INFINITE-DIMENSIONAL LMI APPROACH TO ANALYSIS AND SYNTHESIS FOR LINEAR TIME-DELAY SYSTEMS

KOJIRO IKEDA, TAKEHITO AZUMA AND KENKO UCHIDA

This paper considers an analysis and synthesis problem of controllers for linear time-delay systems in the form of delay-dependent memory state feedback, and develops an Linear Matrix Inequality (LMI) approach. First, we present an existence condition and an explicit formula of controllers, which guarantee a prescribed level of L^2 gain of closed loop systems, in terms of infinite-dimensional LMIs. This result is rather general in the sense that it covers, as special cases, some known results for the cases of delay-independent/dependent and memoryless/memory controllers, while the infinity dimensionality of the LMIs makes the result difficult to apply. Second, we introduce a technique to reduce the infinite-dimensional LMIs to a finite number of LMIs, and present a feasible algorithm for synthesis of controllers based on the finite-dimensional LMIs.

1. INTRODUCTION

The fact that the state space of time-delay systems is infinite-dimensional leads generally to infinite-dimensional characterizations for analysis and synthesis in time-delay systems. For example, it is well known that the optimal LQ control for time-delay systems is given in the memory, i. e. infinite-dimensional, state feedback form whose feedback gains are characterized by the infinite-dimensional Riccati equations; as for state feedback control synthesis, we could say that the memory state feedback form is general and natural for time-delay systems, and can expect that memory state feedback controllers achieve better performance than memoryless state feedback controllers [2, 11, 16, 17]. Of course, the infinite-dimensional characterizations give us contrary hard problems in computations and implementations [6]. Our concern is to find a feasible approach to such infinite-dimensional tasks in analysis and synthesis for linear time-delay systems.

Recently the Linear Matrix Inequality (LMI) approach [4] has been developed in analysis and synthesis problems for linear time-delay systems and its advantages in numerical computations are presented [6, 8, 9, 11, 14, 15]; however, the approach is mostly developed under some finite-dimensional assumptions assured by a special form of Lyapunov functional in analysis and/or a memoryless controller form in synthesis. One exception which does not require such finite-dimensional assumptions is a series of the works by Gu [8, 9]; he proposes a discretization technique which can characterize a general Lyapunov functional with a finite number of LMIs. As more recent references on LMI for time-delay systems, which we learned after submitting this paper, [5] (and references inside) and [7] should be mentioned; a synthesis problem of state feedback with delay is discussed in [5], and a memoryless state feedback is designed for a system with distributed time-delays in [7].

In this paper, focusing on input-output L^2 gain performance, we consider an analysis and synthesis problem of memory state feedback controllers for linear systems with time-delay via an LMI approach which is an extension of the LMI approach developed in [2] where stability and stabilizability is focused. First we show a result of L^2 gain analysis of linear systems with time-delay and make a comparison with some previous works in some special cases. Next we discuss a controller synthesis problem based on this result of L^2 gain analysis. We also consider a synthesis problem of controllers with constrained feedback gains. We derive the results of L^2 gain analysis and controller synthesis in the form of infinite-dimensional LMIs, and present a procedure to reduce the infinite-dimensional LMIs to a finite number of LMIs. Finally we show a numerical example.

2. SYSTEM DESCRIPTION

Consider the following linear time-delay system defined on the time interval $[0, \infty)$,

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + A_1 x(t-h) + Bu(t) + Dw(t), \\ z(t) &= Cx(t), \\ x(\beta) &= 0, \ -h \le \beta \le 0, \end{aligned} \tag{2.1}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^{m_u}$ is the input, $w(t) \in R^{m_w}$ is the disturbance, and $z(t) \in R^l$ is the output. $A_0 \in R^{n\times n}$, $A_1 \in R^{n\times n}$, $B \in R^{n\times m_u}$, $C \in R^{l\times n}$ and $D \in R^{n\times m_w}$ are constant matrices. The parameter h denotes the time delay and $h > 0$.

The input $u(t)$ is given by the following state feedback controller,

$$u(t) = K_0 x(t) + \int_{-h}^{0} K_{01}(\beta)\, x(t+\beta)\, \mathrm{d}\beta, \tag{2.2}$$

where $K_0 \in R^{m_u\times n}$ is a constant matrix and $K_{01}(\beta) \in L^2([-h,0]; R^{m_u\times n})$ is a square integrable matrix function.

In this paper, we use a notation,

$$L(\alpha,\beta) = \begin{bmatrix} P_0 & P_1(\beta) \\ P_1'(\alpha) & P_2(\alpha,\beta) \end{bmatrix} > (<)\ 0, \quad {}^\forall \alpha \in [-h,0],\ {}^\forall \beta \in [-h,0],$$

which means that P_0 and $P_2(\alpha,\beta)$ are symmetric, that is $P_0' = P_0$ and $P_2'(\alpha,\beta) =$

$P_2(\beta, \alpha)$, and the symmetrized matrix,

$$\frac{1}{2}(L(\alpha,\beta) + L'(\alpha,\beta)) = \begin{bmatrix} P_0 & \frac{1}{2}(P_1(\alpha) + P_1(\beta)) \\ \frac{1}{2}(P_1'(\alpha) + P_1'(\beta)) & \frac{1}{2}(P_2(\alpha,\beta) + P_2(\beta,\alpha)) \end{bmatrix},$$

is positive definite (negative definite) for each $(\alpha, \beta) \in [-h, 0] \times [-h, 0]$, where "$'$" denotes transposition of vector and matrix. The notation, $L(\alpha, \beta) \geq (\leq)\ 0$, is similarly defined. Note that, if a matrix function $L(\alpha, \beta) > 0$ is continuous in (α, β), there exists a positive number λ such that $L(\alpha, \beta) \geq \lambda I$ for all $(\alpha, \beta) \in [-h, 0] \times [-h, 0]$, where I denotes identity matrix.

3. L^2 GAIN ANALYSIS

3.1. General result

From (2.1),(2.2),the closed loop system can be written in the following form,

$$\begin{aligned} \dot{x}(t) &= \widetilde{A}_0 x(t) + \widetilde{A}_1 x(t-h) + \int_{-h}^{0} \widetilde{A}_{01}(\beta)\, x(t+\beta)\, \mathrm{d}\beta + Dw(t), \\ z(t) &= Cx(t), \end{aligned} \tag{3.1}$$

where

$$\widetilde{A}_0 = A_0 + BK_0, \quad \widetilde{A}_1 = A_1,$$
$$\widetilde{A}_{01}(\beta) = BK_{01}(\beta).$$

First we analyze L^2 gain for the closed loop system (3.1). The L^2 gain of the system (3.1) is defined as follows,

$$G = \sup_{w \in L^2, w \neq 0} \frac{\|z\|_{L^2}}{\|w\|_{L^2}},$$

where $\|\cdot\|_{L^2}$ denotes L^2 norm.

Now we introduce the following functional,

$$\begin{aligned} V(x_t) \ &= x'(t)\, Px(t) + \int_{-h}^{0} x'(t+\beta)\, Qx(t+\beta)\, \mathrm{d}\beta \\ &\quad + x'(t) \int_{-h}^{0} R(\beta)\, x(t+\beta)\, \mathrm{d}\beta + \int_{-h}^{0} x'(t+\alpha) R'(\alpha)\, \mathrm{d}\alpha x(t) \\ &\quad + \int_{-h}^{0}\int_{-h}^{0} x'(t+\alpha) S(\alpha,\beta)\, x(t+\beta)\, \mathrm{d}\alpha \mathrm{d}\beta, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} & x_t = \{x(t+\beta) \mid -h \leq \beta \leq 0\}, \\ & P, Q \in R^{n \times n}, \\ & R(\beta) \in L^2([-h,0]; R^{n\times n}), \\ & S(\alpha,\beta) \in L^2([-h,0] \times [-h,0]; R^{n\times n}). \end{aligned}$$

By using this functional, we have a result for L^2 gain analysis of the time-delay system (3.1).

Theorem 3.1. If there exist constant matrices P, Q and continuously differentiable matrix functions $R(\beta)$, $S(\alpha,\beta)$ which satisfy the following inequalities,

$$L_1(\alpha,\beta) = \begin{bmatrix} \begin{pmatrix} \widetilde{A}_0'P + P\widetilde{A}_0 + Q \\ +R'(0) + R(0) + C'C \end{pmatrix} & P\widetilde{A}_1 - R(-h) & \begin{pmatrix} P\widetilde{A}_{01}(\beta) + \widetilde{A}_0'R(\beta) \\ -\frac{\partial}{\partial\beta}R(\beta) + S(0,\beta) \end{pmatrix} & PD \\ \widetilde{A}_1'P - R'(-h) & -Q & \widetilde{A}_1'R(\beta) - S(-h,\beta) & 0 \\ \begin{pmatrix} \widetilde{A}_{01}'(\alpha)P + R'(\alpha)\widetilde{A}_0 \\ -\frac{\partial}{\partial\alpha}R'(\alpha) + S(\alpha,0) \end{pmatrix} & R'(\alpha)\widetilde{A}_1 - S(\alpha,-h) & \begin{pmatrix} R'(\alpha)\widetilde{A}_{01}(\beta) + \widetilde{A}_{01}'(\alpha)R(\beta) \\ -(\frac{\partial}{\partial\beta} + \frac{\partial}{\partial\alpha})S(\alpha,\beta) \end{pmatrix} & R'(\alpha)D \\ D'P & 0 & D'R(\beta) & -\gamma^2 I \end{bmatrix} < 0, \quad (3.3)$$

$$L_2(\alpha,\beta) = \begin{bmatrix} P & R(\beta) \\ R'(\alpha) & S(\alpha,\beta) \end{bmatrix} > 0, \quad (3.4)$$

$$Q > 0, \quad (3.5)$$

$$^\forall\alpha \in [-h,\ 0],\ ^\forall\beta \in [-h,\ 0],$$

then the time-delay system (3.1) is internally, asymptotically, stable and the L^2 gain of (3.1) is less than γ.

Proof. (Stability) We shall show that the functional (3.2) is a Lyapnov functional for the system(3.1), that is $V(x_t) > 0$ and $\frac{\mathrm{d}}{\mathrm{d}t}V(x_t) > 0$ for $x(t) \neq 0$. $V(x_t) > 0$ follows from (3.4),(3.5) and the expression,

$$V(x_t) = \int_{-h}^{0}\int_{-h}^{0} \begin{bmatrix} h^{-1}x(t) \\ x(t+\alpha) \end{bmatrix}' L_2(\alpha,\beta) \begin{bmatrix} h^{-1}x(t) \\ x(t+\beta) \end{bmatrix} \mathrm{d}\alpha\mathrm{d}\beta + \int_{-h}^{0} x'(t+\beta)\,Qx(t+\beta)\,\mathrm{d}\beta.$$

Differentiating both sides of (3.2) with respect to t along the trajectory of the system (3.1) with $w(t) \equiv 0$ and rearranging terms, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x_t) = \int_{-h}^{0}\int_{-h}^{0} \begin{bmatrix} h^{-1}x(t) \\ h^{-1}x(t-h) \\ x(t+\alpha) \end{bmatrix}' L_0(\alpha,\beta) \begin{bmatrix} h^{-1}x(t) \\ h^{-1}x(t-h) \\ x(t+\beta) \end{bmatrix} \mathrm{d}\alpha\mathrm{d}\beta$$

where

$$L_0(\alpha,\beta) =$$
$$\begin{bmatrix} \begin{pmatrix} \tilde{A_0}'P + P\tilde{A_0} \\ +Q \\ +R(0)' + R(0) \end{pmatrix} & P\tilde{A_1} - R(-h) & \begin{pmatrix} P\tilde{A_{01}}(\beta) + \tilde{A_0}'R(\beta) \\ -\frac{\partial}{\partial\beta}R(\beta) + S(0,\beta) \end{pmatrix} \\ \tilde{A_1}'P - R'(-h) & -Q & \tilde{A_1}'R(\beta) - S(-h,\beta) \\ \begin{pmatrix} \tilde{A_{01}}'(\alpha)P \\ +R'(\alpha)\tilde{A_0} \\ -\frac{\partial}{\partial\alpha}R'(\alpha) \\ +S(\alpha,0) \end{pmatrix} & R'(\alpha)\tilde{A_1} - S(\alpha,-h) & \begin{pmatrix} R'(\alpha)\tilde{A_{01}}(\beta) \\ +\tilde{A_{01}}'(\alpha)R(\beta) \\ -(\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta})S(\alpha,\beta) \end{pmatrix} \end{bmatrix}.$$

Using Schur Complement, we can show that the inequality (3.3), that is $L_1(\alpha,\beta) < 0$ is equivalent to the following inequality,

$$L_0(\alpha,\beta) + \begin{bmatrix} C' & PD \\ 0 & 0 \\ 0 & R'(\alpha)D + R'(\beta)D \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \gamma^2 I \end{bmatrix}^{-1} \begin{bmatrix} C' & PD \\ 0 & 0 \\ 0 & R'(\alpha)D + R'(\beta)D \end{bmatrix}' < 0.$$

Thus, from (3.3) we have $L_0(\alpha,\beta) < 0$, and from the above expression of $\frac{\mathrm{d}}{\mathrm{d}t}V(x_t)$ we can see $\frac{\mathrm{d}}{\mathrm{d}t}V(x_t) < 0$ for $x(t) \neq 0$. Then, the internal, asymptotic, stability of the system (3.1) follows from the well-known stability result [10].

(L^2 gain) First note that the internal, asymptotic, stability of the system (3.1) implies $z \in L^2([-h,0];R^l)$ and, in particular, $x(\infty) = 0$ for any $w \in L^2([-h,0];R^{m_w})$. Hence, from $x(\beta) = 0, -h \le \beta \le 0$, we have the identity,

$$\|z\|_{L^2}^2 - \gamma^2\|w\|_{L^2}^2 = \int_0^\infty \left[z'(t)z(t) - \gamma^2 w'(t)\,w(t) + \frac{\mathrm{d}}{\mathrm{d}t}V(x_t)\right] \mathrm{d}t,$$

for $w \in L^2([-h,0];R^l)$. Calculating $\frac{\mathrm{d}}{\mathrm{d}t}V(x_t)$ with (3.2) along the trajectory of the system (3.1) and substituting it into the above identity, we obtain

$$\|z\|_{L^2}^2 - \gamma^2\|w\|_{L^2}^2 = \int_0^\infty \left(\int_{-h}^0 \int_{-h}^0 \begin{bmatrix} h^{-1}x(t) \\ h^{-1}x(t-h) \\ x(t+\alpha) \\ h^{-1}w(t) \end{bmatrix}' L_1(\alpha,\beta) \begin{bmatrix} h^{-1}x(t) \\ h^{-1}x(t-h) \\ x(t+\alpha) \\ h^{-1}w(t) \end{bmatrix} \mathrm{d}\alpha \mathrm{d}\beta \right) \mathrm{d}t.$$

Then $G < \gamma$ follows from (3.3). □

3.2. Results in special cases

It is known that the existence of Lyapunov functional of the form (3.2) is a necessary and sufficient condition for internal stability of linear time-delay systems. From this fact and the analogy of L^2 gain analysis in linear systems with no delay, we suspect that the functional (3.2) might lead to a necessary and sufficient condition, and the LMI conditions in Theorem 3.1 might be rather less-conservative. Instead of pursuing this issue, here, we observe that, for particular choices of structure of the solution $(P, Q, R(\beta), S(\alpha, \beta))$, the LMI conditions (3.3), (3.4), (3.5) in Theorem 3.1 is reduced to the well known condition of delay-independent types [11] or delay-dependent types [14]. To simplify the discussion, we focus on the case of the following system,

$$\begin{aligned} \dot{x}(t) &= \widetilde{A}_0 x(t) + \widetilde{A}_1 x(t-h) + Dw(t), \\ z(t) &= Cx(t). \end{aligned}$$

First note that the positive definiteness of inequalities (3.4) and (3.5) in Theorem 3.1, which are required for (3.2) to be a Lyapunov functional of this system, can be relaxed to positive semidefiniteness except $P > 0$. In view of this, let $R(\beta) \equiv 0$ and $S(\alpha, \beta) \equiv 0$ in the inequality (3.3), we can rewrite (3.3) as

$$\begin{bmatrix} \widetilde{A}_0' P + P\widetilde{A}_0 + Q + C'C & P\widetilde{A}_1 & PD \\ \widetilde{A}_1' P & -Q & 0 \\ D'P & 0 & -\gamma^2 I \end{bmatrix} < 0, \tag{3.6}$$

and obtain the next result from Theorem 3.1.

Corollary 3.2. If there exists positive definite P and Q which satisfy the LMI condition (3.6), then the time-delay system is internally, asymptotically, stable and the L^2 gain is less than γ.

The LMI condition (3.6) is equivalent to the Riccati inequality condition derived by Lee et al in [14].

Next let $R(\beta) = PU(\beta)$ and $S(\alpha, \beta) = U'(\alpha)PU(\beta)$, where $U(\beta)$ is a matrix function defined by the following functional differential equation,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\beta} U(\beta) &= (\widetilde{A}_0 + U(0))U(\beta), \\ U(-h) &= \widetilde{A}_1, \ -h \le \beta \le 0. \end{aligned} \tag{3.7}$$

We have a sufficient condition for the inequality (3.3), which is given by

$$\begin{bmatrix} M + C'C & M & PD \\ M & M & PD \\ D'P & D'P & -\gamma^2 I \end{bmatrix} < 0, \tag{3.8}$$

where $M = (\widetilde{A}_0 + U(0))'P + P(\widetilde{A}_0 + U(0))$. Thus we can obtain the next result from Theorem 3.1.

Corollary 3.3. If there exist a positive definite matrix P and a matrix function $U(\beta)$ which is the solution to the equation (3.7) and satisfy the LMI condition (3.8), then the time-delay system is internally, asymptotically, stable and the L^2 gain is less than γ.

Corollary 3.3 is the result derived by He et al in [11] where the LMI condition (3.8) is expressed in the equivalent Riccati inequality form.

The LMI condition (3.6) is independent of the time-delay h and is finite-dimensional. On the other hand, the LMI condition (3.8), which seems the finite-dimensional one at first sight, is infinite-dimensional in actual, since it requires to solve the infinite-dimensional equation (3.7) that depends on the time-delay h.

As shown in Theorem 3.1 and observed above, the Lyapunov functional (3.2) leads generally to infinite-dimensional and delay-dependent conditions or finite-dimensional and delay-independent conditions. In some special cases, however, our approach with a generalization of the functional (3.2) leads us to finite-dimensional and delay-dependent conditions. To illustrate this fact, consider the system with only distributed delay,

$$\begin{aligned} \dot{x}(t) &= \widetilde{A}_0 x(t) + \int_{-h}^{0} \widetilde{A}_{01}(\beta)\, x(t+\beta)\,\mathrm{d}\beta, \\ z(t) &= Cx(t), \end{aligned} \tag{3.9}$$

and consider the following functional,

$$V(x_t) = x'(t)\,Px(t) + \int_{-h}^{0} x'(t+\beta)\,Q(\beta)\,x(t+\beta)\,\mathrm{d}\beta. \tag{3.10}$$

Note that $Q(\beta)$ is here allowed to depend on β. Then calculating the time derivative of (3.10) and rearranging terms as in the proof of Theorem 3.1, we have a sufficient condition for $\frac{\mathrm{d}}{\mathrm{d}t}V(x_t) + z'(t)z(t) - \gamma^2 w'(t)\,w(t) < 0$, which is given as $Q(-h) \geq 0$ and

$$\begin{bmatrix} \widetilde{A}_0'P + P\widetilde{A}_0 + Q(0) & P\widetilde{A}_{01}(\beta) & PD \\ \widetilde{A}_{01}'(\beta)P & -h^{-1}\frac{\mathrm{d}}{\mathrm{d}\beta}Q(\beta) & 0 \\ D'P & 0 & -\gamma^2 I \end{bmatrix} < 0,$$

$$\forall\, \beta \in [-h,\ 0].$$

This LMI condition is the infinite-dimensional one. However, in the special case of $\widetilde{A}_{01}(\beta) = \widetilde{A}_{01}$, setting $Q(\beta) = (\beta + h)I$ yields the following finite-dimensional LMI condition of delay-dependence,

$$\begin{bmatrix} \widetilde{A}_0'P + P\widetilde{A}_0 + hI & P\widetilde{A}_{01} & PD \\ \widetilde{A}_{01}'P & -h^{-1}I & 0 \\ D'P & 0 & -\gamma^2 I \end{bmatrix} < 0. \tag{3.11}$$

Thus we obtain the next result.

Corollary 3.4. If there exists the positive definite matrix P which satisfies the LMI condition (3.11), then the time-delay system (3.9) with $\tilde{A}_{01}(\beta) = \tilde{A}_{01}$ is internally, asymptotically, stable and the L^2 gain is less than γ.

In [15], Li and DeSouza derived a finite-dimensional and delay-dependent LMI condition for robust stability and stabilization based on a Lyapunov functional. We can see that their LMI has a similar structure to (3.11), and expect that our framework described by (3.9) and (3.10) presents an essential point of their procedure consisting of a sophisticated system transformation and a special Lyapunov functional.

4. CONTROLLER SYNTHESIS

4.1. Synthesis of controller gain

Now we consider the synthesis of controllers which attain a prescribed level of L^2 gain of the closed loop system (3.1). The problem is to find a gain $(K_0, K_{01}(\beta))$ of the controller (2.2) based on the analysis result of Theorem 3.1.

Theorem 4.1. If there exist constant matrices W, X, Z_0 and continuously differentiable matrix function $Z_{01}(\beta)$ and $Y(\alpha,\beta)$ which satisfy the following inequalities,

$$L_3(\alpha,\beta) = \begin{bmatrix} \begin{pmatrix} WA_0' + WA_0 \\ +X + 2W \\ +BZ_0 + Z_0'B' \end{pmatrix} & A_1W - W & \begin{pmatrix} BZ_{01}(\beta) \\ +WA_0' \\ +Z_0'B' \\ +Y(0,\beta) \end{pmatrix} & WC' & D \\ WA_1' - W & -X & \begin{pmatrix} WA_1' \\ -Y(-h,\beta) \end{pmatrix} & 0 & 0 \\ \begin{pmatrix} Z_{01}'(\alpha)B' \\ +A_0W + BZ_0 \\ +Y(\alpha,0) \end{pmatrix} & \begin{pmatrix} A_1W \\ -Y(\alpha,-h) \end{pmatrix} & \begin{pmatrix} BZ_{01}(\beta) \\ +Z_{01}'(\alpha)B' \\ -(\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta}) \\ Y(\alpha,\beta) \end{pmatrix} & 0 & D \\ CW & 0 & 0 & -I & 0 \\ D' & 0 & D' & 0 & -\gamma^2 I \end{bmatrix} < 0, \quad (4.1)$$

$$L_4(\alpha,\beta) = \begin{bmatrix} W & W \\ W & Y(\alpha,\beta) \end{bmatrix} > 0, \quad (4.2)$$

$$X > 0, \quad (4.3)$$

$$\forall \alpha \in [-h,\ 0],\ \forall \beta \in [-h,\ 0],$$

then the time-delay system (2.1) with the state feedback controller (2.2)

$$K_0 = Z_0W^{-1}, \quad K_{01}(\beta) = Z_{01}(\beta)\,W^{-1}, \quad (4.4)$$

is internally, asymptotically, stable and the L^2 gain is less than γ.

Proof. Assume that the conditions of Theorem 4.1 are satisfied and consider the closed loop system (3.1) with the feedback gain given by (4.4). Then, using the inequalities (4.1), (4.2) and (4.3) together with Schur Complement, we can show that the inequalities (3.3), (3.4) and (3.5) admit the following solutions,

$$P = W^{-1}, \quad R(\beta) = W^{-1}, \quad S(\alpha, \beta) = W^{-1}Y(\alpha, \beta)W^{-1}, \quad Q = W^{-1}XW^{-1},$$

that is, the conditions of Theorem 3.1 are satisfied. Thus, Theorem 4.1 follows from Theorem 3.1. □

4.2. Constraint on controller gain

To simplify the discussion, we assume that the controlled output $z(t)$ in (2.1) does not directly depend on the control input $u(t)$. This may lead to large control inputs which are synthesized by Theorem 4.1. One conventional way to make such a possibility small is to impose some constraints on the feedback gain.

Now we constrain the feedback gain as follows,

$$K_0'K_0 < \gamma_1 I, \; K_{01}'K_{01}(\beta) < \gamma_2 I, \; {}^\forall \beta \in [-h, \; 0], \tag{4.5}$$

where γ_1 and γ_2 are given in advance, and consider the same synthesis problem as in Section 4.1. Based on Theorem 4.1, we have the following theorem.

Theorem 4.2. For given positive numbers p_1, p_2 and q, if there exist W, X, Z_0 and continuously differentiable matrix function $Z_{01}(\beta)$ and $Y(\alpha, \beta)$ which satisfy the following inequalities,

$$L_3(\alpha, \beta) < 0, \; L_4(\alpha, \beta) > 0, \; X > 0, \tag{4.6}$$

$$\begin{bmatrix} p_1 I & Z_0' \\ Z_0 & I \end{bmatrix} > 0, \tag{4.7}$$

$$\begin{bmatrix} p_2 I & Z_{01}'(\beta) \\ Z_{01}(\beta) & I \end{bmatrix} > 0, \tag{4.8}$$

$$\begin{bmatrix} qI & I \\ I & W \end{bmatrix} > 0, \tag{4.9}$$

$${}^\forall \alpha \in [-h, \; 0], \; {}^\forall \beta \in [-h, \; 0],$$

where $L_3(\alpha, \beta)$ and $L_4(\alpha, \beta)$ are given as (4.1) and (4.2) respectively, then the time-delay system (2.1) with the state feedback controller (2.2)

$$K_0 = Z_0 W^{-1}, \quad K_{01}(\beta) = Z_{01}(\beta) W^{-1}, \tag{4.10}$$

is internally, asymptotically, stable and the L^2 gain is less than γ. Here K_0 and $K_{01}(\beta)$ are constrained as follows,

$$K_0'K_0 < p_1 q^2 I, \quad K_{01}'(\beta)K_{01}(\beta) < p_2 q^2 I.$$

Proof. (4.7), (4.8) and (4.9) are equivalent to the following conditions respectively,

$$Z_0' Z_0 < p_1 I, \quad Z_{01}'(\beta)\, Z_{01}(\beta) < p_2 I, \quad W^{-1} < qI.$$

By using the above conditions, we have the following results,

$$\begin{aligned} K_0' K_0 &= W^{-1} Z_0' Z_0 W^{-1} \\ &< p_1 W^{-1} W^{-1} \\ &< p_1 q^2 I \\ K_{01}'(\beta) K_{01}(\beta) &= W^{-1} Z_{01}'(\beta)\, Z_{01}(\beta)\, W^{-1} \\ &< p_2 W^{-1} W^{-1} \\ &< p_2 q^2 I. \end{aligned}$$

□

Thus by using this theorem and choosing p_1, p_2 and q appropriately, we can obtain the controllers with feedback gains satisfying (4.5) and assuring $G < \gamma$. Next we show an algorithm to choose p_1, p_2 and q.

Algorithm:

Step 1: Let p_{10}, p_{20} and q_0 be initial values of p_1, p_2 and q respectively.

Step 2: Solve inequalities in Theorem 4.2 and the following inequalities,

$$p_1 < p_{10}, \quad p_2 < p_{20}, \quad q < q_0.$$

— If *Step 2* has no solution, the algorithm has no solution for the initial values p_{10}, p_{20}, q_0.

Step 3: Check the next conditions for p_1, p_2 and q of *Step 2.*

$$p_1 q^2 < \gamma_1, \quad p_2 q^2 < \gamma_2. \tag{4.11}$$

— If (4.11) is satisfied, the algorithm is finished. The controller designed in *Step 2* satisfies (4.5).

— If (4.11) is not satisfied, go back to *Step 1.*

When we come back from *Step 3* to *Step 1*, p_{10}, p_{20} and q_0 are generally modified into smaller ones, so that p_1, p_2 and q can be chosen smaller in *Step 2* and satisfy (4.11) in *Step 3.* Note that it is generally more difficult to solve the inequalities of Theorem 4.2 for smaller p_1, p_2 and q. The solvability condition of the inequalities of Theorem 4.2, which might be characterized by open loop properties, e. g. stabilizability, of the system (3.1), is our future task of interest.

To illustrate this algorithm, a design example is presented in Section 6. It is a matter of course that smaller gains $(K_0, K_{01}(\beta))$, which are realized by taking p_1, p_2 and q smaller, do not necessarily guarantee smaller control inputs. One possible way to handle constraints on control inputs such as $u(t)'u(t) \leq \mu$ is to introduce a step of state-reachable set analysis, which is characterized with infinite-dimensional LMIs [13]. As for the existing results on constrained control input, see Chapter 14 of [6] and references inside.

5. REDUCTION TO A FINITE NUMBER OF LMI CONDITIONS

Inequalities in Theorem 4.1 depend on parameters α and β. It seems difficult to solve these infinite-dimensional (parameter-dependent) inequalities directly. In our approach, we reduce these infinite-dimensional inequalities to a finite number of LMIs by using the technique in [3, 2], and obtain the solution of the infinite-dimensional inequalities by computing the finite number of LMIs.

Here we restrict solutions in Theorem 4.1 to the following forms,

$$\begin{aligned} Y(\alpha,\beta) &= Y_0 + g_1(\alpha,\beta)\,Y_1 + g_2(\alpha,\beta)\,Y_2 + \cdots + g_{l_Y}(\alpha,\beta)\,Y_{l_Y}, \\ Z_{01}(\beta) &= Z_0^{01} + h_1(\beta)Z_1^{01} + h_2(\beta)Z_2^{01} + \cdots + h_{l_z}(\beta)\,Z_{l_z}^{01}, \end{aligned} \tag{5.1}$$

where $g_i : R^2 \to R$ is a continuous differentiable function of α and β such that

$$g_i(\alpha,\beta) = g_i(\beta,\alpha),$$

$h_i : R \to R$ is a continuous differentiable function of β, and the unknown matrices satisfy

$$\begin{gathered} Y_i \in R^{n\times n},\ Y_i' = Y_i\ (i = 0,1,\cdots,l_Y), \\ Z_i^{01} \in R^{m_u\times n}\ (i = 0,1,\cdots,l_Z). \end{gathered}$$

Note that (5.1) satisfies matrix inequalities (4.1), (4.2). Then inequalities in Theorem 4.1 can be written in the form of the following parameter dependent LMI condition,

$$F_0(M) + f_1(\theta)\,F_1(M) + \cdots + f_r(\theta)\,F_r(M) < 0, \tag{5.2}$$

where

$$\theta \in \Theta = \{[\alpha\ \beta]' \mid \alpha \in [-h,\ 0], \beta \in [-h,\ 0]\},$$

and $f_i : R^2 \to R$ is a continuous function of α and β, and a symmetric matrix function F_i depends affinely on the unknown matrix $M = [Y_0, \cdots, Y_{l_Y}, Z_0^{01}, \cdots, Z_{l_Z}^{01}]$. The parameter dependent LMI condition (5.2) can be reduced to a finite number of LMI conditions as follows.

Theorem 5.1. [3] Let $\{p_1, p_2, \cdots, p_q\}$ be vertices of a convex polyhedron which includes the curved surface T,

$$T = \{[f_1(\theta)\ f_2(\theta)\ \cdots\ f_r(\theta)]' \mid \theta \in \Theta\}. \tag{5.3}$$

Assume that there exists M which satisfies the following LMI condition for all $p_i (i = 1, 2, \cdots, q)$,

$$F_0(M) + p_{i1} F_1(M) + \cdots + p_{ir} F_r(M) < 0, \tag{5.4}$$

where p_{ij} is the jth element of p_i. Then M satisfies (5.2) for all $\theta \in \Theta$.

A general technique to construct a convex polyhedron which includes the curved surface T is proposed in [3].

In the special case that $r = 2s$,

$$f_i(\alpha, \beta) = \begin{cases} f_i(\alpha), & i = 1, 2, \cdots, s, \\ f_i(\beta), & i = s+1, s+2, \cdots, 2s, \end{cases}$$

and $f_i(\alpha)$ and $f_i(\beta)$ are polynomial functions of α and β, respectively, we can use a simple technique to construct such a convex polyhedron, which is given by

Theorem 5.2. [12] Let $p^{ij} \in R^{2s}$ be defined such that

$$p^{ij} = \begin{bmatrix} p^i \\ p^j \end{bmatrix}, \quad i, j = 0, 1, \cdots, s,$$

where

$$\begin{aligned} p^0 &= [h_1 \ h_1^2 \ \cdots h_1^s]' \in R^s, \\ p^1 &= [h_2 \ h_1^2 \ \cdots h_1^s]' \in R^s, \\ &\vdots \\ p^s &= [h_2 \ h_2^2 \ \cdots h_2^s]' \in R^s. \end{aligned}$$

Then the convex polyhedron whose vertices are given by $p^{ij}, i, j = 0, 1, \cdots, s$ includes the curved surface $T = \{[\alpha \ \alpha^2 \ \cdots \ \alpha^s \ \beta \ \beta^2 \ \cdots \ \beta^s]' | \alpha \in [h_1, h_2], \beta \in [h_1, h_2]\}$

Actually taking $h_1 = -h$ and $h_2 = 0$ in Theorem 5.2, we have a desired convex polyhedron. To make the volume of the convex polyhedron smaller for less conservative solutions, we may divide the interval $[-h, 0]$ into sub-intervals $[h_a, h_b]$, $[h_c, h_d]$, $\cdots$, and apply repeatedly Theorem 5.2 in each sub-interval.

6. NUMERICAL EXAMPLE

Consider the next time-delay system,

$$\Sigma_p : \begin{aligned} & \dot{x}(t) = x(t) + 0.3x(t-1) + u(t), \\ & y(t) = x(t), \\ & x(\beta) = 0, \ -h \le \beta \le 0. \end{aligned} \tag{6.1}$$

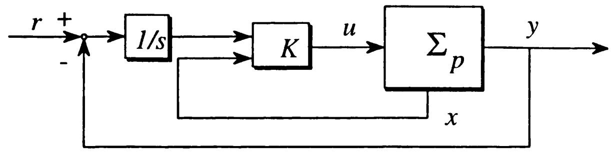


Fig. 1. The closed loop system.

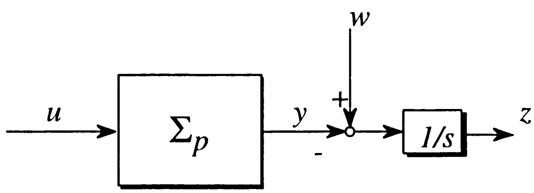


Fig. 2. Generalized plant.

Now we design a state feedback controller K of the form (2.2) such that the error, $r-y$, is asymptotically zero. As shown in Figure 1, an integrator is added in order to assure the asymptotically-zero error for step references. When we use the technique of Section 5, we restrict solutions of Theorem 4.1 and Theorem 4.2 as follows,

$$\begin{aligned} Z_{01}(\beta) &= Z_0 + \beta Z_1 + \beta^2 Z_2, \\ Y(\alpha,\beta) &= Y_0 + (\alpha+\beta)Y_1 + (\alpha^2+\beta^2)\,Y_2. \end{aligned}$$

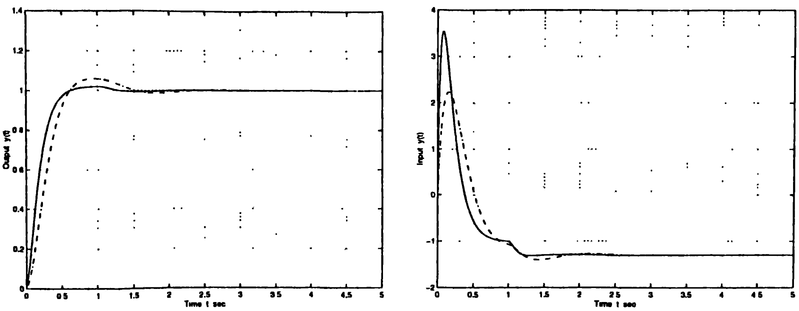
First we apply Theorem 4.1 to Figure 2 and obtain the state feedback controller with the next feedback gains,

$$\begin{aligned} K_0 &= [\ 115.48 \quad -24.94\ ], \\ K_{01}(\beta) &= [\ 75.79 \quad -12.45\ ] + \beta\,[\ -9.31 \quad -3.09\ ] + \beta^2\,[\ 15.14 \quad -3.53\ ]. \end{aligned} \tag{6.2}$$

Second setting $p_1 = 3.49 \times 10^4$, $p_2 = 1.28 \times 10^2$, $q = 2.56$ and using Theorem 4.2, we obtain the state feedback controller (2.2) with the next feedback gains,

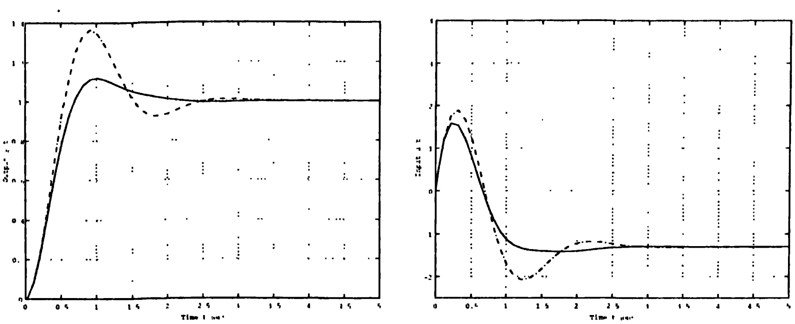
$$\begin{aligned} K_0 &= [\ 36.16 \quad -11.74\ ], \\ K_{01}(\beta) &= [\ 23.49 \quad -4.01\ ] + \beta\,[\ 1.71 \quad -0.53\ ] + \beta^2\,[\ -0.07 \quad -0.19\ ]. \end{aligned} \tag{6.3}$$

The simulation results are shown in Figure 3, where the reference is 1 ($r = 1$). In this figure, the solid line and the dashdot line denote the simulation result of the case (6.2) and (6.3) respectively. The error $r-y$ is asymptotically zero at both cases. Note that the asymptotically-zero error is assured for arbitrary L^2 type references, since both feedback schemes provide finite L^2 gain from reference r to error $r-y$.



(a) Output y (b) Input u

Fig. 3. Simulation result (Memory feedback case).



(a) Output y (b) Input u

Fig. 4. Simulation result (Memoryless feedback case).

We see also that, by using Theorem 4.2, we can make the maximum of the control input small.

Our approach with corresponding specializations, which is equivalent to the approach of [14] for unconstrained gain case, provides memoryless controllers. On this example, the feedback gain for unconstrained gain case is calculated as

$$K_0 = \begin{bmatrix} 14.49 & -5.27 \end{bmatrix}, \tag{6.4}$$

and the feedback gain for the constrained gain case is calculated as

$$K_0 = \begin{bmatrix} 12.94 & -3.20 \end{bmatrix}. \tag{6.5}$$

The simulation results are shown in Figure 4, where the solid line and the dashdot line denote the result of (6.4) and (6.5) respectively. Compared with the results shown in Figure 3, we see worse tracking properties for both cases. We also see that the maximum of the control input given by (6.5) is larger than that of the control input given by (6.4), that is, in the memoryless feedback case, our algorithm cannot succeed in making the maximum control input small.

7. CONCLUSION

In this paper, we considered L^2 gain analysis and control synthesis problems for linear systems with time-delay via an LMI approach. We derived conditions for analysis and synthesis in the form of infinite-dimensional LMIs and showed a technique to reduce the infinite-dimensional LMIs to a finite number of LMIs which provide feasible formulas. We demonstrated the efficacy of our approach by a numerical example.

The LMI approach presented in this paper requires the exact value of the time-delay h. This may make us anxious that the constructed controller is sensitive to any variation of time-delay. However, the closed loop system which is formed by the controller of Theorem 4.1 is robustly stable against sufficiently small variation of time-delay, which is discussed in [1].

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Mr. Kojiro Ikeda, Dr. Kenko Uchida, Department of Electrical, Electronics and Computer Engineering, Waseda University, Okubo 3-4-1, Shinjuku, 169-8555, Tokyo. Japan. e-mail:ikeda@uchi.elec.waseda.ac.jp

Dr. Takehito Azuma, Department of Electrical and Electronic Engineering, Faculty of Engineering Kanazawa University, Kodatsuno 2-40-20, Kanazawa, 920-8667. Ishikawa. Japan.