KYBERNETIKA — VOLUME 37 (2001), NUMBER 3, PAGES 229–238

NEW QUALITATIVE METHODS FOR STABILITY OF DELAY SYSTEMS

ERIK I. VERRIEST

A qualitative method is explored for analyzing the stability of systems. The approach is a generalization of the celebrated Lyapunov method. Whereas classically, the Lyapunov method is based on the simple comparison theorem, deriving suitable candidate Lyapunov functions remains mostly an art. As a result, in the realm of delay equations, such Lyapunov methods can be quite conservative. The generalization is here in using the comparison theorem directly with a different scalar equation with known qualitative behavior. It leads to criteria for stability of general difference and delay differential equations.

1. INTRODUCTION

The past decades have seen an explosive growth in delay systems research. Foremost in the analysis is the problem of stability for such systems, and many alternative methods have been explored. Many survey papers have been written where this information may be found (see [4] and references therein and the sessions in recent ACC, CDC, ECC and MTNS conferences.) Most results are for linear systems of the form

$$\dot{x}(t) = Ax(t) + Bx(t-\tau). \tag{1}$$

For time-invariant delays and constant system matrices, complex analytic and algebraic (2-D) methods have been used. Another approach is based on Kharitonov's ideas. Qualitative methods based on Lyapunov theory have also enjoyed great success. There are basically two different approaches here, the Lyapunov–Krasovskii method and the Lyapunov–Razumikhin method. The latter has a higher degree of complexity, but merits in its ability to yield delay-dependent stability criteria. It has been pointed out that delay equations in the above form (1) are mainly of an academic interest, because of the imprecision with which they model real phenomena. However, there is great benefit in knowledge of the behavior of such systems, since the theory may always be 'robustified'. This is a route taken by many authors [8, 17]. Since the Lyapunov–Krasovskii theory yields delay independent conditions, precise values for delays do not need to be known, and the obtained conditions are in fact conditions for *robust* stability. The Riccati type condition (the existence of $P, Q, R > 0$ for which $A'P + PA + Q + PBQ^{-1}B'P = 0$), stated in terms of the system matrices, is therefore of great value: Because of its simplicity (it is equivalent to an LMI [2]) it should be the first condition to test. If one is successful, it would have been overkill to have investigated the less restrictive, but very complex conditions suggested by other methods.

Recently, a special class of systems with deviating argument (i. e., delay or advance) of the form ($\alpha_0 \leq \alpha \leq 1$)

$$\dot{x} = Ax(t) + Bx(\alpha t) \tag{2}$$

has been investigated. Instability and asymptotic stability conditions were also presented in [10, 18], where the equation is referred to as a *linear-delay equation*. Some of the general theorems on non-oscillatory behavior in Erbe et al [5] apply to this system.

Sufficient conditions for asymptotic stability based on the Lyapunov–Krasovskii theory, were presented in [14]. A modified Riccati equation

$$A'P + PA + Q + \frac{PBQ^{-1}B'P}{\alpha} + R = 0 \tag{3}$$

was shown to play a significant role in this case.

All this prompts the question: Is it not possible to obtain less conservative conditions but without the expense of added complexity? Now, whereas this may seem like we are asking to get something from nothing, it is shown below that indeed this is possible, if one shifts the focus away from classical Lyapunov theory. A generalization, which really is in the same spirit as the original Lyapunov theory, is presented in Section 2. In Section 3, we explore the ideas on a simple difference equation, but with infinite dimensional state space. In Section 4, the delay differential system is approached with this new tool.

2. GENERALIZED LYAPUNOV THEORY

In this section we set the stage for a new class of Lyapunov functions by presenting first some background on a generalized Lyapunov theory [9, 12]. The point of departure is the comparison theorem. Specifically in the context of interconnected systems the use of *vector Lyapunov functions* has been successful. See also [13]. However, for reasons explained further we choose to stay in the domain of scalar comparison systems.

Let the autonomous dynamics of a system be described by

$$\dot{x} = f(x) \tag{4}$$

and assume that $x = 0$ is an equilibrium. The classical theorem of Lyapunov in its simplest form states that if there exist a function $V : \mathbb{R}^n \to \mathbb{R}$ such that for any $x \in \mathbb{R}^n$, it holds that $V(x) \geq 0$ with equality iff $x = 0$, then the equilibrium, $x = 0$, of the autonomous system is stable if, along solutions of the system, $\dot{V} \leq 0$. It is asymptotically stable if the right hand side may be replaced by $-\alpha V$.

This is based on the comparison theorem, which states that if the positive differentiable function $v(t)$ satisfies for some $\alpha > 0$,

$$\dot{v}(t) = -\alpha v(t), \quad v(0) = v_0 > 0, \tag{5}$$

and

$$\dot{V}(t) \leq -\alpha V(t), \quad V(0) = v_0 > 0, \tag{6}$$

then $V(t) \leq v(t)$ for all t. Of course the solution to (5) is well known: $v(t) = v_0 \exp(-\alpha t)$, so that one actually has *exponential* stability.

The novel idea in this paper is to replace (5) by another equation, for which the solution $v(t)$ is positive and either asymptotically stable with explicitly known behavior of decay, or at least is known to be asymptotically stable.

We select a particular instance: Consider the equation $\dot{v}(t) = av(t) + bv(t-\tau)$. It is known [7, p. 135] that this scalar system is asymptotically stable if the parameters (a, b) are inside the region whose boundary is given in parametric form by

$$a = b \cos \omega\tau, \quad b \sin \omega\tau = -\omega, \quad 0 < \omega < \frac{\pi}{\tau} \tag{7}$$

and the line $a = -b$. Moreover, the Riccati condition shows that the region $|b| < -a$ is precisely the region of *robust asymptotic stability* (stability for all τ) [15, 16].

The whole idea is equivalent to mapping, in a certain sense, the higher dimensional state space onto a one dimensional submanifold passing through the equilibrium. This line must not possess any self intersections. It follows that its points are well ordered, for instance, by using the ordering inherited by the arc length distance function. In fact any function, topologically conjugate to it will serve the same purpose. Let us call the point on the line, obtained by such a projection of a state x, the shadow $s(x)$. If now the system and its shadow are such that for all initial states in the neighborhood of the equilibrium, its shadow converges to the origin, then the null solution is asymptotically stable. Such convergence is implied if the shadow system satisfies a comparison system. For ordinary differential equations this states:

Comparison Theorem. Consider the scalar ordinary differential systems, defined on $[0, T)$.

$$\dot{x} = f_i(x); \quad i = 1, 2, \tag{8}$$

with common equilibrium, $x = 0$, and assume that for all x in the neighborhood of the equilibrium, $f_1 \leq f_2$. Assume further that both equations have the same initial condition, x_0. If the solution to these equations is defined in an interval $0 \leq t < T$, then $x_1(t) \leq x_2(t)$ holds for all t in the interval $(0, T)$, where x_1 and x_2 are respectively the solutions to the above equations with the given common initial condition.

Typically, one compares the induced one-dimensional shadow dynamics with the scalar equation $\dot{v} = -\alpha v$, for some positive α, to derive conditions for *exponential* stability since the solutions to the comparison system are indeed exponential. The extension of this result for ordinary differential systems is:

Theorem 1. Suppose that a positive definite function $V : \mathbb{R}^n \to \mathbb{R}$ exists, such that along solutions of the dynamical equation,

$$\dot{x}(t) = f(x) \tag{9}$$

with equilibrium $x = 0$, the Lie derivative $L_f V$ of V at t is upper bounded by a function $\psi(V)$ and if the null solution of the scalar system described by

$$\dot{v} = \psi(v) \tag{10}$$

is asymptotically stable, and the nonzero solutions of (10) do not change sign, then the null solution of the dynamical system (9) is asymptotically stable.

Proof. It follows from the assumption

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) \leq \psi(V), \qquad \frac{\mathrm{d}}{\mathrm{d}t}v(t) = \psi(v)$$

and the comparison theorem that $0 \leq V(x(t)) \leq v(t)$, and since $v(t) \to 0$ if $t \to \infty$, $V(x(t)) \to 0$. □.

The power of this theorem lies in the fact that it may be easier to find a suitable function ψ for which (10) holds, than trying to find a classical Lyapunov functional satisfying (5) for the specific right hand side $-\alpha v(t)$.

Example. Given the system

$$\dot{x} = -x^3 + \frac{1}{2}\sin^3 x.$$

The (simple) generalized Lyapunov function $V(x) = x^2$ gives

$$\begin{aligned} \dot{V} &= 2x\left(-x^3 + \frac{1}{2}\sin^3 x\right) \\ &= \left[-2 + \left(\frac{\sin x}{x}\right)^3\right] x^4 \;\leq\; -x^4. \end{aligned}$$

The comparison system is

$$\dot{v} = -v^2$$

with solution

$$v(t) = \frac{1}{t + v(0)^{-1}}.$$

Hence the given nonlinear system with initial condition x_0 satisfies

$$x^2(t) \leq \frac{1}{t + x_0^{-2}}$$

and is asymptotically stable, but convergence is not exponential.

The following two sections give applications to delay systems. Whereas [1] presented the general methodology with vector Lyapunov functions for functional equations (see also [3, 9]), the present paper exploits the fact that *explicit necessary and sufficient conditions* for robust stability exist in the scalar case.

3. DIFFERENCE EQUATION

In this section we consider *difference* equations of the form

$$x(t+1) = Ax(t) + Bx(t+\epsilon), \quad 0 < \epsilon < 1. \tag{11}$$

Such equations were discussed in [6, 11]. They model effects of delays in computer controlled systems. Note that if ϵ is rational, then the behaviour of (11) follows from standard discrete system techniques. Indeed, let $\epsilon = p/q$ with $0 < p < q$ both integer. Define the qn-dimensional vector $\xi(t) = [x'(t), x'(t+1/q), x'(t+2/q), \ldots, x'(t+(q-1)/q)]'$. It is readily seen that the original system is now embedded in the delay-free, higher order model, (a sampled data system with sampling rate q).

$$\xi(t+1/q) = \begin{bmatrix} 0 & I & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & I \\ A & 0 & 0 & B & 0 \end{bmatrix} \xi(t). \tag{12}$$

The full state space of the original difference system consists of the infinite Cartesian product of finite dimensional subspaces (one for each $\theta \in (0,1)$), but these are all decoupled. That is why in this case the stability is completely determined by the eigenvalues of the finite dimensional block companion matrix.

In the general (irrational ϵ) scalar case, the solution to (11) satisfies for $a, b > 0$,

$$x(t+n) = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k x_{t+k\epsilon},$$

which is easily shown by induction. Consequently,

$$|x(t+n)| \le \sum_{k=0}^{n} \binom{n}{k} |a|^{n-k} |b|^k \max_{0<k<n} |x(t+k\epsilon)|,$$

and thus

$$|x(t+n)| \le \left[\sum_{k=0}^{n} \binom{n}{k} |a|^{n-k} |b|^k\right] \max_{0<k<n} |x(t+k\epsilon)|.$$

Enlarging the set over which one maximizes further,

$$\begin{aligned} \max_{0\le\theta\le1} x(\theta+n) &\le \sum_{k=0}^{n} \left[\binom{n}{k} a^{n-k} b^k\right] \sup_{0<t<1+n\epsilon} x(t) \\ &\le (|a|+|b|)^n \sup_{0<t<1+n\epsilon} x(t). \end{aligned}$$

Thus $\sup_{n<\theta<n+1} |x(\theta)| \le \sup_{0<t<n\epsilon} |x|(|a|+|b|)^n$, and combining for each integer n

$$\sup_{0\le\theta\le n+1} |x(\theta)| \le \sup_{0<t<n\epsilon} |x(t)| \max_{0\le k\le n} \{(|a|+|b|)^k\}.$$

Now one can extract again the integer part of $n\epsilon < n$, and use the same inequality again. By iterating this way, one obtains a bound for $|x(t)|$ at all t in terms of the initial data, involving a power of $(|a| + |b|)$. Consequently, the equation is asymptotically stable if $|a| + |b| < 1$.

The characteristic equation for the difference equation is

$$e^s - a - be^{s\epsilon} = 0 \tag{13}$$

and $s = \sigma + j\omega$ is a root if and only if

$$e^\sigma \cos\omega - a - be^{\sigma\epsilon}\cos\omega\epsilon = 0 \tag{14}$$

$$e^\sigma \sin\omega - be^{\sigma\epsilon}\sin\omega\epsilon = 0 \tag{15}$$

or equivalently

$$e^{\sigma(1-\epsilon)} = b\frac{\sin\omega\epsilon}{\sin\omega} \tag{16}$$

$$e^\sigma = \frac{a}{\sin\omega}\,\frac{\tan\omega\epsilon\tan\omega}{\tan\omega\epsilon - \tan\omega}. \tag{17}$$

We prove the following:

Lemma 1. The characteristic equation of the difference system has isolated roots. Moreover the roots do not contain cluster points.

P r o o f. Suppose this condition were not true. For a connected set of roots, there must exist a path such that each point on the path is a root. This means that $(e^s - b\epsilon e^{s\epsilon})\,\mathrm{d}s$ must vanish identically along such a path. Taking real and imaginary parts, the condition is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\omega} = \frac{e^\sigma\sin\omega - b\epsilon e^{\sigma\epsilon}\sin\omega\epsilon}{e^\sigma\cos\omega - b\epsilon e^{\sigma\epsilon}\cos\omega\epsilon} = -\frac{\mathrm{d}\omega}{\mathrm{d}\sigma}. \tag{18}$$

But this gives

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\omega}\right)^2 = -1 \tag{19}$$

along such a path, which is nonsense. The same argument shows that near any root, there cannot be an other which is arbitrarily close, thus prohibiting cluster points in the spectrum. □

Define thus the generalized Lyapunov function as a positive definite function $V(x)$, satisfying along its solutions $V(x(t+1)) < aV(x(t)) + bV(x(t+\epsilon))$ for some positive scalars, a and b satisfying $a + b < 1$. We now derive a new Riccati equation condition for the robust stability of difference systems based on this Lyapunov function.

Theorem 2. The system (11) is robustly asymptotically stable if there exist positive definite matrices P, Q, R and S, and positive numbers α, β with $\alpha + \beta < 1$ such that the Lyapunov equations

$$A'PA + Q = \alpha P \tag{20}$$

$$B'PB + R = \beta P, \tag{21}$$

and the Riccati equation

$$A' \left[P^{-1} - \frac{1}{\beta} BP^{-1}B' \right]^{-1} A - \alpha P + S = 0, \tag{22}$$

hold.

Proof. Consider the Lyapunov function $V(\{x\}) = x'Px$. Define $v(t) = V(x(t))$. Along trajectories of (11), one has

$$\begin{aligned} V(t+1) - \alpha V(t) - \beta V(t+\epsilon) &= [x'(t)A' + x'(t+\epsilon)B'] \, P[Ax(t) + Bx(t+\epsilon)] \\ &\quad - \alpha x'(t) \, Px'(t) - \beta x'(t+\epsilon) \, Px(t+\epsilon) \\ &\leq \; (x'(t), x'(t+\epsilon)) \begin{pmatrix} A'PA - \alpha P & A'PB \\ B'PA & B'PB - \beta P \end{pmatrix} \begin{pmatrix} x(t) \\ x(t+\epsilon) \end{pmatrix}. \end{aligned}$$

If the weight matrix of the quadratic form is negative definite, the difference is asymptotically stable for all $\epsilon \in (0,1)$. This implies that there exist positive definite Q and R such that the Lyapunov equations

$$A'PA + Q = \alpha P \tag{23}$$

$$B'PB + R = \beta P \tag{24}$$

hold, and that further

$$Q > A'PBR^{-1}B'PA. \tag{25}$$

The discrete Lyapunov equations are equivalent to the Schur–Cohn stability of both $\frac{1}{\sqrt{\alpha}}A$ and $\frac{1}{\sqrt{\beta}}B$. The condition (25) is equivalent to the Riccati condition

$$A'PA - \alpha P - A'PB(B'PB - \beta P)^{-1}B'PA < 0 \tag{26}$$

or, invoking Woodbury's lemma

$$A' \left[P^{-1} - \frac{1}{\beta} BP^{-1}B' \right]^{-1} A - \alpha P < 0, \tag{27}$$

thus proving the assertion. □

4. DELAY–DIFFERENTIAL EQUATION

In this section the generalized Lyapunov theory is extended to functional differential equations, and then applied to a linear time-invariant delay system. The known Riccati condition is retrieved for the robust stability of the delay differential equation, but *without invoking a Lyapunov–Krasovskii functional.* In what follows, $\Phi_+ = C([-\tau, 0], \mathbb{R}^+)$ is the set of nonnegative continuous functions on $[-\tau, 0]$.

Lemma 2. Consider the scalar FDE

$$\dot{x}(t) = ax(t) + bx(t - \tau)$$

with $b > 0$. If the initial data $\phi_0 \in \Phi_+$, then $x(t) \geq 0$ for all $t \geq 0$.

Proof. A solution exists for all $t > 0$. Assume that $t_0 = \text{lub}\,\{t \,|\, x(t) \geq 0\}$. By continuity, $x(t_0) = 0$, and $\dot{x}(t_0) = bx(t_0 - \tau) > 0$. Hence $x(t) \geq 0$ in a neighborhood of t_0, contradicting that t_0 was the least upper bound. □

The lemma says that under the given conditions, a solution, starting positive, remains positive throughout.

Lemma 3. If in addition to the conditions of Lemma 2, $a < -b$, then the solution starting in Φ_+ satisfies:

$$\begin{aligned} x(t) &\geq 0 \quad \forall\, t > 0 \\ x(t) &\to 0 \quad \text{if } t \to \infty. \end{aligned}$$

Proof. Trivial. □

Lemma 4. Consider the scalar equations

$$\begin{aligned} \dot{x}(t) &= ax(t) + bx(t - \tau) \\ \dot{y}(t) &= f(y(t)) + by(t - \tau) \end{aligned}$$

with $b > 0$, and $f(y) < ay$ for $y > 0$, then if $y(\theta) = x(\theta) = \phi(\theta)$; $-\tau < \theta < 0$, with $\phi(\cdot) \in \Phi_+$, it follows that

$$y(t) \leq x(t) \quad \forall\, t.$$

Proof. By Lemma 2, $x(t) \geq 0$ for all $t > -\tau$. Let now $t_0 = \text{lub}\,\{t \,|\, x(t) \geq y(t)\,\}$. Then $x(t_0) = y(t_0)$ and

$$\begin{aligned} \dot{x}(t_0) &= ax(t_0) + bx(t_0 - \tau) \\ &\geq f(x(t_0)) + bx(t_0 - \tau) \\ &\geq f(y(t_0)) + by(t_0 - \tau) \\ &= \dot{y}(t_0). \end{aligned}$$

Thus in a neighborhood of t_0, we have $x(t) \geq y(t)$, contradicting the assertion. □

Lemma 5. If in addition to Lemma 4, we have $a < -b$, then $y(t) \to 0$.

Remark. Note that Lemma 2 can be generalized to the scalar equation $\dot{y}(t) = f(y(t)) + by(t - \tau)$ with $b > 0$, and f Lipschitz (to guarantee the existence of a solution) and $f(0) = 0$, yielding the conclusion that y starting in Φ_+ remains nonnegative.

One now readily derives from the above lemmas:

Theorem 3. If $V(x)$ is a positive definite Lyapunov function, such that along the trajectories of a FDE (1), $v(t) = V(x(t))$ satisfies

$$\dot{v}(t) - \alpha v(t) - \beta v(t-\tau) < 0,$$

where $\beta > 0$ and $\alpha < -\beta$, then the FDE is asymptotically stable.

Example. We consider for the linear system

$$\dot{x}(t) = Ax(t) + Bx(t-\tau).$$

the candidate Lyapunov function $V(x) = x'Px$, where P is positive definite. Along solutions of the system:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = x'(t)\,(PA + A'P)\,x(t) + 2x'(t)\,PBx(t-\tau).$$

It follows that

$$\frac{\mathrm{d}V}{\mathrm{d}t} - \alpha V(t) - \beta V(t-\tau) = [x'(t), x'(t-\tau)] \begin{bmatrix} PA + A'P - \alpha P & B'P \\ PB & -\beta P \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}.$$

The weight matrix is negative definite if

$$A'P + PA - \alpha P + PB(\beta P)^{-1}B'P < 0.$$

Note that this is the Riccati equation condition (3) for the special case $Q = \beta P$, $R = -(\alpha + \beta)P > 0$.

5. CONCLUSION

We explored a new class of Lyapunov functions. Their application in the stability analysis for delay differential and difference systems is illustrated. Specifically, the fact that necessary and sufficient criteria for robust stability are explicitly known in the scalar case has been exploited.

ACKNOWLEDGEMENT

Research for this paper was made possible thanks to the NSF-CNRS Cooperative Research Grant # 9818312.

(Received November 22, 2000.)

REFERENCES

[1] P. Borne, M. Dambrine, W. Perruquetti, and J. P. Richard: Vector Lyapunov functions for nonlinear time-varying, ordinary and functional differential equations. In: Stability at the End of the XXth Century (Martynyuk, ed.), Gordon and Breach. To appear.

[2] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan: Linear Matrix Inequalities in System and Control Theory. SIAM Studies in Applied Mathematics, Philadelphia 1994.
[3] M. Dambrine and J. P. Richard: Stability and stability domains analysis for nonlinear differential-difference equations. Dynamic Systems and Applications *3* (1994), 369–378.
[4] L. Dugard and E. I. Verriest: Stability and Control of Time-Delay Systems. Springer–Verlag, London 1998.
[5] L. H. Erbe, Qingkai Kong, and B. G. Zhang: Oscillation Theory for Functional Differential Equations. Marcel Dekker, New York 1995.
[6] J. K. Hale: Effects of Delays on Stability and Control. Report CDNS97-270, Georgia Institute of Technology.
[7] J. K. Hale and S. M. Verduyn Lunel: Introduction to Functional Differential Equations. Springer–Verlag, New York 1993.
[8] D. Ivanescu, J.-M. Dion, L. Dugard, and S.-I. Niculescu: Delay effects and dynamical compensation for time-delay systems. In: Proc. 38th IEEE Conference on Decision and Control, Phoenix 1999, pp. 1999–2004.
[9] V. Laksmikantham and S. Leela: Differential and Integral Inequalities. Vol. I and II. Academic Press, New York 1969.
[10] A. A. Laktionov and A. P. Zhabko: Method of difference transformations for differential systems with linear time-delay. In: Proc. IFAC Workshop on Linear Time Delay Systems, Grenoble 1998, pp. 201–205.
[11] H. Logemann and S. Townley: The effect of small delays in the feedback loop on the stability of neutral systems. Systems Control Lett. *27* (1996), 267–274.
[12] A. N. Michel and R. K. Miller: Qualitative Analysis of Large Scale Dynamical Systems. Academic Press, New York 1977.
[13] W. Perruquetti, J. P. Richard, and P. Borne: Estimation of nonlinear time-varying behaviours using vector norms. Systems Anal. Modelling Simulation *11* (1993), 167–184.
[14] E. I. Verriest: Robust stability, adjoints, and LQ control of scale-delay systems. In: Proc. 38th IEEE Conference on Decision and Control, Phoenix 1999, pp. 209–214.
[15] E. I. Verriest and A. F. Ivanov: Robust stabilization of systems with delayed feedback. In: Proc. 2nd International Symposium on Implicit and Robust Systems, Warsaw 1991, pp. 190–193.
[16] E. I. Verriest and A. F. Ivanov: Robust stability of systems with delayed feedback. Circuits Systems Signal Process. *13* (1994), 2/3, 213–222.
[17] L. Xie and C. E. de Souza: Robust stabilization and disturbance attenuation for uncertain delay systems. In: Proc. 2nd European Control Conference, Groningen 1993, pp. 667–672.
[18] A. P. Zhabko, A. A. Laktionov, and V. I. Zubov: Robust stability of differential-difference systems with linear time-delay. In: Proc. IFAC Symposium on Robust Control Design, Budapest 1997, pp. 97–101.

Prof. Dr. Ir. Erik I. Verriest, School of Electrical and Computer Engineering, Georgia Institute of Technology, Atlanta, GA 30332-0250. U. S. A.
e-mail: erik.verriest@ee.gatech.edu