}{\partial z_1} \tag{35}$$

for $i = 3$

$$\begin{aligned} b_{3,1} &= 1 + c_2(b_{2,1} - \psi_1) + (b_{2,1} - \psi_1)\left(\frac{\partial g_1}{\partial z_1} - \psi_1\right) + \frac{\mathrm{d}}{\mathrm{d}t}(b_{2,1} - \psi_1) + K_1\frac{\partial g_2}{\partial z_1} \\ b_{3,2} &= K_1c_2 + a_1b_{2,1} + \frac{\mathrm{d}K_1}{\mathrm{d}t} + K_1\frac{\partial g_2}{\partial z_2} \end{aligned} \tag{36}$$

for $i = 4$

$$\begin{aligned} b_{4,1} &= b_{2,1} - \psi_1 + c_3(b_{3,1} - K_1\psi_2) + (b_{3,1} - K_1\psi_2)\left(\frac{\partial g_1}{\partial z_1} - \psi_1\right) + \frac{\mathrm{d}}{\mathrm{d}t}(b_{3,1} - K_1\psi_2) \\ &\quad + (b_{3,2} - K_1\psi_1)\left(\frac{\partial g_2}{\partial z_1} - \psi_2\right) + K_2\frac{\partial g_3}{\partial z_1} \\ b_{4,2} &= a_1 + c_3(b_{3,2} - K_1\psi_1) + K_1b_{3,1} + (b_{3,2} - K_1\psi_1)\frac{\partial g_2}{\partial z_2} + \frac{\mathrm{d}}{\mathrm{d}t}(b_{3,2} - K_1\psi_1) + K_2\frac{\partial g_3}{\partial z_2} \\ b_{4,3} &= c_3K_2 + a_2b_{3,2} + \frac{\mathrm{d}}{\mathrm{d}t}(K_2) + K_2\frac{\partial g_3}{\partial z_3} \end{aligned} \tag{37}$$

for $4 < i \leq n + 1$

$$\begin{aligned} b_{i,1} &= b_{i-2,1} - K_{i-4}\psi_{i-3} + c_{i-1}(b_{i-1,1} - K_{i-3}\psi_{i-2}) + \frac{\mathrm{d}}{\mathrm{d}t}(b_{i-1,1} - K_{i-3}\psi_{i-2}) \\ &\quad + \sum_{k=1}^{i-2}(b_{i-1,k} - K_{i-3}\psi_{i-k-1})\left(\frac{\partial g_k}{\partial z_1} - \psi_k\right) + K_{i-2}\left(\frac{\partial g_{i-1}}{\partial z_1}\right) \end{aligned}$$

$$b_{i,j} = b_{i-2,j} - K_{i-j-3}K_{j-1}\psi_{i-j-2} + K_{i-2}\left(\frac{\partial g_{i-1}}{\partial z_j}\right) + a_{j-1}b_{i-1,j-1}$$
$$+ c_{i-1}\left(b_{i-1,j} - K_{i-j-2}K_{i-2}\psi_{i-j-1}\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(b_{i-1,j} - K_{i-j-2}K_{j-1}\psi_{i-j-1}\right)$$
$$+ \sum_{k=j}^{i-2}\left(b_{i-1,k} - K_{i-k-2}K_{k-1}\psi_{i-k-1}\right)\left(\frac{\partial g_k}{\partial z_1}\right)$$

$$b_{i,i-2} = K_{i-3} + c_{i-1}\left(b_{i-1,i-2} - K_{i-3}\psi_1\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(b_{i-1,i-2} - K_{i-3}\psi_1\right)$$
$$+ a_{i-3}b_{i-1,i-3} + \left(b_{i-1,i-2} - K_{i-3}\psi_1\right)\left(\frac{\partial g_{i-2}}{\partial z_{i-2}}\right) + K_{i-2}\left(\frac{\partial g_{i-1}}{\partial z_{i-2}}\right)$$

$$b_{i,i-1} = K_{i-2}c_{i-1} + a_{i-2}b_{i-1,i-2} + K_{i-2}\left(\frac{\partial g_{i-1}}{\partial z_{i-1}}\right) + \frac{\mathrm{d}}{\mathrm{d}t}K_{i-2}.$$

When $l = n$, where n is the dimension of the system, it is easy to see that

$$s_{n+1} = \sum_{i=1}^{n}\left(b_{n+1,i} - K_{n-i}K_{i-1}\psi_{n-i+1} + K_{n-1}\left(\frac{\partial f_n}{\partial z_i}\right)\right)e_i. \tag{38}$$

In order to determine the gains of the observer we make the last above equation equal to zero, i.e.

$$b_{n+1,i} - K_{n-i}K_{i-1}\psi_{n-i+1} + K_{n-1}\left(\frac{\partial f_n}{\partial z_i}\right) = 0, \quad \text{for } i = 1,\dots,n.$$

Then, it follows that

$$\psi_{n-i+1} = \frac{b_{n+1,i}}{K_{n-i}K_{i-1}} + \frac{K_{n-1}}{K_{n-i}K_{i-1}}\left(\frac{\partial f_n}{\partial z_i}\right), \quad \text{for } i = 1,\dots,n;$$

or equivalently

$$\psi_j = \frac{b_{n+1,n-j+1}}{K_{n-j}K_{j-1}} + \frac{K_{n-1}}{K_{n-j}K_{j-1}}\left(\frac{\partial f_n}{\partial z_{n-j+1}}\right), \quad \text{for } j = 1,\dots,n. \tag{39}$$

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