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ON MICHÁLEK'S FUZZY TOPOLOGICAL SPACES

FRANCISCO GALLEGO LUPIÁÑEZ

The aim of this paper is to study some properties of Michálek's fuzzy topology which are quite different of the classic properties of the Chang's topology.

The first notion of fuzzy topological space has been defined by C. L. Chang, in 1968. This definition is the natural translation to fuzzy sets of the ordinary definition of topological space. Indeed, a fuzzy topology (in the Chang's sense) is a family T of fuzzy sets in X, such that T is closed with respect to arbitrary union and finite intersection, and empty subset, and the whole set X belong to T [2]. In the paper [3] J. Michálek defined and studied another concept of fuzzy topological space which is quite different of the classic C. L. Chang's definition. The aim of our paper is to show the divergences between these two kinds of fuzzy topological spaces. This study has been suggested by C. K. Wong in his review of the J. Michálek paper.

They are other kinds of fuzzy topology define in last decades (Lowen, Höhle, Šostak, Kotzé, Kubiak, etc.) but I shall not study these other concepts.

Definition 1. (Michálek [3]) Let X be a non-empty set, let $\mathcal{P}(X)$ be the system of all subsets of the set X, and let $\mathcal{F}$ be the system of all fuzzy sets in X. A pair $\langle X, u\rangle$ is called fuzzy topological space supposing that u is a mapping from $\mathcal{P}(X)$ into $\mathcal{F}$ satisfying the following three axioms:

1. if $A \subset X$, then $uA(x) = 1$ for all $x \in A$,
2. if $A \subset X$ contains at most one element, then $uA(x) = \chi_A(x)$, where χ_A is the characteristic function of the set A,
3. if $A_1 \subset X$, $A_2 \subset X$, then $u(A_1 \cup A_2)(x) = \max\{uA_1(x), uA_2(x)\}$.

Remark 1. All non-empty set has trivially a fuzzy topological structure in the Michálek's sense.

Proof. Let X be a non-empty set. We define $u : \mathcal{P}(X) \longrightarrow \mathcal{F}$, as $uA = \chi_A$. Then $\langle X, u\rangle$ is a Michálek's fuzzy topological space. □

Definition 2. (Michálek [3]) Let $\langle X, u\rangle$ be a Michálek's fuzzy topological space. A subset $A \subset X$ is called to be fuzzy closed (in the Michálek sense) if $\chi_A(x) \geq \min\{uA(x), uA^c(x)\}$ for all $x \in X$ (He has proved that this is equivalent to be $uA = \chi_A$).

Remark 2. X is fuzzy closed set in every Michálek's fuzzy topological space $\langle X, u\rangle$.

Proof. $\min\{uX(x), uX^c(x)\} = \min\{uX(x), u\phi(x)\} = \min\{uX(x), 0\} = 0 < \chi_X(x)$, for all $x \in X$. □

Definition 3. (Chang [2]) Let X and Y be two sets and let φ be a map from X to Y. Let μ be a fuzzy set in Y, then the inverse of μ written as $\varphi^{-1}(\mu)$, is defined by $\varphi^{-1}(\mu)(x) = \mu(\varphi(x))$ for all x in X. Conversely, if ν is a fuzzy set in X, the image of ν, written as $\varphi(\nu)$, is a fuzzy set in Y given by

$$\varphi(\nu)(y) = \begin{cases} \sup_{x \in \varphi^{-1}(y)}\{\nu(x)\}, & \text{if } \varphi^{-1}(y) \neq \phi \\ 0, & \text{otherwise.} \end{cases}$$

We will use the above definition for to introduce an important concept on Michálek's fuzzy topological spaces.

Definition 4. Let $\langle X, u\rangle$ and $\langle Y, v\rangle$ be two Michálek's fuzzy topological spaces and let φ be a map from X to Y. We will say that φ is compatible with u and v if, for all $B \in \mathcal{P}(Y)$, we have that $u(\varphi^{-1}(B)) = \varphi^{-1}(vB)$.

Proposition 1. Let $\langle X, u\rangle$ and $\langle Y, v\rangle$ be two Michálek's fuzzy topological spaces, and let $\varphi : X \longrightarrow Y$ be a compatible map with u and v, then for every fuzzy closed (in the Michálek's sense) C in $\langle Y, v\rangle$, we have that $\varphi^{-1}(C)$ is fuzzy closed in $\langle X, u\rangle$.

Proof. C is fuzzy closed in $\langle Y, v\rangle$ if, and only if, $v(C) = \chi_C$. Then, for all $x \in X$, we have

$$u(\varphi^{-1}(C))(x) = (vC)(\varphi(x)) = \begin{cases} 1, & \text{if} \quad \varphi(x) \in C \\ 0, & \text{if} \quad \varphi(x) \notin C \end{cases}$$

i.e. $u(\varphi^{-1}(C)) = \chi_{\varphi^{-1}(C)}$ and $\varphi^{-1}(C)$ is fuzzy closed in $\langle X, u\rangle$. □

Definition 5. (Michálek [3]) Let $\langle X, u\rangle$ be a Michálek's fuzzy topological space, and if $A \subset X$, let μ_{A^o} and $\mu_{\partial A}$ the fuzzy sets given by $\mu_{A^o}(x) = 1 - (uA^c)(x)$ and $\mu_{\partial A}(x) = \min\{uA(x), uA^c(x)\}$. The subset A is called to be fuzzy open (in the Michálek's sense) if for every $x \in X$, $\min\{\chi_A(x), \mu_{\partial A}(x)\} = 0$. (He has proved that this is equivalent to be $\mu_{A^o} = \chi_A$, or the set A^c is fuzzy closed).

Proposition 2. Let $\langle X, u\rangle$ and $\langle Y, v\rangle$ be two Michálek's fuzzy topological spaces, and let $\varphi : X \to Y$ be a compatible map with u and v, then for every fuzzy open (in the Michálek's sense) A in $\langle Y, v\rangle$, we have that $\varphi^{-1}(A)$ is fuzzy open in $\langle X, u\rangle$.

Proof. A is fuzzy open in $\langle Y, v\rangle$ if and only if A^c is fuzzy closed in $\langle Y, v\rangle$, this yield that $\varphi^{-1}(A^c) = (\varphi^{-1}(A))^c$ is fuzzy closed in $\langle X, u\rangle$, and $\varphi^{-1}(A)$ is fuzzy open in $\langle X, u\rangle$. □

Definition 6. (Michálek [3]) Let $\langle X, u\rangle$ be a Michálek's fuzzy topological space, a subset $U \subset X$ is called to be fuzzy neighbourhood of an element $a \in X$ if $u\{a\}(x) \leq \mu_{U^o}(x)$ for every $x \in X$.

A family $\Sigma(a) \subset \mathcal{P}(X)$ is called a system of fuzzy neighbourhoods of a, if:

1. $\Sigma(a) \neq \phi$,
2. $a \in U$ for every $U \in \Sigma(a)$,
3. if $x \neq a$, then there exists $U \in \Sigma(a)$ such that $x \in U^c$,
4. if $U, V \in \Sigma(a)$, then there exists $W \in \Sigma(a)$ such that $W \subset U \cap V$.

Proposition 3. Let $\langle X, u\rangle$ be a Michálek's fuzzy topological space and let $a \in X$. Then, the set

$$\mathcal{S}(a) = \{A \subset X \mid A \text{ is fuzzy open in } \langle X, u\rangle, \text{ and } uA^c(a) = 0\}$$

is a system of fuzzy neighbourhoods of a.

Proof.

1. $X^c = \phi$, then $uX^c(a) = 0$.
   For each $z \in X$, we have that $(1 - uX^c)(z) = (1 - u\phi)(z) = 1$, thus $1 - uX^c = \chi_X$ and X is fuzzy open in $\langle X, u\rangle$. So, $X \in \mathcal{S}(a)$.
2. For every $A \in \mathcal{S}(a)$, $1 - uA^c = \chi_A$ and $(uA^c)(a) = 0$, then $\chi_A(a) = 1$ and $a \in A$.
3. If $x \neq a$, let $U = X\backslash\{x\}$, then $x \in U^c$. We have $1 - uU^c = 1 - u\{x\} = 1 - \chi_{\{x\}} = \chi'_{\{x\}} = \chi_U$, and $uU^c(a) = u\{x\}(a) = \chi_{\{x\}}(a) = 0$. Then $U \in \mathcal{S}(a)$.
4. If $U, V \in \mathcal{S}(a)$, then $1 - uU^c = \chi_U$, $1 - uV^c = \chi_V$ and $uU^c(a) = uV^c(a) = 0$.

Thus,

$$\begin{aligned}(1 - u(U \cap V)^c)(z) &= 1 - u(U \cap V)^c(z) = 1 - u(U^c \cup V^c)(z) \\ &= 1 - \max\{uU^c(z), uV^c(z)\} \\ &= \begin{cases} 0, & \text{if} \quad uU^c(z) \text{ or } uV^c(z) = 1 \ (\Leftrightarrow z \notin U \text{ or } z \notin V) \\ 1, & \text{if} \quad uU^c(z) = uV^c(z) = 0 \ (\Leftrightarrow z \in U \cap V) \end{cases}\end{aligned}$$

then, $1 - u(U \cap V)^c = \chi_{U \cap V}$

$$u(U \cap V)^c(a) = u(U^c \cup V^c)(a) = \max\{uU^c(a), uV^c(a)\} = 0. \text{ So, } U \cap V \in \mathcal{S}(a).$$

□

Definition 7. Let $\langle X, u \rangle$ be a Michálek's fuzzy topological space and let $a \in X$. We will say that $\mathcal{S}(a) = \{A \subset X \mid A \text{ is fuzzy open in } \langle X, u \rangle, \text{ and } uA^c(a) = 0\}$ is the system of fuzzy open neighbourhoods of a.

Proposition 4. Let $\langle X, u \rangle$ and $\langle Y, v \rangle$ be two Michálek's fuzzy topological spaces and let $\varphi : X \to Y$ be a compatible map with u and v, then for each $a \in X$ and for each $V \in \mathcal{S}(\varphi(a))$, there exists $U \in \mathcal{S}(a)$ such that $\varphi(U) \subset V$.

Proof. $V \in \mathcal{S}(\varphi(a))$ if and only if V is fuzzy open in $\langle Y, v \rangle$ and $(vV^c)(\varphi(a)) = 0$. Then $\varphi^{-1}(V)$ is fuzzy open in $\langle X, u \rangle$ and $u\varphi^{-1}(V)^c(a) = u\varphi^{-1}(V^c)(a) = \varphi^{-1}(vV^c)(a) = vV^c(\varphi(a)) = 0$. So $\varphi^{-1}(V) \in \mathcal{S}(a)$ and it is $\varphi(\varphi^{-1}(V)) \subset V$. □

Remark 3. J. Michálek uses his fuzzy topological space $\langle X, u \rangle$ for to define certain topological spaces: let $\langle X, u \rangle$ be a Michálek's fuzzy topological space, for every $A \subset X$, he takes $vA = \{x \in X \mid uA(x) > 0\}$ and he proves that $\langle X, v \rangle$ is a topological space.

It is necessary to remark that, in Michálek's sense, a "topological space" is a couple $\langle X, v \rangle$ supposing that v is a mapping of $\mathcal{P}(X)$ into itself satisfying the following two axioms:

I. if $A \subset X$ contains at most one element, then $vA = A$;

II. if $A_1 \subset X$, $A_2 \subset X$, then $v(A_1 \cup A_2) = vA_1 \cup vA_2$.

Then, this closure operator is not a topological closure operator (in the Kuratowski's sense), because is not necessarily idempotent.

This fact motives the surprising properties of Michálek's fuzzy topological spaces, for example: all compatible map verifies that the preimage of every fuzzy open is fuzzy open, and this is an obstacle for a straight definition of continuity of maps between these fuzzy topological spaces.

Nevertheless the Michálek's fuzzy topological spaces are interesting because they have applications to probability theory [3, Theorem 5].

Remark 4. Using the Chang's concept of topology we can assign to each ordinary subset A in X the fuzzy closure of its characteristic function $\overline{\chi_A}$, which is the least closed fuzzy set in X containing χ_A. Let us define an operator $u : \mathcal{P}(X) \to \mathcal{F}$ as, $A = \overline{\chi_A}$. Clearly it is not a fuzzy closure operator in the sense of Michálek, because the demand 2 of Definition 1 is not verified. Nevertheless, in the paper [3], he defined also certain "generalized fuzzy topological spaces" in which demand 2 is dropped.

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Dr. Francisco Gallego Lupiáñez, Departamento de Geometría y Topología, Facultad de Matemáticas, Universidad Complutense de Madrid, 28040 Madrid. Spain.
e-mail: fg_lupianez@mat.ucm.es