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DELAY DIFFERENTIAL SYSTEMS WITH TIME–VARYING DELAY: NEW DIRECTIONS FOR STABILITY THEORY[1]

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In this paper we give an example of Markus–Yamabe instability in a constant coefficient delay differential equation with time-varying delay. For all values of the range of the delay function, the characteristic function of the associated autonomous delay equation is exponentially stable. Still, the fundamental solution of the time-varying system is unbounded. We also present a modified example having absolutely continuous delay function, easily calculating the average variation of the delay function, and then relating this average to earlier work of the author on preservation of the stability exponent in delay differential equations with time-varying delay. In this way we suggest one possible viewpoint on the conditions for Markus–Yamabe instability. Finally, we give a very brief sketch of an example of quenching of instability. To suggest a view on conditions for quenching phenomena, we relate this to earlier work of Cooke on preservation of spectral dynamics in delay systems having time-varying delay.

1. INTRODUCTION

In this paper we demonstrate a new instability phenomenon involving delay systems having time-varying delays. Following this we give an outline of the workings of a quenching phenomenon which is presented in detail elsewhere [9]. Connections with sampled data control will be very valuable, and our main theorems are preceded by some formulas establishing connections between sampled data control theory and delay differential equations with time-varying delays. We will try to convince the reader that the street runs two ways, i. e. that sampled data control theory has interesting techniques and insights to offer the area of delay differential equations, and that sampled data systems can often be recast in the language of delay equations having time-varying delays.

The notion of time dependence in the delay as something which has importance in itself has not fared well in the control literature. When usually considered, time-varying delays are investigated in the context of stability robustness. One starts

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with a time-invariant system of linear or nonlinear type, and then using some kind of Lyapunov or operator theory [19, 20] or classical analysis [17], one finds bounds such that perturbations with time-varying delay, if kept within these bounds, will not destabilize the system. Sun et al [19] use a state transformation along with Lyapunov analysis to examine the stability of a linear delay system with nonlinear delay perturbation. In this way they manage to reduce allowable conservatism of the nonlinear system. Verriest [20] considers a linear delay system with time-varying coefficients and time-varying delay. It is shown that a certain time-varying candidate Lyapunov functional leads naturally to a Riccati differential equation which is known to occur in robust control theory, and in fact a variety of robustness results immediately follow. Niculescu et al [17] have considered nonlinear delay systems with several time-varying delays bounded uniformly by a single function $h(\cdot)$. Rather than engage in Lyapunov analysis, they work more directly with the matrix for the current value of $x(\cdot)$. Via ordinary differential equations techniques, they derive an exponential bound for $|x(t)|$, where the exponent is time-varying of the form $-\sigma \cdot \int_0^t \frac{d\theta}{h(\theta)}$, and σ is the unique positive solution of a certain scalar transcendental equation. All these approaches have their value for stability analysis. In later sections we will try to convince the reader of the interest of a very different view on time-delay time dependence in control.

We begin in Section 2 with a look at some of the author's previous work on preservation of the stability exponent under time-delay time dependence [12]. Looking at equations of the form $x'(t) = A_0x(t) + A_1x(t - h(t))$, the author began with the hypothesis that for every member of the range of $h(\cdot)$, the characteristic function $f_\eta(s) = |sI - A_0 - A_1e^{-s\eta}|$ has all zeros in the set $\{\mathrm{Re}(s) < \gamma\}$. Given a compactness condition and some other bounds to be satisfied, it was shown that there is a time-varying positive definite quadratic functional G, defined along system trajectories, satisfying a bound of the form $G_t \leq G_0e^{t(2\gamma+Ba(t))}$ for $t > 0$, where $a(t)$ is the average variation of the delay function over the interval $[0, t]$. This immediately led to a similar exponential bound for $|x(t)|^2$, as well as some interesting corollaries.

In Section 3 we present formulas showing that typical sampled data systems with control input can be viewed as delay differential equations with periodic delays. Although they did not express their formulas in terms of periodic delays and sampled data control systems, Cooke and Wiener [3] have given formulas for the solutions of the kind of delay differential equation we are interested in. We will see the value of these formulas in the section which follows.

In Section 4 we consider delay systems of the form $x'(t) = a_0x(t) + a_1x(t - h(t))$, having range$(h(\cdot))$ contained in a compact interval for which the system characteristic function $f_\eta(s) = s - a_0 - a_1e^{-s\eta}$ is asymptotically stable. Using a formula from Section 3, we give a new example of such a delay system which has unbounded fundamental solution. This type of instability is frequently referred to as Markus–Yamabe instability, recalling a similar instability in the area of time-varying ordinary differential equations presented by Markus and Yamabe [14]. The first example we give, owing to the connection with sampled data control, will have discontinuous $h(\cdot)$. We continue in this section by proving that the example can be modified to have continuous $h(\cdot)$. We conclude Section 4 by considering the instability with continuous delay in light of the growth bounds previously given by the author in terms of the average variation of the delay function, described in Section 2.

In Section 5 we begin with an outline of an example of quenching of instability in a second order delay differential equation of the form $x''(t)+a_1x'(t-h(t))+a_0x(t)=0$. For each member of range($h(\cdot)$), the characteristic function has an exponentially unstable zero. Even so, the formulas of Section 3 allow one to see that this delay equation is exponentially stable. The details of the proof, as well as the proof that quenching can also occur with continuous delay, are given in [9]. We conclude Section 5, and the paper, by considering how an early paper of Cooke [2] can give us an idea of where to look, and where not to look, for examples of quenching. In this paper, on delay equations which Cooke refers to as asymptotically autonomous, it is shown that the spectral dynamics of many autonomous delay equations are preserved under certain time-varying delay perturbations, and under these conditions there will be no quenching phenomena.

2. STABILITY, THE DELAY FUNCTION'S RANGE AND AVERAGE VARIATION, AND LYAPUNOV THEORY

In this section we introduce the reader to research on preservation of the growth exponent in differential equations having time-varying delays. As a vantage point, we will use the maximum growth exponent found in a collection of linear autonomous delay systems parametrized by the delay. We will consider the delay differential system

$$x'(t) = A_0x(t) + A_1x(t-h(t)) \tag{1}$$

having time-varying delay, where A_0, $A_1 \in \mathbb{R}^{n\times n}$. For any real γ, we let $H_\gamma = \{\eta \geq 0 : f_\eta(s) = |sI - A_0 - A_1e^{-s\eta}|$ has no complex zeros with $\mathrm{Re}(s) \geq \gamma\}$. The author has shown [12] that if $h(\cdot)$ varies at a slow enough rate within some compact subset of H_γ, then this growth or decay exponent γ will be preserved within positive ε in the time-varying system (1). The primary idea of this line of research has been that the information on the stability abscissa coming from the characteristic function can be recast in terms of a very precise Lyapunov functional. This functional has its roots in the theory of autonomous linear delay equations [4, 6, 18], and still has some interest there. Furthermore, it is flexible enough so that a modification gives it a valuable place in the theory of time-varying systems [12, 13].

We will try to present the idea of this Lyapunov functional without spending too much effort on somewhat complicated technical details. Consider the autonomous delay equation

$$x'(t) = A_0x(t) + A_1x(t-\eta). \tag{2}$$

Recalling the notation x_t for the function $x_t(u) = x(t+u)$, $-\eta \leq u \leq 0$, the initial data for (2) is then $x_0 = \phi$, where $\phi \in C[-\eta, 0]$, the space of vector functions which are continuous over $[-\eta, 0]$. To emphasize initial data, we often refer to the solution $x(\cdot)$ as $x(\phi, t)$.

If $\eta \in H_\gamma$, then there is some positive ε with all solutions $x(\cdot)$ of (2) exponentially bounded with exponent no greater than $\gamma - \varepsilon$. In this case we note that $y(\phi, t) =$

$e^{-\gamma t}x(\phi,t)$ decays exponentially, and we define $V_1(\phi) = \int_0^\infty y^T(\phi,u)\,Wy(\phi,u)\,du$ for any positive definite matrix W. We note that $V_1(x_t) = \int_0^\infty y^T(x_t,u)\,Wy(x_t,u)\,du$, and changing variables, this is written $V_1(x_t) = e^{2\gamma t}\cdot \int_t^\infty x^T(u)\,Wx(u)\,e^{-2\gamma u}\,du$, from which one sees that $\frac{d}{dt}(V_1(x_t)) = 2\gamma V_1(x_t) - x^T(t)\,Wx(t)$. This simple relation is the same as found in linear ordinary differential equations, i.e. in the case $A_1 = 0$.

There are remainder terms which occur in carrying the analysis over to the case of time-varying delay. To make adjustments, the author settled on $V(\phi) = V_1(\phi) + \phi^T(0)\,M\phi(0) + e^{-\gamma\eta}\cdot \int_{-\eta}^0 \phi^T(u)\,R\phi(u)\,e^{-2\gamma u}\,du$ for positive symmetric matrices W, M, R, a functional introduced by Infante and Castelan [6]. When emphasizing dependence on η, we write $V = V(\eta,\phi)$. Upon calculating $\frac{d}{dt}(V(x_t))$ and making some rearrangements, one will find with well chosen $W = k_W I$, $R = k_R I$, $M = k_M I$ that $\frac{d}{dt}(V(x_t)) \le 2\gamma V(x_t)$ for all $t \ge 0$, which leads after some analysis to $V(x_t) \le V(x_0)\,e^{2\gamma t}$ along solutions $x(t)$ of (2) for all $t \ge 0$. In this form one can see why this quadratic functional would be interesting when extended to time-varying systems.

For a discussion of explicit calculations of the values of this functional in cases of engineering interest, or of the interest this has in feedback stabilization, there is now a reasonably varied literature [6, 7, 11, 15]. We concentrate here on adjusting this functional for the analysis of time-varying systems.

Beginning with a nonnegative function $h(t)$ and a system (1) $x'(t) = A_0x(t) + A_1x(t-h(t))$, we consider the time-varying functional $G(t,x_t) = V(h(t),x_t)$, defined along solutions $x(\cdot)$. Since we will want to differentiate $h(\cdot)$, we will take $h(\cdot)$ to be absolutely continuous. After long calculations, one finds [12] that the derivative along system trajectories is equal to $2\gamma G$ plus a remainder functional, i.e. $\frac{d}{dt}(G(t,x_t)) = 2\gamma G(t,x_t) + E(h(t),\,h'(t),x_t) + h'(t)\,F(h(t),x_t)$, where $E(\eta,\eta',\phi)$ is a quadratic form in $[\,\phi(0)\ \phi(-\eta)\,]$ which depends on the values of η, η', and $F(\eta,\phi)$ is a quadratic functional in ϕ which depends on η. After detailed analysis [12], the following was proven about the remainder $E + \eta' F$.

Theorem 2.1. Let γ be real, let D be a compact subset of H_γ, and let $d_0 = \sup D$. Then there exist μ_1, μ_2 with $-1 < \mu_1 < 0 < \mu_2$, positive symmetric $M = k_M I$, $R = k_R I$, $W = k_W I$, and positive B, all constant, with the following property: for $\eta \in D$, $\mu_1 \le \eta' \le \mu_2$, $\phi \in C[-d_0,0]$, one has $E(\eta,\eta',\phi) + \eta' F(\eta,\phi) \le B|\eta'|\,V(\eta,\phi)$.

Noting the constraints on $h'(t)$ given by μ_1, μ_2, we let $S_\gamma(D,\mu_1,\mu_2)$ be the class of all absolutely continuous $h(\cdot)$, with domain $[0,\infty)$ and range contained in D, having $\mu_1 \le h'(t) \le \mu_2$ a.e. in Lebesgue measure over $[0,\infty)$. A number of stability conclusions follow from Theorem 2.1. We give a sample below [12].

Theorem 2.2. Let D be a compact subset of H_γ, and take the constants as in Theorem 2.1. Let $h(\cdot)$ be any member of $S_\gamma(D,\mu_1,\mu_2)$, and consider the delay equation (1) $x'(t) = A_0x(t) + A_1x(t-h(t))$. For each solution $x(\cdot)$ of (1), let $G(t,x_t) = V(h(t),x_t)$. Then

a) $\frac{d}{dt}(G(t,x_t)) \le 2\gamma G(t,x_t) + B|h'(t)|\,G(t,x_t)$ a.e. for $t \ge 0$.

b) $G(t, x_t) \leq G(0, x_0)\, e^{f(t)}$ for all $t \geq 0$, where $f(t) = 2\gamma t + B \int_0^t |h'(\tau)|\, \mathrm{d}\tau$.

c) $G(t, x_t) \leq G(0, x_0)\, e^{t(2\gamma + Ba(t))}$ for $t > 0$, where $a(t) = \frac{1}{t} \int_0^t |h'(\tau)|\, \mathrm{d}\tau$.

d) $k_M |x(t)|^2 \leq G(0, x_0)\, e^{t(2\gamma + Ba(t))}$ for $t > 0$, with $a(t)$ as given in c).

From d) we easily deduce system asymptotic stability if $\gamma < 0$ and $\zeta = \limsup_{t\to\infty} a(t)$ satisfies $B\zeta < 2|\gamma|$. Finally, it is worth noting that if $h(\cdot)$ has finite variation over $[0, \infty)$, i. e. if $\int_0^\infty |h'(t)|\, \mathrm{d}t = \kappa$ is finite, then we can set $B_0 = e^{B\kappa}$ with B as in the theorem, and immediately see that $G(t, x_t) \leq B_0 G(0, x_0)\, e^{2\gamma t}$ for $t \geq 0$.

One can not escape observing the part played by the average variation of $h(\cdot)$ over $[0, t]$, i. e. by $a(t)$ above. In a subsequent section we will see that counterexamples to stability are possible when $a(t)$ adheres to moderate rather than small values. There are still research questions on the topic of Lyapunov approaches to preservation of stability exponents in delay equations with time-varying delay. One of the most useful, and most difficult, is the question of finding a simple estimate for B in Theorems 2.1, 2.2. In the author's work this constant is constructed from $||A_1||$, from γ, from $\sup D$, from a maximum norm of the solution of a matrix functional equation which serves as a kernel for the quadratic functional V_1, and from k_M, k_R, k_W. The characterization of B makes the question of how to calculate B obvious but not very practical. Now that we know the part played by $a(t)$, it would be valuable to have a more practical way, implicitly or explicitly, to relate the stability of the time-varying system to the average variation of the delay function.

Before proceeding to the next section, the author is eager to mention that there are now known methods for finding the set H_0, i. e. the set H_γ with $\gamma = 0$. This gives us a new incentive to consider the idea of preservation of the stability exponent in time-varying systems. It must be warned that some of the methods are practical only for low order systems, but this situation is changing.

Several authors have made recent contributions. Marshal et al [15] have given a polynomial elimination technique that works well for low order delay equations. Chen et al [1] have given a matrix based technique for finding H_0. In an $n \times n$ retarded system with q commensurate delays, this method involves finding the eigenvalues of a $2n^2q \times 2n^2q$ matrix as well as the unitary generalized eigenvalues of a $qn \times qn$ matrix pencil. Also focusing on retarded systems, using matrices of comparable size, Niculescu [16] has presented a pair of matrix pencils which are checked for generalized eigenvalues on the unit circle, from which the structure of the set H_0 is deduced.

It is worthwhile to mention here that any method for determining H_0 immediately yields a method for determining H_γ for any real γ. First note that for any complex function $f(s)$, the zeros of $g(s) = f(s + \gamma)$ are the translates by $-\gamma$ of the zeros of $f(s)$. For the linear delay equation (2) $x'(t) = A_0 x(t) + A_1 x(t - \eta)$, we have $f_\eta(s) = |sI - A_0 - e^{-s\eta} A_1|$ and $f_\eta(s + \gamma) = |sI - (A_0 - \gamma I) - (e^{-\gamma\eta} A_1)\, e^{-s\eta}|$. The system $x'(t) = (A_0 - \gamma I)\, x(t) + e^{-\gamma\eta} A_1 x(t - \eta)$ has eigenvalues in $\{\mathrm{Re}(s) \geq 0\}$ if and only if (2) $x'(t) = A_0 x(t) + A_1 x(t - \eta)$ has eigenvalues in $\{\mathrm{Re}(s) \geq \gamma\}$. In this paper we will only calculate H_0 for low order delay systems.

3. BASIC FORMULAS

In this section we give some formulas showing that the most commonly encountered sampled data control systems can be represented as delay differential systems with a certain type of time-varying delay. The delay function will be piecewise linear and periodic. We will see the benefit of this in the following section, where the simplicity of sampled data formulas [8] guides us in our investigation of the effect of time-varying delays.

Let us consider the effect of sampling in a linear control system, holding samples from one or several discrete time units in the past. We consider a sample holding time of α, so that inputs are then sampled at times $t_k = k\alpha$. Let A be the matrix for the system free dynamics, and for $r = 0, \dots, m$, let B_r be control matrices. Sampling up to $m\alpha$ units in the past, we write $x'(t) = Ax(t) + \sum_0^m B_r u(t_{k-r})$ for $t_k \le t < t_{k+1}$.

Now write $x'(t) - Ax(t) = c$ with $c = \sum_0^m B_r u(t_{k-r})$. Multiplying both sides by $M(t) = e^{-tA}$, we have $\frac{\mathrm{d}}{\mathrm{d}t}(M(t)\,x(t)) = M(t)\,c$. Integrating and multiplying all terms by e^{tA}, we have $x(t) - e^{(t-t_k)\,A}x(t_k) = \int_{t_k}^t e^{(t-\tau)\,A}\mathrm{d}\tau \cdot c = S_{t-t_k}(A) \cdot c$ with $S_\beta(A) = \int_0^\beta e^{uA}\,\mathrm{d}u$. We now write this with the vector c decomposed.

Lemma 3.1. Consider the sampled input equation $x'(t) = Ax(t) + \sum_0^m B_r u((k-r)\,\alpha)$ over the interval $k\alpha \le t < (k+1)\,\alpha$. The solution is given by the following formula with $t_q = q\alpha : x(t) = e^{(t-t_k)\,A}x(t_k) + S_{t-t_k}(A)\sum_0^m B_r u(t_{k-r})$.

In the case that A is invertible, the solution $x(t)$ can be written as follows:

$$x(t) = e^{(t-t_k)\,A}x(t_k) + \left(e^{(t-t_k)\,A} - I\right) A^{-1} \sum_0^m B_r u(t_{k-r}).$$

We can gain insight by relating these sampled data notions to the idea of periodic delay. This will be useful in the following sections, where we consider delay differential equations having free dynamics, as in the case of direct control using past inputs of the system state. For any positive α, consider the sawtooth function $h(\cdot)$ given by $h(t) = t$ for $0 \le t < \alpha$, with $h(\cdot)$ extended periodically for $t \ge \alpha$, i.e. $h(t) = t - k\alpha$ for $k\alpha \le t < (k+1)\,\alpha$, with k any nonnegative integer. We set $h_0(t) = h(t)$, and $h_r(t) = r\alpha + h(t)$ for $r = 1, \dots, m$. For $k\alpha \le t < (k+1)\,\alpha$, we have $t - h_r(t) = (k-r)\,\alpha = t_{k-r}$. Thus the above lemma could be expressed equally well in terms of delay equations with t_{k-r} replaced by $t - h_r(t)$.

We let $x_k = x(k\alpha)$, and set $B_r = A_r$ for $r = 0, \dots, m$. Then Lemma 3.1 gives us the following formula for the solution $x(t)$ to $x'(t) = Ax(t) + \sum_0^m A_r x(t - h_r(t))$, which is really just a reworking, in delay equation terms, of the formula for a sampled system with data held for time duration α, sensitive to information $m\alpha$ time units in the past, and controlled from its present and past states.

Lemma 3.2. Consider the delay differential equation $x'(t) = Ax(t) + \sum_0^m A_r x(t - h_r(t))$ with the above notation. For $t_k \le t < t_{k+1}$, the solution satisfies the following

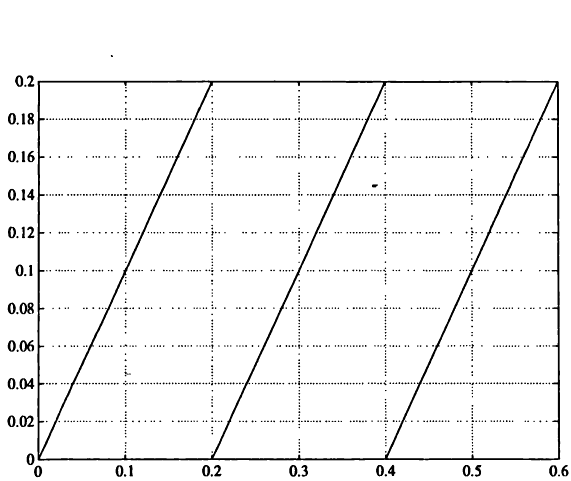

Fig. 3.1. Three periods of $h(t)$ with $\alpha = .2$.

formula:

$$x(t) = \left(e^{(t-t_k)\,A} + S_{t-t_k}(A)\,A_0\right) x_k + S_{t-t_k}(A) \sum_1^m A_r x_{k-r}.$$

In the case that A is invertible, the solution $x(t)$ can be written as follows:

$$x(t) = \left(e^{(t-t_k)\,A}(I + A^{-1}A_0) - A^{-1}A_0\right) x_k + \left(e^{(t-t_k)\,A} - I\right) A^{-1} \sum_1^m A_r x_{k-r}.$$

Since $e^{\beta A}$ and $S_\beta(A)$ are both bounded over $[0, \alpha]$, we can conveniently deduce the stability or instability of the above delay equation directly from the behavior of the difference equation for x_k. Noting that $x_k = x(k\alpha)$, we have $x_{k+1} = \sum_0^m L_r x_{k-r}$ with $L_0 = e^{\alpha A} + S_\alpha(A)\,A_0$ and $L_r = S_\alpha(A)\,A_r$ for $r > 0$. The solutions of this difference equation will converge geometrically to zero if and only if the solutions of the delay differential equation converge exponentially to zero.

Although the above difference equation is all one needs to know the stability of the delay equation, the behavior of the matrices $e^{\beta A}$ and $S_\beta(A)$ for $0 < \beta < \alpha$ still does have importance in practical control, since they tell us the intersample behavior, which is used for performance analysis [8].

In a paper on differential equations with piecewise continuous arguments, Cooke and Wiener [3] have given formulas which reduce to the above in our case. An interesting feature of their approach is that both forward and delay effects can be included, i. e. information coming from advance functions of the form $h_{-r}(t) = -r\alpha + h(t)$ can be included in their approach.

4. AN INSTABILITY COUNTEREXAMPLE

In this section we present a counterexample of Markus–Yamabe type for delay differential equations with constant coefficients and time-varying delays. For each fixed member of the range of the delay function, the associated autonomous delay system will be exponentially stable, and yet the time-varying system considered will have its fundamental solution unbounded. This will be especially surprising in light of the order of the system, since the example given will be first order. We hope in this way to encourage the notion that time-delay time dependence can reasonably be considered in its own right.

We begin with a look at the associated autonomous system used in the counterexample.

Lemma 4.1. Let $a = -1$, $a_0 = -1.5$, $\eta^+ = 2.05$. Then the system

$$x'(t) = ax(t) + a_0 x(t - \eta) \tag{3}$$

is exponentially stable for each $\eta \in [0, \eta^+]$.

Proof. For system characteristic function we have $f_\eta(s) = s + 1 + 1.5e^{-s\eta}$, so that $f_\eta(s) = s + 2.5$ for $\eta = 0$, and the system (3) is stable with zero delay. Thus we know [5] that either a) the system (3) is exponentially stable for all $\eta \geq 0$, or b) for $\eta' = \min\{\eta \geq 0 : f_\eta(s)$ has an imaginary axis zero$\}$, the system (3) is exponentially stable for each $\eta < \eta'$. Now $f_\eta(i\omega) = i\omega + 1 + 1.5e^{-i\omega\eta} = 1 + 1.5\cos(\omega\eta) + i(\omega - 1.5\sin(\omega\eta))$. Setting the real part equal to zero, we find that $-\frac{2}{3} = \cos(\omega\eta)$, so we must have $\sin(\omega\eta) = \pm\frac{1}{3}\sqrt{5}$ for a zero of $f_\eta(i\omega)$ to exist. Now setting the imaginary part equal to zero, we find that $\omega = 1.5\sin(\omega\eta) = \pm\frac{1}{2}\sqrt{5}$, which in turn gives us $-\frac{2}{3} = \cos\left(\frac{1}{2}\eta\sqrt{5}\right)$. The minimum nonnegative value of η satisfying this last equation is $\eta = \frac{2}{\sqrt{5}}\arccos\left(-\frac{2}{3}\right)$. Thus $\eta' \geq \frac{2}{\sqrt{5}}\arccos\left(-\frac{2}{3}\right)$.

Now using these values $\omega = \pm\frac{1}{2}\sqrt{5}$ and $\eta = \frac{2}{\sqrt{5}}\arccos\left(-\frac{2}{3}\right)$, we will find that $f_\eta(i\omega) = 0$, so that $\eta' = \frac{2}{\sqrt{5}}\arccos\left(-\frac{2}{3}\right)$. Finally, note that $\eta' \approx 2.05765$, and certainly $\eta' > 2.05$. □

The instability of the time-varying system that we use for the counterexample will be easiest to see using the formula from the simplest case of Lemma 3.2 of the preceding section. The delay function is simply given by $h(t) = t - t_k$ for $t_k \leq t < t_{k+1}$, with $t_k = k\alpha$. Writing $x_k = x(t_k)$, we have $x_{k+1} = \left(e^{a\alpha}(1 + a^{-1}a_0) - a^{-1}a_0\right)x_k$, so that exponential stability of the time-varying system $x'(t) = ax(t) + a_0x(t - h(t))$ is equivalent to $|T| < 1$ with $T = e^{a\alpha}(1 + a^{-1}a_0) - a^{-1}a_0$.

Lemma 4.2. Again let $a = -1$, $a_0 = -1.5$. Then for $T = e^{a\alpha}(1 + a^{-1}a_0) - a^{-1}a_0$, we have $|T| > 1$ for $\alpha > \ln(5)$.

Proof. We have $T = 2.5e^{-\alpha} - 1.5$. Thus $\frac{dT}{d\alpha} < 0$. Now $T = 1$ at $\alpha = 0$, and $T \downarrow -1.5$ as $\alpha \uparrow \infty$. Solving for $T = -1$, we have $2.5e^{-\alpha} = 0.5$, so that $e^\alpha = 5$, and $\alpha = \ln(5)$. Thus $T < -1$ for $\alpha > \ln(5)$. □

Noting that $\ln(5) \approx 1.6094 < 2.05 = \eta^+$, we immediately obtain the counterexample.

Theorem 4.3. Consider delay differential equations of the form (1) $x'(t) = ax(t) + a_0x(t - h(t))$, where $h(t) = t - k\alpha$ over the interval $[k\alpha, k\alpha + \alpha)$ for integers $k \geq 0$. There exist $\eta^+ > 0$ and real a, a_0 making the system (3) $x'(t) = ax(t) + a_0x(t - \eta)$ exponentially stable for each $\eta \in [0, \eta^+]$, and yet having an interval J contained in $(0, \eta^+)$ giving (1) unbounded fundamental solutions for every $\alpha \in J$. Particularly, for $a = -1$, $a_0 = -1.5$, and $\eta^+ = 2.05$, any choice of the constant $\alpha \in (\ln(5), \eta^+)$ will give (1) an unbounded fundamental solution.

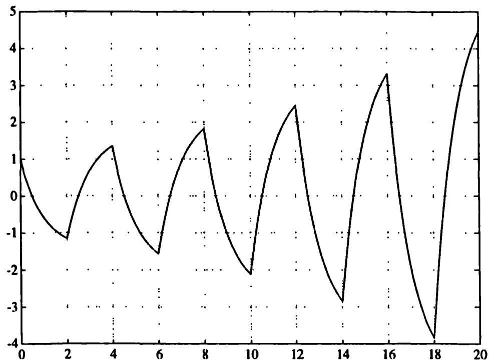

Fig. 4.1. The solution to $x'(t) = -1x(t) - 1.5x(t - h(t))$, $x(0) = 1$ with $h(t)$ as in Theorem 4.3 and $\alpha = 2$.

Now that we have established the instability theorem for a piecewise continuous delay function $h(t)$, it is straightforward although somewhat tedious to give a theorem for the case of a continuous delay. For this purpose we have the following lemma giving a bound for solutions of equations having constant coefficients and time-varying delays. To encourage further exploration of the ideas in this paper, we give the lemma for matrix delay systems. The notation $||\cdot||$ is used for the operator norm of a square matrix, i. e. $||F|| = \sup_{|x|=1} |Fx|$. The proofs of a), b) below, which use contradiction arguments, are not given here, since they are provided in [9].

Lemma 4.4. Consider (1) $x'(t) = Ax(t) + A_0x(t - h(t))$, where $h_s = \sup_{0 \leq t \leq r} h(t)$ and $h(t)$ is piecewise continuous over $[0, r]$ and uniformly nonzero over $[0, r)$. Then with $0 < C = \sup_{-h_s \leq u \leq 0} |x(u)|$, $L = ||A|| + ||A_0||$ we have both a), b) for $0 \leq t \leq r$:

a) $|x(t)| \leq Ce^{Lt}$,

b) $|x(t) - x(0)| \leq C(e^{Lt} - 1)$.

To construct the appropriate continuous delay function, we again let $\alpha \in (\ln(5), 2.05)$, set $t_k = k\alpha$ and $x_k = x(t_k)$, let ε be contained in $(0, \alpha)$, and consider the delay function $h_\varepsilon(t)$ defined as follows: $h_\varepsilon(t) = t$ for $0 \leq t < \alpha - \varepsilon$, $h_\varepsilon(t) = \frac{\varepsilon - \alpha}{\varepsilon}(t - \alpha)$ for

$\alpha - \varepsilon \le t < \alpha$ and $h_\varepsilon(t) = h_\varepsilon(t - t_k)$ for $t_k \le t < t_{k+1}$, where k is any positive integer. To put this geometrically, the graph of $h_\varepsilon(t)$ for $0 \le t \le \alpha$ connects the points $(0,0)$ and $(\alpha-\varepsilon, \alpha-\varepsilon)$ by a line segment, and the points $(\alpha-\varepsilon, \alpha-\varepsilon)$ and $(\alpha, 0)$ by another, and $h_\varepsilon(\cdot)$ is extended periodically for $t > \alpha$.

Theorem 4.5. Let $\eta^+ = 2.05$ and let $\alpha \in (\ln(5), \eta^+)$. Consider the delay differential equation

$$x'(t) = -1x(t) - 1.5x(t - h_\varepsilon(t)), \tag{4}$$

where $h_\varepsilon(\cdot)$ is as given immediately above. The system

$$x'(t) = -1x(t) - 1.5x(t - \eta) \tag{5}$$

is again exponentially stable for each $\eta \in [0, \eta^+]$, yet there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, the fundamental solution of (4) is unbounded.

Proof. Set $a = -1$, $a_0 = -1.5$, and $L = |a| + |a_0| = 2.5$. Take ε with $0 < \varepsilon < \alpha$. Recalling Lemma 3.2, we have $x(t) = R_t x(t_k)$ for $t_k \le t \le t_{k+1} - \varepsilon$, where

$$R_t = 2.5e^{-(t-t_k)} - 1.5.$$

Since

$$|x(t_{k+1}) - x(t_{k+1} - \varepsilon)| \le (e^{L\varepsilon} - 1)\left(\max_{k\alpha \le t \le k\alpha + \alpha - \varepsilon} |x(t)|\right),$$

we have

$$|x(t_{k+1}) - x(t_{k+1} - \varepsilon)| \le (e^{L\varepsilon} - 1)\left(\max_{k\alpha \le t \le k\alpha + \alpha - \varepsilon} |R_t|\right)(|x_k|).$$

Now $\frac{\mathrm{d}R_t}{\mathrm{d}t} < 0$ for $t_k \le t \le t_{k+1} - \varepsilon$, so that R_t decreases over the interval $J_k(\varepsilon) = [t_k, t_{k+1} - \varepsilon]$, and we have

$$R_{k\alpha} = 1 = \max_{J_k(\varepsilon)} R_t > R_{k\alpha+\alpha-\varepsilon} = \min_{J_k(\varepsilon)} R_t = 2.5e^{-(\alpha-\varepsilon)} - 1.5.$$

Write

$$R_{k\alpha+\alpha-\varepsilon} = G_\varepsilon,$$

since this is independent of k, and notice that as $\varepsilon \downarrow 0$, we have $G_\varepsilon \downarrow 2.5e^{-\alpha} - 1.5$, which is strictly less than -1 since $\alpha > \ln(5)$. Thus we have ε' with $G_\varepsilon < -1$ for $0 < \varepsilon < \varepsilon'$. For such ε we know that

$$\max(|G_\varepsilon|, 1) = |G_\varepsilon|,$$

and this tells us that

$$|x(t_{k+1}) - x(t_{k+1} - \varepsilon)| \le (e^{L\varepsilon} - 1)(|G_\varepsilon|)(|x_k|).$$

We can write this as

$$|x_{k+1} - G_\varepsilon x_k| \le (e^{L\varepsilon} - 1)(|G_\varepsilon|)(|x_k|) \quad \text{for } 0 < \varepsilon < \varepsilon', \text{ with } G_\varepsilon < -1.$$

Note that $x(0) \neq 0$, and assume inductively that $x_m \neq 0$ for $m = 0, \ldots, k$. Divide by $|x_k|$ in the immediately above inequality, obtaining

$$\left|\frac{x_{k+1}}{x_k} - G_\varepsilon\right| \leq (e^{L\varepsilon} - 1)\,(|G_\varepsilon|).$$

Again noting that G_ε decreases to a real number strictly less than -1 as $\varepsilon \downarrow 0$, we can easily show that there exist $\varepsilon_0 > 0, T > 1$ with $\left|\frac{x_{k+1}}{x_k}\right| \geq T$ for $0 < \varepsilon < \varepsilon_0$. There is no dependence of T on k, and now certainly $x_{k+1} \neq 0$, completing the induction. The fundamental solution of (4) is unbounded for ε in $(0, \varepsilon_0)$, since $|x_k| \geq T^k |x(0)|$. □

Since the delay function in the above instability counterexample is absolutely continuous, it is interesting to examine this counterexample in light of Section 2, where the average variation of the delay function was related to the preservation of the exponent of growth or decay. Recall the inequality $k_M|x(t)|^2 \leq G(0, x_0)\, e^{t(2\gamma + Ba(t))}$ for $t > 0$, with $a(t) = \frac{1}{t}\int_0^t |h'(\tau)|\,\mathrm{d}\tau$. It is interesting to calculate the value of $a(k\alpha)$, the average of $|h'_\varepsilon(\cdot)|$ after full periods. Since $h_\varepsilon(\cdot)$ is periodic, and thus so is $h'_\varepsilon(\cdot)$, we know that $\int_0^{k\alpha} |h'_\varepsilon(t)|\,\mathrm{d}t = k \cdot \int_0^\alpha |h'_\varepsilon(t)|\,\mathrm{d}t$, and thus $a(k\alpha) = a(\alpha)$ for all k. Now $h'_\varepsilon(t) = 1$ if $0 < t < \alpha - \varepsilon$, and $h'_\varepsilon(t) = \frac{\varepsilon - \alpha}{\varepsilon}$ if $\alpha - \varepsilon < t < \alpha$. From this we see that $a(\alpha) = \frac{1}{\alpha}\left(1(\alpha - \varepsilon) + \varepsilon\left(\frac{\alpha - \varepsilon}{\varepsilon}\right)\right) = 2\frac{\alpha - \varepsilon}{\alpha}$, i.e. $a(\alpha) = 2\left(1 - \frac{\varepsilon}{\alpha}\right)$. Here we see that no matter how one chooses α, ε which work for the counterexample, the value of $a(\alpha)$ never exceeds two. On one hand, the inequalities in Section 2 tell us that there is a bound for $a(t)$ below which there can be no Markus–Yamabe instability in the time-varying system. On the other hand, one need not resort to high values of $a(t)$ to have this kind of instability. An interesting, but possibly very open ended research question, would be to investigate the kinds of variability in the delay function which can lead to Markus–Yamabe instabilities. Some special attention here could be paid to characterizing this variability in terms of Fourier analysis.

5. QUENCHING IN SECOND ORDER SYSTEMS

In this section we use second order delay equations to sketch an example of how systems having right half-plane eigenvalues can have their instabilities quenched by time-delay time dependence. Since the proofs are given elsewhere [9], we will present just enough of the idea, we hope, to allow readers to pursue the details and further ramifications on their own. To present our example of quenching, we begin with a system which has an exponential instability for all constant values of the delay.

Example 5.1. Consider the delay equation (2) $x''(t) - 2x'(t - \eta) + x(t) = 0$ having characteristic function $f_\eta(s) = s^2 - 2se^{-s\eta} + 1$. Using a polynomial elimination procedure as in Marshal et al [15] to find the imaginary axis zeros, one finds that there are two zeros on the positive imaginary axis. Calculating $\frac{\mathrm{d}s}{\mathrm{d}\eta}$ implicitly at these zeros, and proceeding with a somewhat detailed analysis considering $\mathrm{Re}\left(\frac{\mathrm{d}s}{\mathrm{d}\eta}\right)$ and the spacing of the η-values associated with the positive imaginary axis zeros, it can be proven [9] that the number of open right half-plane zeros of $f_\eta(s)$ is at least two for all $\eta \geq 0$.

Now consider (1) $x''(t) - 2x'(t - h(t)) + x(t) = 0$, where $h(t) = t - k\alpha$, $0 \leq k\alpha \leq t < (k+1)\,\alpha$ as in Section 3. Converting the system to first order form using 2×2 matrices A_0, A_1 in $x'(t) = A_0x(t) + A_1x(t - h(t))$, and noting that the characteristic function for the associated autonomous system is preserved, we calculate $T = T(\alpha) = e^{\alpha A_0}(I + A_0^{-1}A_1) - A_0^{-1}A_1$. One can show that both members of $\mathrm{Eig}(T)$ have absolute value strictly less than one with $\alpha = \frac{7\pi}{4}$, and noting Lemma 3.2 and the comments following, we have $x_{k+1} = Tx_k$ for $x_k = x(k\alpha)$, and we see that the system (1) is asymptotically stable. Details of the proof, along with a proof that the system can also be quenched with a certain continuous delay function, can be found in [9].

If we relax the hypothesis on the instability, allowing the system to be stable for small values of the delay, we can give an example of a system which is unstable for all large constant delays, in which sampling at all slow rates will quench instability. In fact, in [10] we give an example in which the system has an exponential instability for each constant delay η greater than a certain fixed η', and every nonzero sampling rate gives asymptotic stability.

It is worth mentioning that Cooke has an early paper on equations with time-varying delays which, in retrospect, is relevant to the question of when quenching can occur [2]. The differential equation of interest is $x'(t) = A_0x(t) + A_1x(t - h_0) + B(t)\,(x(t) - x(t - h(t)))$, where h_0 and $\mathrm{range}(h(\cdot))$ both lie in a given interval. Cooke is interested in the hypothesis that both $\|h(t)\,B(t)\| \to 0$ as $t \to \infty$, and $\int_0^\infty \|h(t)\,B(t)\|\,\mathrm{d}t < \infty$. These together are called asymptotically autonomous delay in the paper. The main question is how well the spectral solutions of the autonomous system (2) $x'(t) = A_0x(t) + A_1x(t - h_0)$ are carried over to the dynamics of the time-varying system. Defining $s(t,\theta) = \int_\theta^t B(\tau)\,(1 - e^{-h(\tau)})\,\mathrm{d}\tau$, he shows that for every simple zero μ of the characteristic equation of (2), there is a nonzero real c, a real $\theta \geq 0$, a nonzero vector d, an $\varepsilon(t) \to 0$ as $t \to \infty$, and a trajectory $x(\cdot)$ of the nonautonomous system having $x(t) = (e^{(t-\theta)\,\mu + cs(t,\theta)})\,(d + \varepsilon(t))$ for $t \geq \theta$. If, for instance, the function $s(t,\theta)$ converges to zero as t increases without bound, the original spectral dynamics will be carried over quite well, and there can be no quenching phenomena. This, however, does not exclude the possibility of Markus–Yamabe instability.

Finally, the reader may be interested in an example of a Markus–Yamabe instability for a system with time-varying delay, in which the associated autonomous equations are damped second order delay equations. Such an example is given in [9].

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