KYBERNETIKA — VOLUME 37 (2001), NUMBER 1, PAGES 21 – 37

SOLUTION SET IN A SPECIAL CASE OF GENERALIZED NASH EQUILIBRIUM GAMES[1]

JOSEF CACH

A special class of generalized Nash equilibrium problems is studied. Both variational and quasi-variational inequalities are used to derive some results concerning the structure of the sets of equilibria. These results are applied to the Cournot oligopoly problem.

1. INTRODUCTION

In this paper we use first-order analysis to compute a *generalized Nash equilibrium* (GNE) in a game. There does not exist any universal technique how to solve this problem analytically. In addition, one cannot expect uniqueness of its solutions in general. To obtain deeper analytic results we make use of a special structure of constraints and utility functions of the game.

An economic background of equlibrium problems can be found in [1, 5, 9, 12, 15]. We refer to Harker [7] to learn more information about history of formulation of Nash and generalized Nash equilibria in the form of so-called *variational* (VI) and *quasi-variational inequalities* (QVI) and about some basic results in this area. Concerning the mathematical theory of VI and QVI, the reader is referred to [2, 4, 8, 10]. In the later works Outrata and Zowe [14] and Outrata, Kočvara and Zowe [13] GNE is modeled via so-called *mathematical program with equilibrium constraints* (MPEC) and in the latter work a corresponding mathematical theory about MPECs is developed.

The aims of this paper are

(a) to analyze the structure of the solution set of generalized Nash equilibrium problems under some special assumptions;

(b) to give an efficient algorithm for computing these special GNE;

(c) to apply obtained results to the Cournot oligopoly problem.

The outline of this paper is as follows. In the next section we introduce basic definitions and results about variational and quasi-variational inequalities. The third

[1]This research has been partially supported by the grant No. A1075005 from the Academy of Sciences of the Czech Republic.

section is devoted to general Nash equilibria which have only one constraint "across all players". In Section 4 these results are extended to the case, where there exist more "across all players" constraints, but each of them holds only for an isolated group of players. Section 5 presents applications of previous results to the Cournot oligopoly.

The following notation is employed: x_i is the ith component of a vector $x \in \mathbb{R}^n$ and $\mathbb{R}^n_+$ denotes the nonnegative orthant of $\mathbb{R}^n$. For a index set $I \subset \{1,\dots,n\}$, x_I is a subvector of a vector x with components x_i, $i \in I$. For x, $y \in \mathbb{R}^n$ the inequality $x \le y$ means $x_i \le y_i$, for all $i \in \{1,\dots,n\}$. For a convex set $\Pi \subset \mathbb{R}^n$ and $x \in \Pi$, $N_\Pi(x)$ is the normal cone to Π at x in the sense of convex analysis.

2. PRELIMINARIES

Consider a classical problem from the game theory:

$$\begin{aligned}&\text{Find } y^* \in \mathbb{R}^n \quad \text{such that}\\ &y_i^* \in \operatorname*{arg\,max}_{\alpha_i \le y_i \le \beta_i} u_i\left(y_i, y_{-i}^*\right) \quad \text{for all } i \in \{1,\dots,n\};\end{aligned} \tag{2.1}$$

where $y_{-i} = (y_1,\cdots,y_{i-1},y_{i+1},\cdots,y_n)$.

The function $u_i\,[\mathbb{R}^n \to \mathbb{R}]$ is the utility function of the ith player which is maximized subject to box constraints $\alpha_i \le y_i \le \beta_i$ (y_i is a strategy of ith player). Solution y^* of this problem is the so-called *Nash equilibrium.*

Let us denote by Γ the set of feasible strategies

$$\Gamma := \{y \in \mathbb{R}^n \,|\, \alpha \le y \le \beta\}, \tag{2.2}$$

where $\alpha = (\alpha_1,\dots,\alpha_n)^T$ and $\beta = (\beta_1,\dots,\beta_n)^T$.
We pose now the following assumption about utilities:

(U1) the functions $u_i\left(\cdot, y_{-i}^*\right)$ are concave and differentiable for all $i \in \{1,\dots,n\}$ and for all $y^* \in \Gamma$;

Then this problem can equivalently be replaced by a *variational inequality* (VI):

$$\begin{aligned}&\text{Find } y^* \in \Gamma \text{ such that}\\ &\sum_{i=1}^n -\nabla_{y_i} u_i(y^*) \cdot (y_i - y_i^*) \ge 0 \quad \text{for all } y \in \Gamma;\end{aligned} \tag{2.3}$$

with Γ given by (2.2).

A standard existence and uniqueness result concerning the VI above is stated in the next lemma.

Lemma 2.1. Suppose that in VI (2.3) the feasible set Γ is compact. Then this VI has a solution. If, moreover, the Jacobian of the mapping F defined by

$$F : y \to \left(-\nabla_{y_1} u_1(y), -\nabla_{y_2} u_2(y), \dots, -\nabla_{y_n} u_n(y)\right)$$

is positive definite on the set Γ, then this solution is exactly one.

Remark 1. From the preceding lemma it is clear that if condition (U1) holds, then problem (2.1) has a solution.

Suppose further that there exists one constraint which concerns all single strategies of the players, the so-called "across all players" constraint. Then we have the following problem:

$$\text{(S)} \qquad \begin{aligned} &\text{Find } y^* \in \mathbb{R}^n \quad \text{such that} \\ &y_i^* \in \underset{\substack{\alpha_i \leq y_i \leq \beta_i \\ g\left(y_i, y_{-i}^*\right) \leq 0}}{\arg\max}\ u_i\left(y_i, y_{-i}^*\right) \quad \text{for all } i \in \{1, \dots, n\}; \end{aligned}$$

where $g\,[\mathbb{R}^n \to \mathbb{R}]$ defines the above mentioned "across all players" constraint. Any solution of this problem is entitled a *generalized Nash equilibrium* (GNE).

The set of feasible strategies Γ_S, given by

$$\Gamma_S := \{y \in \mathbb{R}^n \,|g\,(y) \leq 0; \alpha \leq y \leq \beta\}; \tag{2.4}$$

is now a subset of set Γ given by (2.2). Suppose in the sequel that Γ_S is nonempty. Further assume that assumption (U1) holds and pose the following assumption about the constraint function g:

(C1) the functions $g\left(\cdot, y_{-i}^*\right)$ are convex or monotone and continuous for all $i \in \{1, \dots, n\}$ and for all $y^* \in \Gamma_S$.

Then this problem can equivalently be replaced by a *quasi-variational inequality* (QVI):

$$\begin{aligned} &\text{Find } y^* \in \times_{i=1}^n K_i\left(y_{-i}^*\right) \text{ such that} \\ &\sum_{i=1}^n -\nabla_{y_i} u_i\left(y^*\right) \cdot \left(y_i - y_i^*\right) \geq 0 \quad \text{for all } y \in \times_{i=1}^n K_i\left(y_{-i}^*\right); \end{aligned} \tag{2.5}$$

where $K_i\left(y_{-i}^*\right) = \left\{x \in \mathbb{R} \middle| \alpha_i \leq x \leq \beta_i;\ g\left(x, y_{-i}^*\right) \leq 0\right\}$.

Further equivalent reformulation of this problem which we will be using is the so-called *generalized equation* (GE):

Find $y^* \in \mathbb{R}^n$ such that

$$0 \in \begin{bmatrix} -\nabla_{y_1} u_1\left(y^*\right) \\ \vdots \\ -\nabla_{y_n} u_n\left(y^*\right) \end{bmatrix} + N_{\times_{i=1}^n K_i\left(y_{-i}^*\right)}\left(y^*\right), \tag{2.6}$$

where $N_{\times_{i=1}^n K_i\left(y_{-i}^*\right)}\left(y^*\right)$ is the normal cone to the set $\times_{i=1}^n K_i\left(y_{-i}^*\right)$ at y^* in the sense of convex analysis.

Now we state two results; the former concerns the existence of solutions to QVI (2.5) and the latter concerns the relation between solutions of a special VI and QVI (2.5).

Lemma 2.2. Each solution of the VI of the type (2.3) with Γ replaced by Γ_S is a solution of QVI (2.5). These solutions are termed *normal equilibria* (cf. Rosen [15]). The converse is not true in general.

Suppose now that there exists more "across all players" constraints $g_1(y), \ldots$ $\ldots, g_m(y)$. The set of feasible strategies is given by the formula

$$\Gamma_M := \{y \in \mathbb{R}^n \mid g_j(y) \le 0 \text{ for all } j \in \{1, \ldots, m\}; \alpha \le y \le \beta\}.$$

Lemma 2.3. Suppose that we have more "across all players" constraints. Let y^* be a solution of QVI (2.5) with $K_i(y^*_{-i})$ given by

$$K_i(y^*_{-i}) = \{x \in \mathbb{R} \,|\, \alpha_i \le x \le \beta_i;\; g_j(x, y^*_{-i}) \le 0 \text{ for all } j \in \{1, \ldots, m\}\};$$

and let y^* satisfies condition

$$g_j(y^*) < 0 \quad \text{for all } j \in \{1, \ldots, m\}. \tag{2.7}$$

Then y^* is a solution of VI (2.3). Conversely, if a solution of VI (2.3) fulfills (2.7), then it is also a solution of mentioned QVI.

Lemma 2.1 combined with Lemma 2.2 tell us that QVI (2.5) has always a solution. Lemma 2.3 is a slight modification of Theorem 5 from [7] and implies that the solution set of QVI (2.5) is composed from solutions to VI (2.3) which are inactive to all "across all players" constraints and from solutions which are active to some such a constraint. Let us call the last mentioned solutions, i.e. those in which the inequality $g_j(y) \le 0$ becomes an equality for any j, *active* solutions of the QVI (2.5).

3. SINGLE–CONSTRAINT CASE

To find active solutions of a QVI is a difficult task, but in the case described below this set can be suitably characterized. Before we state the next lemma, let us pose another assumption about utilities :

(U2) $\nabla_{y_i} u_i(y) = f_i(y_i, g(y))$ for all $i \in \{1, \ldots, n\}$ and for all $y \in \Gamma_S$, i.e. $\nabla_{y_i} u_i(y)$ depends only on y_i and $g(y)$.

Theorem 3.1. Suppose that conditions (U1), (U2) and (C1) are satisfied, then the set Ω^* of active solutions of the QVI (2.5) has the following form

$$\Omega^* = \left\{ y^* \;\middle|\; \begin{array}{c} g(y^*) = 0;\; \alpha \le y^* \le \beta;\; \exists\, a \in N_{\times_{i=1}^n K_i(y^*_{-i})}(y^*) \\ \text{such that } \forall i \in \{1, \ldots, n\} \text{ one has } \; (y_i^*, 0) \in f_i^{-1}(a_i) \end{array} \right\}. \tag{3.1}$$

Proof. The form of the set Ω^* follows immediately from the GE (2.6). Indeed, we know from this GE that there exists a vector $a \in N_{\times_{i=1}^n K_i(y_{-i}^*)}(y^*)$ such that

$$\begin{bmatrix} \nabla_{y_1} u_1(y^*) \\ \vdots \\ \nabla_{y_n} u_n(y^*) \end{bmatrix} = a.$$

By virtue of condition (U2) and due to equality $g(y^*) = 0$, this can be rewritten to

$$\begin{bmatrix} f_1(y_1^*, 0) \\ \vdots \\ f_n(y_n^*, 0) \end{bmatrix} = a;$$

which completes the proof. □

Now let us consider the following assumption:

(U2') $u_i(y) = f_i(y_i, g(y))$ for all $i \in \{1, \dots, n\}$ and for all $y \in \Gamma_S$, i.e. $u_i(y)$ depends on y_i and $g(y)$.

This assumption is expressed in terms of utility functions instead of their gradients like in the assumption (U2). The next lemma gives us the relation between (U2) and (U2') for some special cases of function g.

Lemma 3.2. Suppose that the constraint function g has the form $g(y) = \sum_{i=1}^n g_i(y_i)$. Then (U2) can equivalently be replaced by (U2'). The same holds for the form $g(y) = \prod_{i=1}^n g_i(y_i)$.

Proof. Suppose that (U2') holds. When computing the derivative of u_i, one gets

$$\nabla_{y_i} u_i(y) = \nabla_1 f_i(y_i, g(y)) + \nabla_2 f_i(y_i, g(y)) \nabla_{y_i} g_i(y_i),$$

where the symbol $\nabla_1 f_i(x_1, x_2)$ means the partial derivative of the function $f_i(x_1, x_2)$ subject to x_1 and analogously $\nabla_2 f_i(x_1, x_2)$ is partial derivative subject to x_2 . The validity of (U2) is now easy to see.

Conversely, if (U2) holds, then

$$u_i(y) = \int \nabla_{y_i} u_i(y) \, dy_i = \int f_i\left(y_i, g_i(y_i) + \sum_{j \neq i} g_j(y_j)\right) dy_i.$$

Let as describe the last indefinite integral as the function F, i. e.

$$u_i(y) = F\left(y_i, \sum_{j\neq i} g_j(y_j)\right) = F\left(y_i, -g_i(y_i) + \sum_j g_j(y_j)\right).$$

The validity of the assumption (U2') follows directly from this formula. The proof for $\prod_{i=1}^n g_i(y_i)$ can be established in a similar manner. □

The solutions set Ω^* can be easily computed provided we add the following assumptions:

(U3) $f_i(\cdot, 0)$ are strictly monotone for all $i \in \{1, \dots, n\}$;

(C2) $g\left(\cdot, y^*_{-i}\right)$ are strictly monotone and continuous for all $i \in \{1, \dots, n\}$ and for all $y^* \in \Gamma_S$.

In this situation one gets from Theorem 3.1 a useful corollary.

Corollary 3.3. Suppose that the condition (U1), (U2), (U3) and (C2) are satisfied. Then the set Ω^* of active solutions of the QVI (2.5) has the form

$$\Omega^* = \left\{ y^* \;\middle|\; \begin{array}{c} g(y^*) = 0;\; y^*_{\{1,\dots,n\}\setminus I^*} \in \left[\alpha^*_{\{1,\dots,n\}\setminus I^*}, \beta^*_{\{1,\dots,n\}\setminus I^*}\right], \\ y^*_{I^*} \in \{\gamma^*_{I^*}\} \cup [\alpha^*_{I^*}, \beta^*_{I^*}] \end{array} \right\} \tag{3.2}$$

with $\alpha^*, \beta^*, I^*, \gamma^*_{I^*}$ computed by the following algorithm:

Algorithm. Introduce the index sets

$$I^g_{\text{inc}} = \{i \in \{1, \dots, n\} \mid g(\cdot, y_{-i}) \text{ are increasing}\}$$

$$I^f_{\text{inc}} = \{i \in \{1, \dots, n\} \mid f_i(\cdot, 0) \text{ are increasing}\},$$

and further introduce I^g_{dec} and I^f_{dec} in an analogous manner. Set $I^* = \emptyset$. To obtain Ω^* let us perform the following steps for all $i \in \{1, \dots, n\}$.

Step 1: Compute the values $a^*_i = f_i(\alpha_i, 0)$ and $b^*_i = f_i(\beta_i, 0)$;

Step 2: If a^*_i and b^*_i are both positive, then put

$$[\alpha^*_i, \beta^*_i] = [\alpha_i, \beta_i] \quad \text{for } i \in I^g_{\text{inc}};$$

$$[\alpha^*_i, \beta^*_i] = [\beta_i, \beta_i] \quad \text{for } i \in I^g_{\text{dec}};$$

Step 3: If a_i^* and b_i^* are both negative, then put

$$[\alpha_i^*, \beta_i^*] = [\alpha_i, \alpha_i] \quad \text{for } i \in I_{\text{inc}}^g ;$$

$$[\alpha_i^*, \beta_i^*] = [\alpha_i, \beta_i] \quad \text{for } i \in I_{\text{dec}}^g ;$$

Step 4: If $\min\{a_i^*, b_i^*\} \le 0 \le \max\{a_i^*, b_i^*\}$, i.e. there exists x_i^* such that $f_i(x_i^*, 0) = 0$, then put

$$[\alpha_i^*, \beta_i^*] = [\alpha_i, x_i^*] \qquad \text{for } i \in \left(I_{\text{inc}}^g \cap I_{\text{dec}}^f\right);$$

$$[\alpha_i^*, \beta_i^*] = [x_i^*, \beta_i] \qquad \text{for } i \in \left(I_{\text{dec}}^g \cap I_{\text{dec}}^f\right);$$

$$\{\gamma_i^*\} \cup [\alpha_i^*, \beta_i^*] = \{\beta_i\} \cup [\alpha_i, x_i^*] \qquad \text{for } i \in \left(I_{\text{dec}}^g \cap I_{\text{inc}}^f\right);$$

$$\{\gamma_i^*\} \cup [\alpha_i^*, \beta_i^*] = \{\alpha_i\} \cup [x_i^*, \beta_i] \qquad \text{for } i \in \left(I_{\text{inc}}^g \cap I_{\text{inc}}^f\right).$$

If $i \in I_{\text{inc}}^f$, add i to I^*.

Proof. Suppose that $y^* \in \Omega^*$ (given by (3.1)). From the convex analysis we know that $N_{\times_{i=1}^n K_i(y_{-i}^*)}(y^*) = \times_{i=1}^n N_{K_i(y_{-i}^*)}(y_i^*)$. This implies that $(y_i^*, 0) \in f_i^{-1}\left(N_{K_i(y_{-i}^*)}(y_i^*)\right)$ for all $i \in \{1, \dots, n\}$. We compute now the cones $N_{K_i(y_{-i}^*)}(y_i^*)$.

Since

$$\begin{aligned} K_i(y_{-i}^*) &= \{x \mid \alpha_i \le x \le \beta_i;\ g(x, y_{-i}^*) \le 0\} \\ &= \{x \mid \alpha_i \le x \le \beta_i;\ g(x, y_{-i}^*) \le g(y_i^*, y_{-i}^*)\}, \end{aligned}$$

one has

$$K_i(y_{-i}^*) = \begin{cases} \{x \mid \alpha_i \le x \le \beta_i;\ x \le y_i^*\} = [\alpha_i, y_i^*] & \text{if } g(\cdot, y_{-i}) \text{ is increasing;} \\ [y_i^*, \beta_i] & \text{if } g(\cdot, y_{-i}) \text{ is decreasing.} \end{cases}$$

Suppose now that $i \in I_{\text{inc}}^g$.

From the relation

$$N_{[\gamma,\delta]}(x) = \begin{cases} [-\infty, 0] & \text{when } x = \gamma \\ 0 & \text{when } \gamma < x < \delta \\ [0, \infty] & \text{when } x = \delta \end{cases}$$

we can deduce

$$N_{K_i(y_{-i}^*)}(y_i^*) = N_{[\alpha_i, y_i^*]}(y_i^*) = \begin{cases} [0, \infty] & \text{when } \alpha_i < y_i^* \\ \mathbb{R} & \text{when } \alpha_i = y_i^*. \end{cases}$$

Now, if the conditions of the second step of the algorithm hold, then $(y_i^*, 0) \in f_i^{-1}([0,\infty])$ for all $y_i^* \in [\alpha_i, \beta_i]$, i.e. $[\alpha_i^*, \beta_i^*] = [\alpha_i, \beta_i]$. If the conditions of the third step hold, then $(y_i^*, 0) \notin f_i^{-1}([0,\infty])$ for all $y_i^* \in [\alpha_i, \beta_i]$ but only for $y_i^* = \alpha_i$ it holds $(\alpha_i, 0) \in f_i^{-1}(\mathbb{R})$, i.e. $[\alpha_i^*, \beta_i^*] = [\alpha_i, \alpha_i]$.

Finally, if the conditions of the fourth step hold, then

$$(y_i^*, 0) \in f_i^{-1}([0,\infty]) \text{ for all } \begin{cases} y_i^* \in [\alpha_i, x_i^*] & \text{when } i \in I_{\text{dec}}^f \\ y_i^* \in [x_i^*, \beta_i] & \text{when } i \in I_{\text{inc}}^f, \end{cases}$$

and $(\alpha_i, 0) \in f_i^{-1}(\mathbb{R})$ in the case when $i \in I_{\text{inc}}^f$.

The proof for $i \in I_{\text{dec}}^g$ can be performed in a similar way. □

Remark 2. In assumption (U3) we can suppose that the functions $f_i(y_i, 0)$ are only monotone; nevertheless the structure of the solutions set Ω^* can again be described analytically, but it is more complicated.

The description of the set Ω^* can be simplified under one additional assumption. Before we state this result, let us introduce a more restrictive variant of assumption (U3).

(U3') $f_i(\cdot, 0)$ are strictly decreasing for all $i \in \{1, \dots, n\}$.

Corollary 3.4. Suppose that the assumptions (U1), (U2), (U3') and (C2) are satisfied. Consider the variational inequality:

$$\text{Find } z^* \in \Gamma \text{ such that}$$
$$\sum_{i=1}^{n} -f_i(z_i^*, 0) \cdot (y_i - z_i^*) \geq 0 \quad \text{for all } y \in \Gamma; \tag{3.3}$$

with Γ given by (2.2).
Then there exists just one solution z^* of the VI above and one can describe Ω^* via z^* in the following way:

$$\Omega^* = \Omega^*(z^*, g, \alpha, \beta) := \left\{ y^* \;\middle|\; \begin{matrix} g(y^*) = 0;\ \alpha_i \leq y_i^* \leq z_i^* \text{ for } i \in I_{\text{inc}}^g; \\ z_i^* \leq y_i^* \leq \beta_i \text{ for } i \in I_{\text{dec}}^g \end{matrix} \right\}. \tag{3.4}$$

Furthermore, Ω^* is nonempty if and only if $g(z^*) \geq 0$.

Proof. The Jacobian of the mapping $F : y \to (-f_1(y_1, 0), \dots, -f_n(y_n, 0))$ from VI (3.3) is positive definite due to (U3'), which means (due to Lemma 2.1) that there exists exactly one solution of the mentioned VI. VI (3.3) can be equivalently rewritten

$$-f_i(z_i^*, 0) \cdot (y_i - z_i^*) \geq 0 \quad \text{for all } y_i \text{ such that } y \in \Gamma, i \in \{1, \dots, n\};$$

i. e., if a_i^* and b_i^* from Corollary 3.3 are both positive, then $z_i^* = \beta_i$, if a_i^* and b_i^* are both negative, then $z_i^* = \alpha_i$ and if there exists an x_i^* such that $f_i(x_i^*, 0) = 0$, then $z_i^* = x_i^*$. The form of Ω^* follows immediately from the algorithm.

From the relations which define z^* we infer that

$$g(z^*) \geq g\left(\alpha_{I_{\text{inc}}^g}, \beta_{I_{\text{dec}}^g}\right).$$

This implies the second statement of this corollary due to the structure of the set Ω^*. □

We illustrate now the above theory by a simple example.

Example 1. Consider the generalized Nash equilibrium problem from Harker [7], where the utility functions and constraints are defined by

$$u_1(y_1, y_2) = -(y_1)^2 - \tfrac{8}{3} y_1 y_2 + 34 y_1;$$

$$u_2(y_1, y_2) = -(y_2)^2 - \tfrac{5}{4} y_1 y_2 + \tfrac{97}{4} y_2;$$

$$[\alpha_1, \beta_1] = [\alpha_2, \beta_2] = [0, 10];$$

$$g(y) = y_1 + y_2 - 15.$$

Then from the derivatives of functions u_1 and u_2 given by

$$\begin{bmatrix} \nabla_{y_1} u_1(y_1, y_2) \\ \\ \nabla_{y_2} u_2(y_1, y_2) \end{bmatrix} = \begin{bmatrix} -2y_1 - \frac{8}{3} y_2 + 34 \\ \\ -2y_2 - \frac{5}{4} y_1 + \frac{97}{4} \end{bmatrix};$$

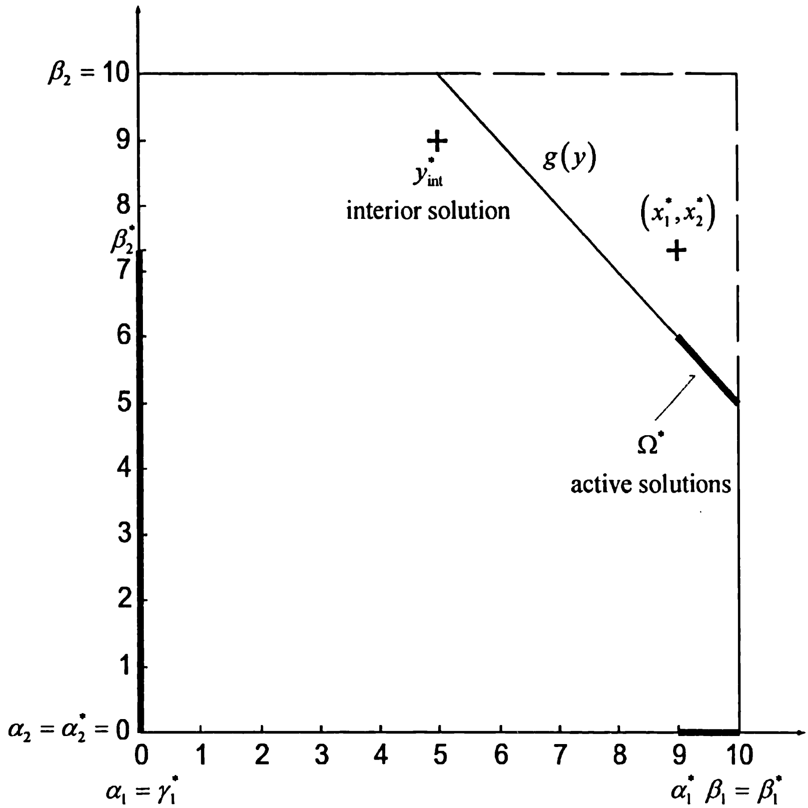
we can see, that they satisfy condition (U2) with f_1 and f_2 given by

$$\begin{bmatrix} f_1(y_1, g(y)) \\ \\ f_2(y_2, g(y)) \end{bmatrix} = \begin{bmatrix} \frac{2}{3} y_1 - \frac{8}{3} g(y) - 6 \\ \\ -\frac{3}{4} y_2 - \frac{5}{4} g(y) + \frac{11}{2} \end{bmatrix}.$$

The Corollary 3.4 cannot be used because the function $f_1(\cdot, 0)$ is strictly increasing, i. e. the condition (U3') does not hold. The conditions (U1), (U3) and (C2) are evidently satisfied; therefore we can use the algorithm from Corollary 3.3.

In its first step we compute

$$a_1^* = -6; \qquad b_1^* = \tfrac{2}{3};$$
$$a_2^* = \tfrac{11}{2}; \qquad b_2^* = -2.$$



The conditions of the second and the third step hold neither for $i = 1$ nor for $i = 2$ and in the fourth step we determine that

$$x_1^* = 9; \quad x_2^* = \tfrac{22}{3};$$

$$[\alpha_1^*, \beta_1^*] = [9, 10];$$

$$[\alpha_2^*, \beta_2^*] = \left[0, \tfrac{22}{3}\right];$$

$$I^* = \{1\}; \quad \gamma_1^* = 0.$$

Consequently, the solution set Ω^* (3.1) has in this case the form

$$\Omega^* = \left\{ (y_1^*, y_2^*) \;\middle|\; \begin{array}{c} y_1^* + y_2^* - 15 = 0;\ 0 \le y_2^* \le \tfrac{22}{3}; \\ 9 \le y_1^* \le 10 \text{ or } y_1^* = 0; \end{array} \right\} = [(9,6), (10,5)];$$

i.e. the line segment which connects the points $(9,6)$ and $(10,5)$.

However, there exists the solution $y^*_{\text{int}} = (5,9)$ of VI (2.3) defined by u_1, u_2 and $[\alpha, \beta]$ which satisfies condition $g(y^*_{\text{int}}) < 0$. Due to Lemma 2.3 y^*_{int} is a generalized Nash equilibrium too. The Jacobian of the mapping

$$F : y \to (-\nabla_{y_1} u_1(y), -\nabla_{y_2} u_2(y));$$

has the form

$$\begin{bmatrix} 2 & \frac{8}{3} \\ \frac{5}{4} & 2 \end{bmatrix};$$

hence it is positive definite, which implies that this interior solution is exactly one due to Lemma 2.1.

4. MULTI–CONSTRAINT CASE

Let us turn now our attention to the case, when there exist more "across all players" constraints than only one, but each of them concerns only an isolated group of players. Suppose that the whole set of players is decomposed in m disjoint sets, i.e.

$$I = \{1, \dots, n\} = \bigcup_{j \in M} I_j \quad I_i \cap I_j = \emptyset \text{ for } i \neq j;$$

where $M := \{1, \dots, m\}$.
Define the mapping $\text{ind} : \{1, \dots, n\} \to M$ such that

$$\text{ind}(i) = j \quad \text{if and only if } i \in I_j.$$

Denote by y_{I_j} the subvector of a vector y with elements $y_i, i \in I_j$. Due to above mentioned structure it holds

$$g_j(y) = g_j(y_{I_j}); \tag{4.1}$$

for all $j \in M$.
Now we can introduce the GNE problem:

$$\text{(MU)} \qquad \begin{aligned} &\text{Find } y^* \in \mathbb{R}^n \quad \text{such that} \\ &y^*_i \in \underset{\substack{\alpha_i \le y_i \le \beta_i \\ g_{\text{ind}(i)}(y_i, y^*_{-i}) \le 0}}{\arg\max} \; u_i(y_i, y^*_{-i}) \quad \text{for all } i \in \{1, \dots, n\}. \end{aligned}$$

Under assumptions (U1) and (C1) (for all functions g_j) this problem can be replaced by the QVI:

$$\begin{aligned} &\text{Find } y^* \in \times_{i=1}^n K_i(y^*_{-i}) \text{ such that} \\ &\sum_{i=1}^n -\nabla_{y_i} u_i(y^*) \cdot (y_i - y^*_i) \ge 0 \quad \text{for all } y \in \times_{i=1}^n K_i(y^*_{-i}); \end{aligned} \tag{4.2}$$

where $K_i\left(y^*_{-i}\right) = \left\{x \left| \alpha_i \le x \le \beta_i;\ g_{\mathrm{ind}(i)}\left(x, y^*_{-i}\right) \le 0\right.\right\}$.

In the next Lemma we specify a form of the subset of the solution set of QVI (4.2). Before we state this Lemma suppose that the index set M is split into two disjoint index sets K and L. At the rest of this section let us suppose, that the set K contains such groups of players which have interior solutions and on the other hand that L contains groups with active solutions. Define by Γ_K the set

$$\Gamma_K := \{y_{I_K} \,|\, \alpha_{I_K} \le y_{I_K} \le \beta_{I_K}\};$$

and the set Γ_L analogously.
Suppose that the conditions (U1) and (C1) (for all functions g_j) hold in the rest of this section and pose the further conditions:

(U2-M) $\nabla_{y_i} u_i(y) = f_i(y_i, g_1(y), \dots, g_m(y))$ for all $i \in \{1, \dots, n\}$ and for all $y \in \Gamma_M$, i. e. $\nabla_{y_i} u_i(y)$ depends on y_i and $g_j(y), j \in \{1, \dots, m\}$;

(U3'-M) the functions $f_i(\cdot, g_K(y^*), 0_L)$ are strictly decreasing for all $i \in \{I_j \,|\, j \in L\}$ and for all $y^* \in \Gamma_K$;

(C2-M) the functions $g_j\left(\cdot, y^*_{-i}\right)$ are strictly monotone for all $j \in L$, for all $i \in I_j$ and for all $y^* \in \Gamma_L$;

(POS) the Jacobian of the mapping

$$F : y_{I_K} \to (-f_i(y_i, g_K(y), 0_L) \,|\, i \in I_K)$$

is positive definite for all $y_{I_K} \in \Gamma_K$;

Consider further the variational inequality:

$$\begin{aligned}&\text{Find } z^* \in \Gamma_K \text{ such that}\\ &\sum_{i \in I_K} -f_i(z_i^*, g_K(z^*), 0_L) \cdot (y_i - z_i^*) \ge 0 \quad \text{for all } y \in \Gamma_K;\end{aligned} \tag{4.3}$$

This VI has due to positive definitness assumption (POS) and due to Lemma 2.1 exactly one solution. Let us denote it by $y^*_{I_K}$.
Now consider another variational inequality

$$\begin{aligned}&\text{Find } z^* \in \Gamma_L \text{ such that}\\ &\sum_{i \in I_L} -f_i\left(z_i^*, g_K\left(y^*_{I_K}\right), 0_L\right) \cdot (y_i - z_i^*) \ge 0 \quad \text{for all } y \in \Gamma_L;\end{aligned} \tag{4.4}$$

where $y^*_{I_K}$ is a solution of the previous VI (4.3).
This VI has again exactly one solution due to assumption (U3'-M) which implies positive definitness of its mapping $-f\left(y, g_K\left(y^*_{I_K}\right), 0_L\right)$. Let us denote it by $y^*_{I_L}$.

Lemma 4.1. Denote by Ω_K^* the set of solutions y of the QVI (4.2) which satisfy the conditions

$$\begin{aligned} g_k(y) &< 0 \quad \text{for all } k \in K; && (4.5)\\ g_l(y) &= 0 \quad \text{for all } l \in L. && (4.6)\end{aligned}$$

Then under conditions (U2-M), (U3'-M), (C2-M) and (POS) Ω_K^* is nonempty if and only if

$$\begin{aligned} g_k\left(y_{I_K}^*\right) &< 0 \quad \text{for all } k \in K; && (4.7)\\ g_l\left(y_{I_L}^*\right) &\geq 0 \quad \text{for all } l \in L. && (4.8)\end{aligned}$$

The form of Ω_K^* is

$$\Omega_K^* = \left\{y \,\middle|\, y_{I_K} = y_{I_K}^*;\, y_{I_l} \in \Omega^*\left(y_{I_l}^*, g_l, \alpha_l, \beta_l\right) \text{ for } l \in L\right\}; \tag{4.9}$$

where the sets $\Omega^*\left(y_{I_l}^*, g_l, \alpha_l, \beta_l\right)$ are defined by (3.4).

Proof. If we suppose that (4.6) holds for a solution of the QVI (4.2) , then we can substitute 0_L in place of $g_L(y)$ in each $f_i(y_i, g_K(y), g_L(y))$. Now if we consider strategies from the subvector y_{I_K} of y , then by (U2-M) and (4.1) derivatives of their utility functions $\nabla_{y_k} u_k(y) = f_k(y_k, g_K(y), 0_L)$, $k \in I_K$, do not depend on strategies from the subvector y_{I_L}. The same holds for their constraints $g_K(y) = g_K(y_{I_K})$, i. e., we can solve separately that part of QVI (4.2) which correspond to subvector y_{I_K}. Due to (4.5) and Lemma 2.3 it suffices to solve VI (4.3) to obtain solutions of the reduced QVI, but VI (4.3) has exactly one solution $y_{I_K}^*$ due to the assumptions imposed.
If we substitute $g_K\left(y_{I_K}^*\right)$ instead of $g_K(y)$ in $\nabla_{y_l} u_l(y) = f_l(y_l, g_K(y), g_L(y))$, $l \in I_l$, we can solve separately that part of QVI (4.2) corresponding to the subvector y_{I_l}, where $l \in L$ is arbitrary. We are thus able to compute the active optimal subvector of strategies of the lth group of players. In addition, the assumptions of Corollary 3.4 are satisfied for each such a subproblem due to conditions (U2-M), (U3'-M) and (C2-M). The rest follows easily from this corollary and from the nonnegativity condition (4.6). □

The preceding lemma informs us, how to find the subset Ω_K^* of the solution set of the QVI (4.2). The whole set Ω_w^* of solutions to this QVI can be obtained via determining each Ω_K^*, $K \subset \{1, \dots, m\}$, i. e.,

$$\Omega_w^* = \bigcup_{K \subset \{1,\dots,m\}} \Omega_K^*. \tag{4.10}$$

Unfortunately, the total number of all subsets of the set $\{1, \dots, m\}$ is 2^m and for each such a subset we have to solve one or two variational inequalities. All in all we have to solve approximately 2^{m+1} problems to obtain the whole set Ω_w^*. This is numerically impossible provided the number of constraints g_j is too large. The preceding result can be applied, however, when we want to know, if some groups of players have active constraints g_j in contrast to others groups of players.

5. COURNOT OLIGOPOLY MODEL EXAMPLE AND GNE

Let us apply the principle from the preceding section to the Cournot oligopoly model. In this model each firm $i \in \{1, \dots, n\}$ furnishes a quantity y_i to a common market. Firm i incurs cost $f_i^c(y_i)$ and obtains revenues $y_i \cdot p\left(\sum_{i=1}^n y_i\right)$. The function $p : \operatorname{int} \mathbb{R}_+ \to \operatorname{int} \mathbb{R}_+$ is usually called the inverse demand curve. The utility functions u_i, $i \in \{1, \dots, n\}$ are given by formula

$$u_i(y) = y_i \cdot p\left(\sum_{i=1}^n y_i\right) - f_i^c(y_i) . \tag{5.1}$$

Consider now the GNE problem (MU) from the beginning of the preceding section with utility functions (5.1), with $[\alpha_i, \beta_i] \subset \operatorname{int} \mathbb{R}_+$ and with constraints g_j of the form

$$g_j(y) = \sum_{i \in I_j} y_i - P_i \quad \text{for all } j \in M = \{1, \dots, m\} ; \tag{5.2}$$

where $P_j > 0$ for all $j \in M$.
These constraints are the joint upper production bounds for isolated groups of firms (for example for firms in different countries) which operate, however, on a common market.

Assume that the following conditions from [11] hold:

(O1) f_i^c are convex and twice continuously differentiable for all $i \in \{1, \dots, n\}$;

(O2) p is strictly convex and twice continuously differentiable on $\operatorname{int} \mathbb{R}_+$;

(O3) $t \cdot p(t)$ is a concave function of t.

In the next lemma we show that these assumptions imply the assumptions of Lemma 4.1 for our Cournot oligopoly model.

Lemma 5.1. Suppose that conditions (O1), (O2) and (O3) hold in the Cournot oligopoly model with g_j given by (5.2). Then conditions (U2-M), (U3'-M), (C2-M) and (POS) are satisfied for arbitrary subsets K and L of the set M.

Proof. Condition (C2-M) is fulfilled obiously. Condition (POS) holds due to Lemma 12.2 from [13]. It remains to prove (U2-M) and (U3'-M).
From (5.1) we compute

$$\nabla_{y_i} u_i(y) = y_i \nabla p\left(\sum_{i=1}^n y_i\right) + p\left(\sum_{i=1}^n y_i\right) - \nabla f_i^c(y_i) ;$$

i. e. the functions f_i from (U2-M) have forms

$$\begin{aligned} & f_i(y_i, g_1(y), \dots, g_m(y)) \\ = \quad & y_i \nabla p\left(\sum_{j=1}^m g_j(y) + \sum_{j=1}^m P_j\right) + p\left(\sum_{j=1}^m g_j(y) + \sum_{j=1}^m P_j\right) - \nabla f_i^c(y_i) . \end{aligned}$$

Hence, (U2-M) is fulfilled.
The function $-\nabla f_i^c(y_i)$ is decreasing due to (O1) and if we show, that

$$\nabla p\left(\sum_{j=1}^{m} g_j(y) + \sum_{j=1}^{m} P_j\right) < 0;$$

then (U3'-M) evidently holds.
From (O3) we can derive

$$\nabla^2 (t \cdot p(t)) = 2\nabla p(t) + t\nabla^2 p(t) \leq 0;$$

i.e.

$$2\nabla p(t) \leq -t\nabla^2 p(t).$$

The right-hand side is, however, negative because $\nabla^2 p(t) > 0$ due to (O2) and $t > 0$ because $t = \sum_{i=1}^{n} y_i$ in our case. This completes the proof. □

Let us illustrate the essence of the preceding lemma on a simple example.

Example 2. Suppose that four firms are acting on the market and incur the following production costs

$$f_1^c(y_1) = \tfrac{1}{18}y_1; \qquad f_2^c(y_1) = \tfrac{1}{16}y_2;$$
$$f_3^c(y_3) = \tfrac{1}{18}y_3; \qquad f_4^c(y_4) = \tfrac{5}{72}y_4;$$

and the inverse demand curve is given by

$$p(T) = T^{-1}.$$

Further, the box constraints have the form

$$[\alpha_1, \beta_1] = [0, 6]; \qquad [\alpha_2, \beta_2] = [0, 6];$$
$$[\alpha_3, \beta_3] = [0, 5]; \qquad [\alpha_4, \beta_4] = [0, 3].$$

There are two groups of firms, the first one is composed from the firms 1 and 2 and the second one from the firms 3 and 4. The constraints g_i have the form

$$g_1(y_1, y_2) = y_1 + y_2 - 10; \quad g_2(y_3, y_4) = y_3 + y_4 - 5.$$

If $K = \{1, 2\}$ and $L = \emptyset$, then $y^*_{I_K} = (3.8792, 2.8212, 3.8792, 1.7633)$ and

$$g_2\left(y^*_{I_K}\right) = 3.8792 + 1.7633 - 5 = 0.6425 > 0;$$

i.e. condition (4.7) does not hold.
If $K = \emptyset$ and $L = \{1, 2\}$, then $y^*_{I_L} = (2.5, 0.9375, 2.5, 0)$ and

$$g_1\left(y^*_{I_L}\right) = 2.5 + 0.9375 - 10 = -6.5625 < 0;$$

i. e. condition (4.8) does not hold.
If $K = \{2\}$ and $L = \{1\}$, then $y^*_{I_K} = (3.2211, 0.5769)$, $y^*_{I_L} = (3.2211, 1.8989)$ and

$$g_1\left(y^*_{I_L}\right) = 3.2211 + 1.8989 - 10 = -4.8800 < 0;$$

i. e. condition (4.8) does not hold.
If $K = \{1\}$ and $L = \{2\}$, then $y^*_{I_K} = (4, 3)$, $y^*_{I_L} = (4, 2)$ and both conditions (4.7) and (4.8) hold, i. e. the whole set Ω^*_w of the GNE in this Cournot oligopoly example has the form

$$\Omega^*_w = \Omega^*_{\{1\}} = (4, 3, [(3, 2), (4, 1)])$$

due to (4.9).

6. CONCLUDING REMARKS

In the single-constraint case it is easy to compute the solution set. Under assumptions of Corollary 3.4 we have to solve two separate VI ((2.3) to obtain interior solution and (3.3) to obtain active solutions). If we suppose further that the functions $f_i(y_i, \cdot)$ are increasing for all $i \in \{1, \dots, n\}$ and that the functions $g(\cdot, y_{-i})$ are decreasing for all $i \in \{1, \dots, n\}$ then the solution set of the QVI (2.5) is composed *either* from the interior solution generated by the VI (2.3) *or* from the active solutions generated by the VI (3.3). Indeed, suppose that there exists a solution y^* of the VI (2.3) such that $g(y^*) < 0$ and a solution z^* of the VI (3.3) such that $0 \leq g(z^*)$. It is easy to deduce from (2.3), (U2) and (3.3), that

$$\left(f_i\left(y^*_i, g(y^*)\right) - f_i\left(z^*_i, 0\right)\right) \cdot \left(y^*_i - z^*_i\right) \geq 0 \quad \text{for all } i \in \{1, \dots, n\}. \tag{6.1}$$

If we suppose, that there exists i such that $y^*_i > z^*_i$, then from (6.1) we deduce that $f_i(y^*_i, g(y^*)) \geq f_i(z^*_i, 0)$, but it contradicts the assumptions that the function $f_i(\cdot, 0)$ is strictly decreasing and $f_i(y_i, \cdot)$ is increasing. Thus $y^*_i \leq z^*_i$ for all $i \in \{1, \dots, n\}$, which means that $g(y^*) \geq g(z^*)$ by virtue of the assumption, that the functions $g(\cdot, y_{-i})$ are decreasing. This, however, contradicts the assumption $g(y^*) < 0 \leq g(z^*)$.

From the above discussion we come to the conclusion that it is possible to simplify the description of the solution set under some additional assumptions imposed on the functions u_i and g in the single-constraint case. It is possible that a similar substantial simplification of the problem occurs also in the multi-constraints case.

ACKNOWLEDGEMENTS

The author would like to thank above all to Jiří V. Outrata for many helpful suggestions. He is also indebted to his wife Valika for patience and to his brother Ondra for TEX support.

(Received March 6, 2000.)

REFERENCES

[1] K. Arrow and G. Debreu: Existence of equilibrium for competitive economy. Econometrica *22* (1954), 265–290.

[2] C. Baiocchi and A. Capelo: Variational and Quasi-Variational Inequalities. Wiley, New York 1984.

[3] J. Cach: A Nonsmooth Approach to the Computation of Equilibria (in Czech). Diploma Thesis, Charles University, Prague 1996.

[4] D. Chan and J.-S. Pang: The generalized quasi-variational problem. Math. Oper. Res. *7* (1982), 211–222.

[5] G. Debreu: A social equilibrium existence theorem. Proc. Nat. Acad. Sci. U. S. A. *38* (1952), 886–893.

[6] P. T. Harker: A variational inequality approach for the determination of oligopolistic market equilibrium. Math. Programming *30* (1984), 105–111.

[7] P. T. Harker: Generalized Nash games and quasi-variational inequalities. European J. Oper. Res. *54* (1991), 81–94.

[8] P. T. Harker and J.-S. Pang: Finite-dimensional variational inequalities and complementarity problems: a survey of theory, algorithms and applications. Math. Programming *60* (1990), 161–220.

[9] T. Ichiishi: Game Theory for Economic Analysis. Academic Press, New York 1983.

[10] V. Mosco: Implicit variational problems and quasi-variational inequalities. In: Nonlinear Operations and the Calculus of Variations – Summer School held in Bruxelles on 8–19 September 1975 (J. P. Gossez et al, ed., Lecture Notes in Mathematics 543.) Springer Verlag, Berlin 1976, pp. 83–156.

[11] F. H. Murphy, H. D. Sherali and A. L. Soyster: A mathematical programming approach for determining oligopolistic market equilibrium. Math. Programming *24* (1982), 92–106.

[12] J. Nash: Non-cooperative games. Ann. of Math. *54* (1951), 286–295.

[13] J. V. Outrata, M. Kočvara and J. Zowe: Nonsmooth Approach to Optimization Problems with Equilibrium Constraints: Theory, Applications and Numerical Results. Kluwer, Dordrecht 1998.

[14] J. V. Outrata and J. Zowe: A numerical approach to optimization problems with variational inequality constraints. Math. Programming *68* (1995), 105–130.

[15] J. B. Rosen: Existence and uniqueness of equilibrium points for concave n-person games. Econometrica *33* (1965), 520–534.

Mgr. Josef Cach, ERA, Průmyslová 387, 530 03 Pardubice and Institute of Information Theory and Automation – Academy of Sciences of the Czech Republic, Pod vodárenskou věží 4, 182 08 Praha 8. Czech Republic.
e-mail: cach@utia.cas.cz