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ON SOME GEOMETRIC CONTROL PROPERTIES OF ACTIVE SUSPENSIONS SYSTEMS

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The geometric control properties of vehicles with active suspensions are analyzed. A special attention is devoted to the problem of disturbance decoupling. Active suspensions of advanced vehicles allow the active rejection of external disturbances exerted on the sprung mass of the vehicle and caused by road surface irregularity. We focus on the road irregularity disturbances with the purpose of isolating the chassis from vibrations transmitted through suspensions. The paper is aimed at the synthesis of a decoupling control law of the regulated outputs, i. e., roll, pitch and chassis height, from the external disturbances. The paper emphasizes that disturbance decoupling can be thought as a structural property of road vehicles with active suspensions. The framework throughout is the geometric approach to the control of dynamic systems. It is shown that a controlled and conditioned invariant subspace exists such that it allows the geometric disturbance localization. The decoupling problem with stability and the algebraic feedback of suspension heights, i. e. the system measurements, are considered. Simulations with real data are included to validate theoretical results. Saturating actuators are also considered in order to model a more realistic case.

1. INTRODUCTION

The geometric approach to the system theory and control is used to derive some structural properties of a mechanical system consisting of a vehicle equipped with active suspensions. This paper mainly focuses on disturbance decoupling properties of such mechanisms.

Active suspensions are employed in advanced vehicles in order to enhance both ride comfort and safety. The actuation of suspensions along with proper sensor systems allows the vehicles controller to actively reject external disturbances. In most of the conventional cars, rejection of disturbances is obtained by passive devices providing a damping force constraint at all frequencies and generally unable to attenuate both low and high frequency vibrations. On the contrary, active suspensions are able to change the damping force according to the sensed vibrations and can improve dynamic performance of the whole system. The control of active suspensions has been widely investigated in the literature. Hrovat [9] studied the problem of optimal design of active suspensions by casting it into an equivalent linear–quadratic (LQG)–optimization problem. H^∞ control theory is applied in [8] to control active suspensions in order to reduce yaw, lateral motion and roll in spite of external disturbances. The problem of estimating suspension parameters was investigated in [14] and [15] where an adaptive observer and an extended Kalman filter were implemented in order to identify parameters.

Two different types of disturbances can influence vehicle dynamics. One acts directly on the sprung mass of the vehicle and can be generated by lateral accelerations, the other type of disturbances is due to road irregularity and is transmitted through the suspensions. In this paper we focus on the last type of disturbances and our purpose is to isolate the chassis from vibrations transmitted through suspensions. The paper is aimed at the synthesis of a decoupling control law making the regulated outputs, consisting of roll, pitch and chassis height, insensitive to the external disturbances. Such a type of regulation is referred to as *ride heights regulation*, see e.g. [17].

The framework throughout is the geometric approach to the control of dynamic systems [3, 4, 5, 18, 19]. The geometric aspects of mechanical system dynamics are strongly emphasized by such an approach. This paper builds upon previous results by the authors [1, 12, 13]. The main result of this paper states that the regulated variables (roll, pitch and height of the chassis) can be always decoupled from external disturbances by means of a state feedback controller. The geometric localization of unaccessible disturbances is shown to be a structural property of vehicles with active suspensions.

In most real applications, the state may be not completely accessible for measurements and the performance of an observer based controller might be unsatisfactory. In this paper we assume that the suspension heights and their time derivatives are accessible for measurements and analyze the possibility of decoupling disturbance through an algebraic feedback of these sensed outputs. This part is based on previous results by the authors [1] which are here specialized for the considered mechanical system. The problem of disturbance decoupling with constant or static measurement feedback attracts a large interest in the literature [1, 2, 6, 7, 10]. However, to the best of our knowledge, the problem has not been completely solved. For instance, open problems exist for systems which do not enjoy nor left nor right invertibility properties. See [7] for an exhaustive presentation of the state of the art on this subject.

Theoretical results are validated by simulations. A realistic case is considered. In particular we will show how introducing a limit on the power of the actuators we achieve, despite the strong limit, good performances.

The paper is organized as follows: Section 2 derives dynamic model of the full car and analyzes the controllability and observability properties. Main results on disturbance decoupling without and with stability requirement are discussed in Section 3. In Section 4 the algebraic output feedback is considered. Finally in Section 5 simulations with real data are reported. The appendix describes the reduced roll/height model of vehicles with active suspensions.

Throughout the paper the following notation is used for a three-map system $(\mathbf{A}, \mathbf{B}, \mathbf{E})$.

ker $\mathbf{E}$ is the nullspace of matrix $\mathbf{E}$,
im $\mathbf{B}$ is the column space of matrix $\mathbf{B}$,
$\max\mathcal{I}(\mathbf{A}, \ker\mathbf{E})$ is the maximal $\mathbf{A}$-invariant subspace contained in $\ker\mathbf{E}$,
$\min\mathcal{I}(\mathbf{A}, \operatorname{im}\mathbf{B})$ is the minimal $\mathbf{A}$-invariant subspace containing subspace $\operatorname{im}\mathbf{B}$,
$\mathcal{V}$ is said $(\mathbf{A}, \operatorname{im}\mathbf{B})$-controlled invariant if $\mathbf{A}\mathcal{V} \subseteq \operatorname{im}\mathbf{B} + \mathcal{V}$,
$\max\mathcal{V}(\mathbf{A}, \operatorname{im}\mathbf{B}, \ker\mathbf{E})$ is the maximal $(\mathbf{A}, \operatorname{im}\mathbf{B})$-controlled invariant contained in $\ker\mathbf{E}$,
$\mathcal{S}$ is said $(\mathbf{A}, \ker\mathbf{E})$-conditioned invariant if $\mathbf{A}(\mathcal{S} \cap \ker\mathbf{E}) \subseteq \mathcal{S}$,
$\min\mathcal{S}(\mathbf{A}, \ker\mathbf{E}, \operatorname{im}\mathbf{B})$ is the minimal $(\mathbf{A}, \ker\mathbf{E})$-conditioned invariant containing $\operatorname{im}\mathbf{B}$.

To simplify notation, symbol "im" is usually omitted, therefore the same symbol may represent a matrix or its column space depending on the context.

2. DYNAMIC MODEL OF THE VEHICLE

The mechanical structures of the vehicle is reported in Figure 1 (front and side view).

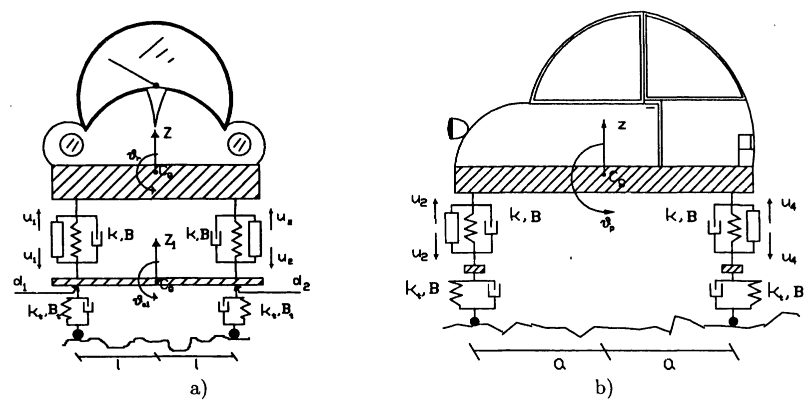

Fig. 1. **Mechanical models of a vehicle with active suspensions. Front and side view.**

The vehicle consists of a rigid chassis and two rigid axes. The sprung mass is linked with these axes by means of four suspensions and actuators. An independent control action is exerted at each active suspensions of the vehicle. The controlled vertical force u_j, $j = 1, \dots, 4$ is generated at the expense of additional energy source such as compressors or pumps. As the aim of the paper is to analyze the structural properties of vehicle mechanisms with active suspensions, the actuator dynamics is not taken into account. Saturating actuators will be considered in Section 5.

Assume that the vehicle is in an equilibrium configuration and that the roll centers, pitch centers and the gravity centers coincide. Moreover assume that tyres are always in contact with the road surface.

According to Figure 1, let us introduce some notation for the model of the vehicle:

θ_r:	variation of the roll angle around the equilibrium;
I_r:	moment of inertia of the chassis about the roll axis;
M_b:	sprung mass;
z:	variation of the height of the M_b center of gravity (CG);
θ_{a1}, I_{a1}:	variation angle and inertia of the axis;
z_1, M_{a1}:	CG height variation and mass of the axis;
θ_{a2}, I_{a2}:	variation angle and inertia of the rear axis;
z_2, M_{a2}:	height and mass of the rear axis.
k, β:	spring and damping coefficients of suspensions;
k_t, β_t:	visco–elastic parameters of tires;
l:	half length of the two axes;
θ_p, I_p:	pitch angle and inertia;
a:	half distance between front and rear axes;
d_j:	independent, unaccessible, external disturbances exerted on the axes at the jth wheel;
u_j:	independent control exerted on the axes at the jth wheel.

The set of strictly positive parameters is defined as

$$\mathcal{P} = \{p \,|\, p = (I_r, I_p, M_b, I_{a1}, I_{a2}, M_{a1}, M_{a2}, a, l, k, \beta, k_t, \beta_t),\ p \in \mathrm{R}^{13}, p_i > 0\}\,. \tag{1}$$

The model has 7 degrees–of–freedom: the roll (θ_r), the pitch (θ_p) angles of the chassis, the rotations of wheel axes (θ_{a1}, θ_{a2}) and the vertical displacements of the sprung mass (z) and of the two axes (z_1, z_2). Lateral and longitudinal dynamics of the sprung mass are not considered.

Equality of visco–elastic parameters of the passive suspensions has been assumed, hence the dynamics of pitch, roll and vertical motions results to be decoupled. Such an assumption can be easily satisfied by means of a proper compensating control for the vertical forces u_i's.

Henceforth, linearized approximation of system dynamics is considered to attack the decoupling problem. This is a reasonable assumption when θ_r, θ_p, z, θ_{a1}, θ_{a2}, z_1 and z_2 are small [9] as in the *ride heights regulation* problem.

Linear approximation of system dynamics around the equilibrium configuration are obtained as

Chassis dynamics (θ_r, θ_p, z):

$$\begin{aligned} I_r\ddot{\theta}_r &= -4kl^2\theta_r - 4\beta l^2\dot{\theta}_r + 2kl^2\theta_{a1} + 2\beta l^2\dot{\theta}_{a1} + 2kl^2\theta_{a2} + 2\beta l^2\dot{\theta}_{a2} + (u_2 - u_1)l \\ &\quad + (u_4 - u_3)l; \\ I_p\ddot{\theta}_p &= -4ka^2\theta_p - 4\beta a^2\dot{\theta}_p - 2kz_1a - 2\beta\dot{z}_1a + 2kz_2a + 2\beta\dot{z}_2a + (u_3 - u_1)a \\ &\quad + (u_4 - u_2)a; \\ M_b\ddot{z} &= -4kz - 4\beta\dot{z} + 2kz_1 + 2\beta\dot{z}_1 + 2kz_2 + 2\beta\dot{z}_2 + (u_1 + u_2 + u_3 + u_4). \end{aligned}$$

Axes dynamics (θ_{ai}, z_i):

$$
\begin{aligned}
I_{a1}\ddot{\theta}_{a1} &= -2(k_t+k)l^2\theta_{a1} - 2(\beta_t+\beta)l^2\dot{\theta}_{a1} + 2kl^2\theta_r + 2\beta l^2\dot{\theta}_r - (u_2-u_1)l - d_1 l + d_2 l;\\
I_{a2}\ddot{\theta}_{a2} &= -2(k_t+k)l^2\theta_{a2} - 2(\beta_t+\beta)l^2\dot{\theta}_{a2} + 2kl^2\theta_r + 2\beta l^2\dot{\theta}_r - (u_4-u_3)l - d_3 l + d_4 l;\\
M_{a1}\ddot{z}_1 &= -2(k_t+k)z_1 - 2(\beta_t+\beta)\dot{z}_1 - 2ka\theta_p - 2\beta a\dot{\theta}_p + 2kz + 2\beta\dot{z} + d_1 + d_2\\
&\quad - (u_1+u_2);\\
M_{a2}\ddot{z}_2 &= -2(k_t+k)z_2 - 2(\beta_t+\beta)\dot{z}_2 + 2ka\theta_p + 2\beta a\dot{\theta}_p + 2kz + 2\beta\dot{z} + d_3 + d_4\\
&\quad - (u_3+u_4).
\end{aligned}
$$

Sign conventions for forces, motion and other parameters of vehicle dynamics are defined in Figure 1.

In this paper we are interested in controlling the chassis posture in spite of disturbances d_j transmitted through the suspensions and generated by road irregularities. Such a type of regulation will be referred to as *ride heights regulation* [17] and consists in controlling the roll and pitch angles and the height of the sprung mass CG. For this regulation problem, the output vector is defined as

$$\mathbf{e} = (\theta_r,\ \theta_p,\ z)^T. \tag{2}$$

2.1. State space model

Vehicle dynamics is here described in the state space domain. Let us define the 14–dimensional state vector $\mathbf{x}$, the 4–dimensional input vector and the 4–dimensional disturbance vector as

$$
\mathbf{x} = (\mathbf{x}_r^T \mathbf{x}_v^T)^T; \quad \text{where} \quad
\begin{cases}
\mathbf{x}_r = (\theta_r\ \theta_{a1}\ \theta_{a2}\ \dot{\theta}_r\ \dot{\theta}_{a1}\ \dot{\theta}_{a2})^T;\\
\mathbf{x}_v = (\theta_p\ z\ z_1\ z_2\ \dot{\theta}_p\ \dot{z}\ \dot{z}_1\ \dot{z}_2)^T;
\end{cases}
$$

$$\mathbf{u} = (u_1\ u_2\ u_3\ u_4); \qquad \mathbf{d} = (d_1\ d_2\ d_3\ d_4).$$

Note that roll dynamics has been grouped in vector $\mathbf{x}_r$, while vector $\mathbf{x}_v$ contains the pitch and vertical dynamics. From the chassis and axes dynamics the state space model of linearized dynamics around the equilibrium configuration is simply obtained as

$$
\begin{cases}
\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{D}\mathbf{d}\\
\mathbf{e} = \mathbf{E}\mathbf{x}
\end{cases} \tag{3}
$$

where the state matrix is

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0}_{(6\times 8)}\\ \mathbf{0}_{(8\times 6)} & \mathbf{A}_{22} \end{bmatrix},$$

with

$$\mathbf{A}_{11} = \begin{bmatrix} \mathbf{0}_3 & \mathbf{I}_3\\ \mathbf{M}_{1k} & \mathbf{M}_{1\beta} \end{bmatrix}; \qquad \mathbf{A}_{22} = \begin{bmatrix} \mathbf{0}_4 & \mathbf{I}_4\\ \mathbf{M}_{2k} & \mathbf{M}_{2\beta} \end{bmatrix};$$

$$\mathbf{M}_{1k} = \begin{bmatrix} \frac{-4kl^2}{I_r} & \frac{2kl^2}{I_r} & \frac{2kl^2}{I_r} \\ \frac{2kl^2}{I_{a1}} & \frac{-2(k_t+k)l^2}{I_{a1}} & 0 \\ \frac{2kl^2}{I_{a2}} & 0 & \frac{-2(k_t+k)l^2}{I_{a2}} \end{bmatrix};$$

$$\mathbf{M}_{1\beta} = \begin{bmatrix} \frac{-4\beta l^2}{I_r} & \frac{2\beta l^2}{I_r} & \frac{2\beta l^2}{I_r} \\ \frac{2\beta l^2}{I_{a1}} & \frac{-2(\beta_t+\beta)l^2}{I_{a1}} & 0 \\ \frac{2\beta l^2}{I_{a2}} & 0 & \frac{-2(\beta_t+\beta)l^2}{I_{a2}} \end{bmatrix};$$

$$\mathbf{M}_{2k} = \begin{bmatrix} \frac{-4ka^2}{I_p} & 0 & \frac{-2ka}{I_p} & \frac{2ka}{I_p} \\ 0 & \frac{-4k}{M_b} & \frac{2k}{M_b} & \frac{2k}{M_b} \\ -\frac{2ka}{M_{a1}} & \frac{2k}{M_{a1}} & \frac{-2(k_t+k)}{M_{a1}} & 0 \\ \frac{2ka}{M_{a2}} & \frac{2k}{M_{a2}} & 0 & \frac{-2(k_t+k)}{M_{a2}} \end{bmatrix};$$

$$\mathbf{M}_{2\beta} = \begin{bmatrix} \frac{-4\beta a^2}{I_p} & 0 & \frac{-2\beta a}{I_p} & \frac{2\beta a}{I_p} \\ 0 & \frac{-4\beta}{M_b} & \frac{2\beta}{M_b} & \frac{2\beta}{M_b} \\ -\frac{2\beta a}{M_{a1}} & \frac{2\beta}{M_{a1}} & \frac{-2(\beta_t+\beta)}{M_{a1}} & 0 \\ \frac{2\beta a}{M_{a2}} & \frac{2\beta}{M_{a2}} & 0 & \frac{-2(\beta_t+\beta)}{M_{a2}} \end{bmatrix},$$

the input matrix is

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix};$$

with

$$\mathbf{B}_1 = \begin{bmatrix} \mathbf{0}_{(3\times 4)} \\ \mathbf{B}_{1L} \end{bmatrix}; \qquad \mathbf{B}_2 = \begin{bmatrix} \mathbf{0}_4 \\ \mathbf{B}_{2L} \end{bmatrix};$$

$$\mathbf{B}_{1L} = \begin{bmatrix} \frac{-l}{I_r} & \frac{l}{I_r} & \frac{-l}{I_r} & \frac{l}{I_r} \\ \frac{l}{I_{a1}} & -\frac{l}{I_{a1}} & 0 & 0 \\ 0 & 0 & \frac{l}{I_{a2}} & -\frac{l}{I_{a2}} \end{bmatrix}; \qquad \mathbf{B}_{2L} = \begin{bmatrix} \frac{-a}{I_p} & \frac{-a}{I_p} & \frac{a}{I_p} & \frac{a}{I_p} \\ \frac{1}{M_b} & \frac{1}{M_b} & \frac{1}{M_b} & \frac{1}{M_b} \\ \frac{-1}{M_{a1}} & \frac{-1}{M_{a1}} & 0 & 0 \\ 0 & 0 & \frac{-1}{M_{a2}} & \frac{-1}{M_{a2}} \end{bmatrix},$$

the disturbance matrix is

$$\mathbf{D} = \begin{bmatrix} \mathbf{D}_1 \\ \mathbf{D}_2 \end{bmatrix};$$

with

$$\mathbf{D}_1 = \begin{bmatrix} \mathbf{0}_{(3\times 4)} \\ \mathbf{D}_{1L} \end{bmatrix}; \qquad \mathbf{D}_2 = \begin{bmatrix} \mathbf{0}_4 \\ \mathbf{D}_{2L} \end{bmatrix};$$

$$\mathbf{D}_{1L} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{-l}{I_{a1}} & \frac{l}{I_{a1}} & 0 & 0 \\ 0 & 0 & \frac{-l}{I_{a2}} & \frac{l}{I_{a2}} \end{bmatrix}; \qquad \mathbf{D}_{2L} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{M_{a1}} & \frac{1}{M_{a1}} & 0 & 0 \\ 0 & 0 & \frac{1}{M_{a2}} & \frac{1}{M_{a2}} \end{bmatrix},$$

and finally the output matrix is

$$\mathbf{E} = [\ \mathbf{E}_1 \quad \mathbf{E}_2\ ] \tag{4}$$

with

$$\mathbf{E}_1 = \left[\begin{array}{c|c} \begin{matrix} 1 \\ 0_{(2\times1)} \end{matrix} & 0_{(3\times5)} \end{array} \right], \quad \mathbf{E}_2 = \left[\begin{array}{c|c} \begin{matrix} 0_{(1\times2)} \\ \mathbf{I}_2 \end{matrix} & 0_{(3\times6)} \end{array} \right].$$

Controllability and observability properties of dynamic system (3) are analyzed in the next proposition.

Proposition 1. Dynamic system $(\mathbf{A}, \mathbf{B}, \mathbf{E})$ in (3) is controllable and observable almost everywhere over the set of parameters $p \in \mathcal{P}$ defined in (1).

Proof. Because of the structure of matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{E}$, after some algebra, one gets that the maximal $\mathbf{A}$-invariant subspace contained in $\ker \mathbf{E}$ is zero:

$$\max \mathcal{I}(\mathbf{A}, \ker \mathbf{E}) = \mathbf{0}.$$

In fact its orthogonal complement results

$$\min \mathcal{I}(\mathbf{A}^T, \mathbf{E}^T) = \begin{bmatrix} \min \mathcal{I}(\mathbf{A}_{11}^T, \mathbf{E}_1^T) & \mathbf{0} \\ \mathbf{0} & \min \mathcal{I}(\mathbf{A}_{22}^T, \mathbf{E}_2^T) \end{bmatrix}$$

where $\min \mathcal{I}(\mathbf{A}_{11}^T, \mathbf{E}_1^T) = \mathrm{R}^6$ and $\min \mathcal{I}(\mathbf{A}_{22}^T, \mathbf{E}_2^T) = \mathrm{R}^8$, almost everywhere over the set of parameters $\mathcal{P}$.

As regards controllability, it holds that the minimal $\mathbf{A}$-invariant subspace containing $\operatorname{im} \mathbf{B}$ results

$$\min \mathcal{I}(\mathbf{A}, \mathbf{B}) = \begin{bmatrix} \min \mathcal{I}(\mathbf{A}_{11}, \mathbf{B}_1) & \mathbf{0} \\ \mathbf{0} & \min \mathcal{I}(\mathbf{A}_{22}, \mathbf{B}_2) \end{bmatrix}$$

with $\min \mathcal{I}(\mathbf{A}_{11}, \mathbf{B}_1) = \mathrm{R}^6$ and $\min \mathcal{I}(\mathbf{A}_{22}, \mathbf{B}_2) = \mathrm{R}^8$. □

Controllability and observability properties hold for system (3) almost everywhere, that is in the whole set of parameters $\mathcal{P}$ less a zero-measure set. For instance, it can be observed that unobservable modes appear if parameters are such that

$$M_{a1} = M_{a2}, \quad I_{a1} = I_{a2},$$

that is in the special symmetry case where the front and rear axes have the same inertia and mass.

From a practical point of view, the full characterization of the zero-measure set where structural properties of Proposition 1 are lost is not worthy of consideration. Henceforth we assume that

$$p \in \bar{\mathcal{P}} = \{p \in \mathcal{P},\ \ \min\mathcal{I}(\mathbf{A},\mathbf{B}) = \mathrm{R}^{14},\ \ \max\mathcal{I}(\mathbf{A},\ker\mathbf{E}) = 0\}$$

or, in other terms, that system (3) is controllable and observable.

3. LOCALIZATION OF DISTURBANCES

According to the state space description of vehicle dynamics derived in Section 2, the *ride heights regulation* can be formalized as a problem of unaccessible disturbance localization [1, 12]:

Problem 1. Given dynamic system (3), determine, if possible, a state feedback $\mathbf{u} = \mathbf{F}\mathbf{x}$ such that, for the feedback system starting at zero state, it holds $\mathbf{e}(t) = 0,\ t \geq 0,$ for all admissible $\mathbf{d}(\cdot)$.

Problem 1 is approached in a geometric control framework [3, 18]. A well known result on unaccessible disturbance localization states that Problem 1 admits solution if and only if the column space of disturbance matrix $\mathbf{D}$ is contained in $\mathcal{V}^*$, the maximal $(\mathbf{A},\mathbf{B})$–controlled invariant contained in $\ker\mathbf{E}$:

$$\mathrm{im}\,\mathbf{D} \subseteq \mathcal{V}^* := \max\mathcal{V}(\mathbf{A},\mathbf{B},\ker\mathbf{E}). \tag{5}$$

The following proposition shows that the unaccessible disturbance localization of regulated output $\mathbf{e}$ for dynamic system (3), i.e. the *ride heights regulation*, is a structural property of vehicles with active suspensions.

Proposition 2. Refer to the dynamic system in eq. (3) of a vehicle with active suspensions. Problem 1 always admits a solution, i.e. there always exists a state feedback gain $\mathbf{F}$ which localizes disturbances $\mathbf{d}(\cdot)$ in the nullspace of the regulated output matrix $\mathbf{E}$.

Proof. Define subspace $\mathcal{J}$, included in the nullspace of matrix $\mathbf{E}$ in (4),

$$\mathcal{J} = \mathrm{im}\,\mathbf{J}; \quad \text{with } \mathbf{J} = \begin{bmatrix} \mathbf{J}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2 \end{bmatrix} \tag{6}$$

where

$$\mathbf{J}_1 = \left[\begin{array}{c|c} \begin{matrix} \mathbf{0}_{(1\times 2)} \\ \mathbf{I}_2 \end{matrix} & \mathbf{0}_{(3\times 2)} \\ \hline \mathbf{0}_{(3\times 2)} & \begin{matrix} \mathbf{0}_{(1\times 2)} \\ \mathbf{I}_2 \end{matrix} \end{array}\right]; \quad \mathbf{J}_2 = \left[\begin{array}{c|c} \begin{matrix} \mathbf{0}_2 \\ \mathbf{I}_2 \end{matrix} & \mathbf{0}_{(4\times 2)} \\ \hline \mathbf{0}_{(4\times 2)} & \begin{matrix} \mathbf{0}_2 \\ \mathbf{I}_2 \end{matrix} \end{array}\right].$$

The proof will show that

$$\begin{aligned} \mathrm{im}\,\mathbf{D} &\subseteq \mathcal{J} && (7) \\ \mathcal{J} &\subseteq \max\mathcal{V}(\mathbf{A},\mathbf{B},\ker\mathbf{E}). && (8) \end{aligned}$$

The proof of inclusion (7) is trivial. As regards condition (8), being $\operatorname{im}\mathbf{J} \subseteq \ker \mathbf{E}$, it is sufficient to prove that $\operatorname{im}\mathbf{J}$ is controlled invariant:

$$\mathbf{A}\,\mathcal{J} \subseteq \mathcal{J} + \operatorname{im}\mathbf{B} \tag{9}$$

that is

$$\operatorname{im}\begin{bmatrix} \mathbf{A}_{11}\mathbf{J}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{22}\mathbf{J}_2 \end{bmatrix} \subseteq \operatorname{im}\begin{bmatrix} \mathbf{J}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2 \end{bmatrix} + \operatorname{im}\begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}.$$

Now, being

$$\mathbf{A}_{11}\mathbf{J}_1 = \left[\begin{array}{c|c} \mathbf{0}_{(3\times 2)} & \begin{array}{c}\mathbf{0}_{(1\times 2)} \\ \mathbf{I}_2\end{array} \\ \hline \mathbf{M}_{1k}^{23} & \mathbf{M}_{1\beta}^{23} \end{array}\right]$$

where $\mathbf{M}_{1k}^{23}$ and $\mathbf{M}_{1\beta}^{23}$ are (3×2)-matrices obtained selecting only the 2nd and the 3rd columns of $\mathbf{M}_{1k}$ and $\mathbf{M}_{1\beta}$, respectively and, being

$$\mathbf{A}_{22}\mathbf{J}_2 = \left[\begin{array}{c|c} \mathbf{0}_{(4\times 2)} & \begin{array}{c}\mathbf{0}_2 \\ \mathbf{I}_2\end{array} \\ \hline \mathbf{M}_{2k}^{34} & \mathbf{M}_{2\beta}^{34} \end{array}\right]$$

where (4×2)-matrices $\mathbf{M}_{2k}^{34}$ and $\mathbf{M}_{2\beta}^{34}$ are built selecting the 3rd and the 4th columns of matrices $\mathbf{M}_{2k}$ and $\mathbf{M}_{2\beta}$ inclusion (9) is proved if there exists two matrices $\mathbf{X}_1$ and $\mathbf{X}_2$ such that

$$\mathbf{A}_{11}\operatorname{im}\mathbf{J}_1 \subseteq \operatorname{im}(\mathbf{J}_1) + \operatorname{im}(\mathbf{B}_1\mathbf{X}_1) \tag{10}$$

$$\mathbf{A}_{22}\operatorname{im}\mathbf{J}_2 \subseteq \operatorname{im}(\mathbf{J}_2) + \operatorname{im}(\mathbf{B}_2\mathbf{X}_2) \tag{11}$$

and

$$\mathbf{B}_2\mathbf{X}_1 = \mathbf{B}_1\mathbf{X}_2 = \mathbf{0}. \tag{12}$$

Let us start to prove that condition (10), with constraint (12), is satisfied. By looking at its structure, matrix $\mathbf{B}_1$ can be rewritten as

$$\mathbf{B}_1 = \left[\begin{array}{c} \mathbf{0}_{(3\times 4)} \\ \hline [\mathbf{V}_{11}, -\mathbf{V}_{11}, \mathbf{V}_{13}, -\mathbf{V}_{13}]_{(3\times 4)} \end{array}\right],$$

where vector $\mathbf{V}_{1j}$ is the jth column vector of $\mathbf{B}_{1L}$.

Now, choose

$$\mathbf{X}_1 = \operatorname{im}\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}.$$

Because of the structure of $\mathbf{B}_2$

$$\mathbf{B}_2 = \left[\begin{array}{c} \mathbf{0}_{(4\times 4)} \\ \hline [\mathbf{V}_{21}, \mathbf{V}_{21}, \mathbf{V}_{23}, \mathbf{V}_{23}]_{(4\times 4)} \end{array}\right],$$

where vectors $\mathbf{V}_{2j}$ are the jth column vectors of $\mathbf{B}_{2L}$, constraint (12) holds.
To verify condition (10), simply compute

$$\mathrm{im}\left[\begin{array}{c|c} \begin{matrix}0_{(1\times 2)}\\ \mathbf{I}_2\end{matrix} & 0_{(3\times 2)} \\ \hline 0_{(3\times 2)} & \begin{matrix}0_{(1\times 2)}\\ \mathbf{I}_2\end{matrix}\end{array}\right] + \mathrm{im}\left(\left[\begin{array}{c} 0_{(3\times 4)} \\ \hline [\mathbf{V}_{11}, -\mathbf{V}_{11}, \mathbf{V}_{12}, -\mathbf{V}_{12}]_{(3\times 4)}\end{array}\right]\begin{bmatrix} 1 & 0\\ -1 & 0\\ 0 & 1\\ 0 & -1\end{bmatrix}\right)$$

$$= \mathrm{im}\left[\begin{array}{c|c} \begin{matrix}0_{(1\times 2)}\\ \mathbf{I}_2\end{matrix} & 0_{(3\times 2)} \\ \hline 0_{(3\times 2)} & \begin{matrix}0_{(1\times 2)}\\ \mathbf{I}_2\end{matrix}\end{array}\right] + \mathrm{im}\left[\begin{array}{c|c} 0 & 0\\ 0 & 0\\ 0 & 0\\ \hline \frac{-2l}{I_r} & \frac{-2l}{I_r}\\ \frac{2l}{I_{a1}} & 0\\ 0 & \frac{2l}{I_{r2}}\end{array}\right] = \mathrm{im}\left[\begin{array}{c|c} \begin{matrix}0_{(1\times 2)}\\ \mathbf{I}_2\end{matrix} & 0_{(3\times 3)} \\ \hline 0_{(3\times 2)} & \mathbf{I}_3\end{array}\right]$$

which certainly includes

$$\mathbf{A}_{11}\mathbf{J}_1 = \mathrm{im}\left[\begin{array}{c|c} 0_{(3\times 2)} & \begin{matrix}0_{(1\times 2)}\\ \mathbf{I}_2\end{matrix} \\ \hline \mathbf{M}_{1k}^{23} & \mathbf{M}_{1\beta}^{23}\end{array}\right].$$

Finally, it is an easy matter to verify that conditions (11) under constraint (12) is satisfied by

$$\mathbf{X}_2 = \mathrm{im}\begin{bmatrix} 1 & 0\\ 1 & 0\\ 0 & 1\\ 0 & 1\end{bmatrix}. \qquad \square$$

The following property is central to the analysis of disturbance decoupling with stability addressed in the next section.

Proposition 3. Refer to dynamic system in (3), for subspace $\mathcal{J}$ defined in (6) it holds

$$\mathcal{J} = \max\mathcal{V}(\mathbf{A}, \mathbf{B}, \ker\mathbf{E}),$$

the maximal $(\mathbf{A}, \mathbf{B})$ controlled invariant contained in $\ker\mathbf{E}$.

Proof. Since, for a triple $(\mathbf{A}, \mathbf{B}, \mathbf{E})$ it holds [3] that

$$\max\mathcal{V}(\mathbf{A}, \mathbf{B}, \ker\mathbf{E}) = \left(\min\mathcal{S}(\mathbf{A}^T, \mathbf{E}^T, \ker\mathbf{B}^T)\right)^{\perp}$$

and being

$$\mathcal{J} \subseteq \max\mathcal{V}(\mathbf{A}, \mathbf{B}, \ker\mathbf{E}) \text{ and } \dim\mathcal{J} = 8, \tag{13}$$

it is sufficient to prove that

$$\dim\min\mathcal{S}(\mathbf{A}^T, \mathbf{E}^T, \ker\mathbf{B}^T) = 6.$$

After some algebra, we obtain

$$\min \mathcal{S}\,(\mathbf{A}^T, \mathbf{E}^T, \ker \mathbf{B}^T)$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 & -1/2\frac{I_r}{I_{a1}} & -1/2\frac{I_r}{I_{a2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1/2\frac{M_b}{M_{a1}} & -1/2\frac{M_b}{M_{a2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1/2\frac{I_p}{aM_{a1}} & 1/2\frac{I_p}{aM_{a2}} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

which is full column rank. □

3.1. Disturbance localization with stability

For the localization problem to be technically sound, it must be required that the state feedback, other than localizing disturbances in the nullspace of the output matrix, strictly stabilizes the whole system.

The disturbance localization problem with stability is formalized in the following statement.

Problem 2. Given dynamic system (3), determine, if possible, a static state feedback $\mathbf{u} = \mathbf{F}\mathbf{x}$ which solves Problem 1 and is such that the dynamic matrix of the feedback system $\mathbf{A} + \mathbf{B}\mathbf{F}$ results asymptotically stable.

From [3], Problem 2 is solvable if and only if an $(\mathbf{A}, \mathbf{B})$–controlled invariant $\mathcal{V}$ exists such that it solves Problem 1 and is internally and externally stabilizable.

The following proposition shows that the *ride heights regulation* problem (Problem 2) for vehicles equipped with active suspensions is solvable.

Proposition 4. Refer to system (3), the $(\mathbf{A}, \mathbf{B})$–controlled invariant $\mathcal{J}$ defined in (6) and solving Problem 1 is internally and externally stabilizable. Thus, $\mathcal{J}$ solves Problem 2.

Proof. The external stabilizability of $\mathcal{J}$ comes directly from the stabilizability of dynamic system (3), cf. Proposition 1.

For the internal stabilizability let us recall that a controlled invariant is internal stabilizable if and only if all its internal unassignable eigenvalues belongs to the left half complex plane, cf. [3]. Being triple $(\mathbf{A}, \mathbf{B}, \mathbf{E})$ in (3) observable and controllable the internal unassignable eigenvalues of $\mathcal{J} = \max \mathcal{V}\,(\mathbf{A}, \mathbf{B}, \ker \mathbf{E})$ correspond to the transmission zeros of dynamic system (3) which can be computed as those $z \in \mathrm{C}$ whereby the system matrix, first introduced in [16],

$$\mathbf{R}(z) = \begin{bmatrix} z\mathbf{I} - \mathbf{A} & \mathbf{B} \\ \mathbf{E} & \mathbf{0} \end{bmatrix}$$

looses rank.

System matrix $\mathbf{R}(z)$ is a (17×18) matrix and determinants of its 18 minors of order 17 annihilate for

$$\begin{aligned} z &= 1/2\frac{-2\beta_t \pm 2\sqrt{\beta_t^2 - 2k_t M_{a1}}}{M_{a1}} \\ z &= 1/2\frac{-2\beta_t \pm 2\sqrt{\beta_t^2 - 2k_t M_{a2}}}{M_{a2}}. \end{aligned} \tag{14}$$

which are the transmission zeros of $(\mathbf{A}, \mathbf{B}, \mathbf{E})$.

Being β_t, k_t, M_{a1} and M_{a2} positive parameters, unaccessible invariant zeros belong to the strict left half plane of C and the proof ends. □

Algorithms for computing the state feedback matrix solving Problem 2 are discussed in [3].

4. ALGEBRAIC OUTPUT FEEDBACK

In most real applications, the state is not completely accessible for measurements and the performance of an observer based controller might be unsatisfactory. From an engineering point of view, the localization of disturbance through algebraic feedback of the sensed outputs is very appealing.

The problem of disturbance decoupling with constant or static measurement feedback has always attracted large interest in the control community also recently [1, 2, 6, 7, 10]. However, to the best of our knowledge, the problem has not been completely solved. For instance, open problems exists for systems which do not enjoy nor left nor right invertibility properties. For a presentation of the state of the art on this subject, the reader is referred to [7].

In this section, the disturbance decoupling problem by algebraic measurement feedback is analyzed for the *ride heights regulation* of vehicles with active suspensions. Let suspension heights and their time derivatives be accessible for measurements. The output vector is defined as

$$\mathbf{y} = \left[\mathbf{y}_h^T, \dot{\mathbf{y}}_h^T\right]^T \tag{15}$$

with

$$\mathbf{y}_h = \begin{bmatrix} (z - \theta_r l - \theta_p a) - (z_1 - \theta_{a1} l) \\ (z + \theta_r l - \theta_p a) - (z_1 + \theta_{a1} l) \\ (z - \theta_r l + \theta_p a) - (z_2 - \theta_{a2} l) \\ (z + \theta_r l + \theta_p a) - (z_2 + \theta_{a2} l) \end{bmatrix}. \tag{16}$$

and in state space

$$\mathbf{y} = \mathbf{C}\mathbf{x} \tag{17}$$

where

$$\mathbf{C} = \left[\begin{array}{cc|cc} \mathbf{C}_H & 0_{(4\times3)} & \mathbf{C}_L & 0_4 \\ \hline 0_{(4\times3)} & \mathbf{C}_H & 0_4 & \mathbf{C}_L \end{array} \right] \tag{18}$$

and

$$\mathbf{C}_H = \begin{bmatrix} -l & l & 0 \\ l & -l & 0 \\ -l & 0 & l \\ l & 0 & -l \end{bmatrix}; \quad \mathbf{C}_L = \begin{bmatrix} -a & 1 & -1 & 0 \\ -a & 1 & -1 & 0 \\ a & 1 & 0 & -1 \\ a & 1 & 0 & -1 \end{bmatrix}.$$

The problem of localizing disturbances by means of an algebraic output feedback is stated as follows.

Problem 3. Given dynamic system (3) with sensed output $\mathbf{y}$ (15), determine, if possible, a static state feedback $\mathbf{u} = \mathbf{K}\mathbf{y}$ such that, for the feedback system starting at zero state, it holds

$$\mathbf{e}(t) = 0, \; t \geq 0, \quad \text{for all admissible } \mathbf{d}(\cdot).$$

The following theorem proved in [3] is basic to solve Problem 3.

Theorem 1. Refer to a general triple $(\mathbf{A}, \mathbf{B}, \mathbf{C})$. There exists a matrix $\mathbf{K}$ such that a given subspace $\mathcal{V}$ is an $(\mathbf{A} + \mathbf{BKC})$-invariant if and only if $\mathcal{V}$ is both an $(\mathbf{A}, \mathbf{B})$–controlled and an $(\mathbf{A}, \ker \mathbf{C})$–conditioned invariant.

From Problem 3, it is an easy matter to show that Problem 3 is solvable if and only if there exists a subspace $\mathcal{V}$ so that, cf. [1]

$$\begin{aligned} &\text{i)} \quad \operatorname{im}\mathbf{D} \subseteq \mathcal{V} \subseteq \ker \mathbf{E}; \\ &\text{ii)} \quad \mathcal{V} \quad \text{is an } (\mathbf{A}, \mathbf{B}) - \text{controlled invariant;} \\ &\text{iii)} \quad \mathcal{V} \quad \text{is an } (\mathbf{A}, \ker \mathbf{C}) - \text{conditioned invariant.} \end{aligned} \tag{19}$$

Starting from conditions (19), the solvability of disturbance decoupling by measurement outputs is proved for the active suspension system.

Proposition 5. Consider vehicle dynamics in (3) with measurement equation $\mathbf{y} = \mathbf{C}\mathbf{x}$ (17). There always exists a feedback gain $\mathbf{K}$ from $\mathbf{y}$ to $\mathbf{u}$ which localizes disturbances $\mathbf{d}(\cdot)$ in the nullspace of the regulated output $\mathbf{e} = (\theta_r, \theta_p, z)$.

Proof. Since subspace $\mathcal{J}$ in (6), satisfies condition i) and ii) in (19), it is sufficient to show that resolvent $\mathcal{J}$, is an $(\mathbf{A}, \ker \mathbf{C})$ conditioned invariant. Simply verify that

$$\mathcal{J} \cap \ker \mathbf{C} = \mathbf{0} \tag{20}$$

in fact being

$$\mathbf{CJ} = \begin{bmatrix} \mathbf{C}_{H1} & \mathbf{0}_4 & \mathbf{C}_{L1} & \mathbf{0}_4 \\ \mathbf{0}_4 & \mathbf{C}_{H1} & \mathbf{0}_4 & \mathbf{C}_{L1} \end{bmatrix}$$

with

$$\mathbf{C}_{H1} = \begin{bmatrix} l & 0 \\ -l & 0 \\ 0 & l \\ 0 & -l \end{bmatrix}; \quad \mathbf{C}_{L1} = \begin{bmatrix} -1 & 0 \\ -1 & 0 \\ 0 & -1 \\ 0 & -1 \end{bmatrix},$$

it ensues that $\mathsf{rank}(\mathbf{CJ}) = \mathsf{rank}(\mathbf{J}) = 8$. and the proof ends. □

4.1. Algebraic output feedback with stability for roll/height dynamics

In [1], authors analyzed in a geometric framework the problem of disturbance decoupling using algebraic output feedback without and with stability requirement. Although the necessary and sufficient conditions for the structural problem without stability are constructive and easily checkable, unfortunately, for the problem with stability, conditions are not constructive anymore and solutions must be find case by case. They must be searched among the output-to-input matrices solving the structural problem without stability.

Consider a general five-map system (A, B, C, D, E):

$$\begin{cases} \dot{x} &= Ax + Bu + Dd \\ y &= Cx \\ e &= Ex, \end{cases} \tag{21}$$

in [1] authors derived conditions to solve the disturbance decoupling problem by static output feedback with stability under the hypothesis that triple (A, B, E) is left invertible or triple (A, B, C) is right invertible. Unfortunately the mechanical model of vehicles with active suspensions described in Section 2 is nor left nor right invertible.

However, in the appendix it is proven that the simplified roll/height model of vehicles with active suspensions enjoys the left invertibility property for triple (A, B, E), see Proposition 6 in the appendix. In such cases we can apply results of [1] which are here reported for completeness.

In order to present next theorem, some further notation must be introduced. Define lattice $\phi((\operatorname{im} B + \operatorname{im} D), \ker E)$ as the lattice of all $(A, (\operatorname{im} B + \operatorname{im} D))$–controlled invariants self bounded with respect to $\ker E$:

$$\begin{aligned} &\phi((\operatorname{im} B + \operatorname{im} D), \ker E) \\ := \; &\{\mathcal{V} \mid A\mathcal{V} \subseteq \mathcal{V} + \operatorname{im} B + \operatorname{im} D;\ \mathcal{V} \subseteq \ker E;\ \bar{\mathcal{V}}^* \cap (\operatorname{im} B + \operatorname{im} D) \subseteq \mathcal{V}\} \end{aligned}$$

whose supremum and infimum are given by

$$\bar{\mathcal{V}}^* := \max \mathcal{V}(A, \operatorname{im} B + \operatorname{im} D, \ker E) \tag{22}$$

$$\bar{\mathcal{V}}_m := \bar{\mathcal{V}}^* \cap \min \mathcal{S}(A, \ker E, \operatorname{im} B + \operatorname{im} D) \tag{23}$$

respectively. Moreover, define lattice $\psi((\ker C \cap \ker E), \operatorname{im} D)$ as the lattice of all $(A, (\ker C \cap \ker E))$-conditioned invariants self hidden with respect to $\operatorname{im} D$:

$$\begin{aligned} &\psi((\ker C \cap \ker E), \operatorname{im} D) \\ := \; &\{\mathcal{S} \mid A(\mathcal{S} \cap \ker C \cap \ker E) \subseteq \mathcal{S},\ \operatorname{im} D \subseteq \mathcal{S},\ \mathcal{S} \subseteq \bar{\mathcal{S}}^* + (\ker C \cap \ker E)\} \end{aligned}$$

whose supremum and infimum are given by, respectively

$$\bar{\mathcal{S}}^* := \min \mathcal{S}(A, (\ker C \cap \ker E), \operatorname{im} D) \tag{24}$$
$$\bar{\mathcal{S}}_M := \bar{\mathcal{S}}^* + \max \mathcal{V}(A, \operatorname{im} D, (\ker C \cap \ker E)). \tag{25}$$

Note that all the above subspaces can be easily determined through the standard geometric approach algorithms.

The following theorem, proved in [1], holds.

Theorem 2. Consider the general five-map system (21). Under the assumption of left invertibility for triple (A, B, E) and stabilizability for couple (A, B), an algebraic feedback matrix K of measurements y ($u = KCy$) solving the disturbance decoupling problem with stability exists if and only if

i) $\bar{\mathcal{V}}_m$ is an $(A, \ker C)$-conditioned invariant

ii) subspace $\bar{\mathcal{V}}_M := \bar{\mathcal{V}}_m + \bar{\mathcal{S}}_M$ is internally stabilizable

iii) $\exists F \mid \quad (A + BF)\bar{\mathcal{V}}_M \subseteq \bar{\mathcal{V}}_M\,, \quad (A + BF)_{\mathcal{X}/\bar{\mathcal{V}}_M}$ is stable, $\quad \ker C \subseteq \operatorname{im} F$

This Theorem, directly apply to the *ride heights regulation problem* for the reduced roll/height model of vehicles with active suspensions derived in the appendix. Note that conditions i) and ii) refer to the problem without the stability requirement and are constructive and easily checkable while condition iii) is not constructive.

Moreover, if the structural part (conditions i) and ii)) of the decoupling problem with static output feedback have only one solution, we have no freedom on choosing matrix K and condition is easily checkable. Otherwise, in a general decoupling problem, it may be possible to choose sensed output such that condition iii) of Theorem 2 is satisfied, in particular the condition $\ker C \subseteq \operatorname{im} F$. This condition can always be guaranteed by using more independent sensor systems, the trivial solution is when all the state space is sensed and thus $\ker C = 0$.

5. SIMULATIONS

A realistic simulation of a road vehicle with active suspensions is reported. The parameters of the vehicle geometry and dynamics are reported in Table 1 and have been taken by the work of Peng and Tomizuka, [11].

Table 1. Parameters of vehicle geometry and dynamics; spring and damping coefficients of tires and suspensions.

l	0.9 m	a	2 m
M_b	1500 kg	I_r	360 kg m^2
I_p	2300 kg m^2	M_{a1}	40 kg
M_{a2}	40 kg	I_{a1}	10.8 Kg m^2
I_{a2}	10.8 Kg m^2	K	18E4 N/m
β	1E3 Ns/m	K_t	1.96E5 N/m
β_t	1.92E3 Ns/m		

Consider the full car dynamics in eq. (3) with measurements $\mathbf{y}$ (15). From Proposition 4, a state feedback exists that solves the *ride heights regulation* problem. By using standard algorithms of the geometric approach [3], the state feedback gain is computed as

$$\mathbf{F} = -10^3 \begin{bmatrix} 160, & -250, & 88, & -4, & -1.7, & 0.77, & -440, & 880, & 190, & 10, & -30, & 43, & 1, & -56 \\ -240, & 250, & -88, & 13, & 1.7, & -0.77, & -520, & 950, & 190, & 10, & -32, & 44, & 1, & -56 \\ -95, & 88, & -250, & -7, & 0.77, & -1.7, & 770, & 1300, & 10, & 190, & 36, & 49, & -56, & 1 \\ -170, & -88, & 250, & 10, & -0.77, & 1.7, & 700, & 1300, & 10, & 190, & 34, & 50, & -56, & 1 \end{bmatrix}.$$

It localizes disturbances $\mathbf{d}$ in the nullspace of the regulated output $\mathbf{e} = (\theta_r, \theta_p, z)$. Geometrically, feedback gain $\mathbf{F}$, makes the resolvent $\mathcal{J}$ (6) invariant and asymptotically stabilizes the closed loop system.

In what follows the influence of external disturbances, due to road surface irregularities, is simulated for the full car model with and without the state feedback $\mathbf{u} = \mathbf{F}\mathbf{x}$.

Suppose that the vehicle has a constant speed of 60 km/h and that the variation of the road surface profile occurs every 16 m on the right side of the car ($d_1 \neq 0; d_3 \neq 0; d_2 = d_4 = 0$). Assuming that the front and rear wheels pass the same path, i.e., $d_1 = d(t)$ and $d_3 = d(t - T_c)$ with $T_c = 0.24$ s (first plot in Figure 2), a ride of 10 seconds has been simulated with and without the decoupling feedback

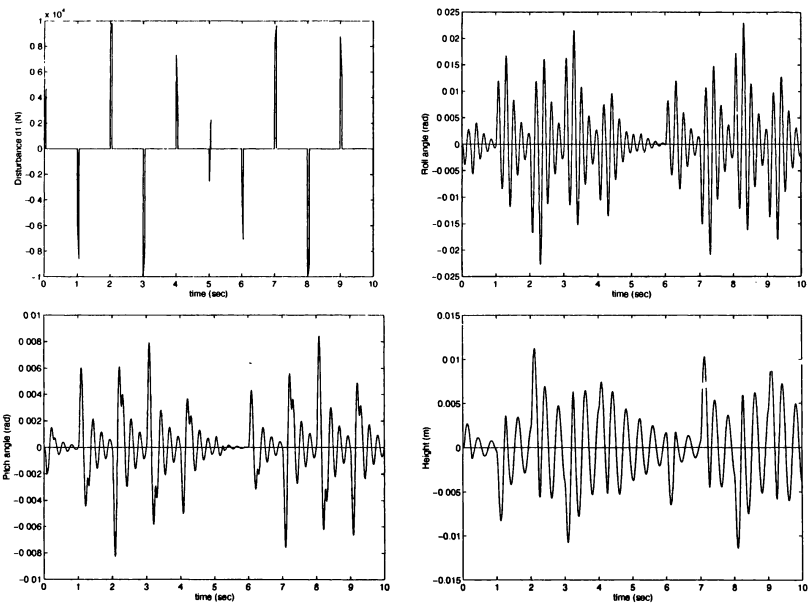

Fig. 2. Disturb, roll, pitch angles and vehicle height for a ride of 10 seconds during which the disturbance d_1 and d_3 are exerted on the vehicle. Both outputs for systems with and without disturbance decoupling are reported. Signals identically zero refer to the vehicle with the decoupling feedback.

The last three plots in Figure 2 refer to the regulated outputs, roll, pitch angles and vehicle height. The outputs are those relative to both system with and without decoupling. As it is expected, variations of roll and pitch angles and of the vehicle height, due to disturbance $\mathbf{d}$, disappear when the disturbance decoupling gain is fed back. The plots in Figure 5. illustrate the behaviour of signals performed by active suspensions and commanded by the disturbance decoupling controller.

The first three plots in Figure 3 report the behaviour of the regulated outputs roll, pitch angles and vehicle height for the decoupled system when an actuator with saturation level at 1200 N is adopted [14]. The control signal is reported as a function of time in Figure 3. It results that the perfect disturbance decoupling cannot be achieved because of saturation of actuators. But, even in presence of a strong saturation, a considerable reduction of the disturbance is obtained, in particular the energy of the disturbance is strongly reduced.

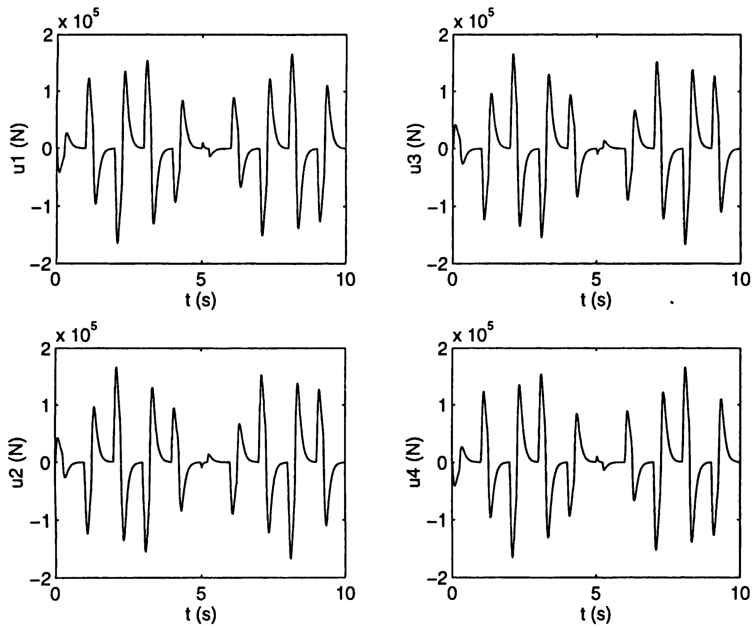

Fig. 3. Active suspensions control outputs.

As regards the reduced model for the roll/height dynamics of the vehicle in eq. (30) with sensed outputs (33), the *ride heights regulation* problem is solved by means of the output feedback gain

$$K = 10^4 \begin{bmatrix} 9.5 & 0.5 & 0.1056 & 0.0056 \\ 0.5 & 9.5 & 0.0056 & 0.1056 \end{bmatrix},$$

which localizes disturbances $\mathbf{d}$ in the nullspace of the regulated output $\mathbf{e} = (\theta_r, z)$. Geometrically, the output feedback gain $\mathbf{K}$, makes the resolvent $\mathcal{J}$ (32) invariant in $(\mathbf{A} + \mathbf{BKC})$.

The stability requirement can be met manipulating the stabilizing state feedback matrix according to the third condition of Theorem 2.

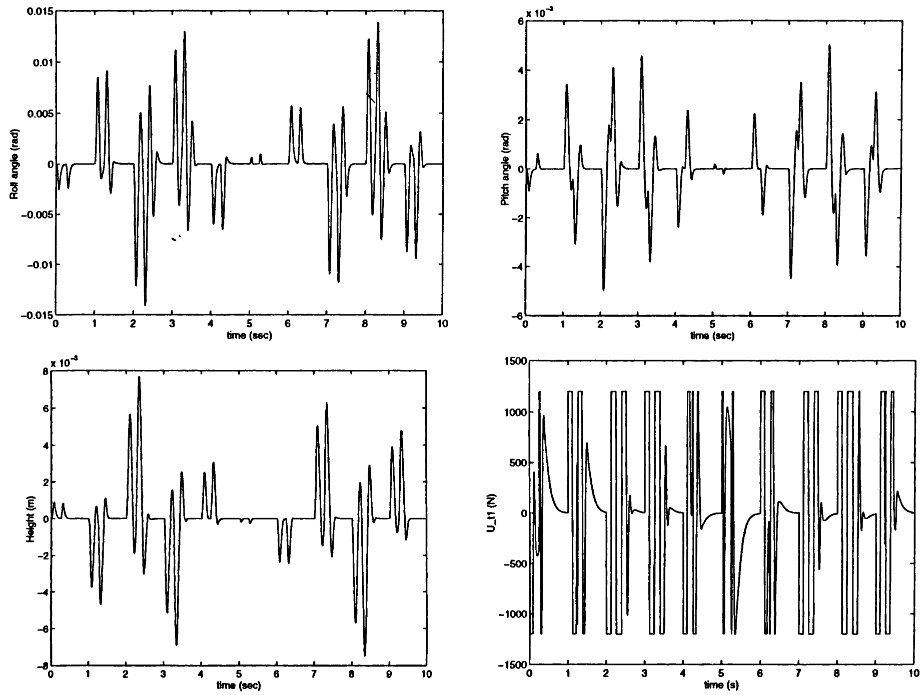

Fig. 4. Roll, pitch angles and vehicle height active suspension control behaviour for disturbance decoupling acting through actuators with saturation level at 1200 N.

6. CONCLUSIONS

This work shows how the geometric approach to the system and control theory can be applied to a class of mechanical systems. Geometric control tools, like the well known conditioned and controlled invariants, allow to emphasize some control properties, like disturbance decoupling, that, because of the generality of the approach, can be viewed as structural properties of the whole system.

The mechanical problem studied, consists in decoupling external disturbances in vehicles with active suspensions. The problem of *ride heights regulation* was considered. The main result of the paper states that there always exists an algebraic feedback able to decouple external disturbances transmitted through suspensions. The aim of this paper is to emphasize that such a decoupling property is a structural property of road vehicles with active suspensions.

The problem of controlling vehicles equipped with active suspensions has been formalized as a decoupling problem and solved in a geometric framework, thus guaranteeing an easy derivation of structural properties as, controllability, observability, left invertibility and decoupling.

The problem of decoupling disturbances through algebraic feedback of sensed outputs was also investigated. In practical applications, the solution of the disturbance decoupling problem through algebraic output feedback strongly simplifies the structure of the control systems and enhances the whole system robustness.

A case study with a realistic simulation has been reported and some aspects of control implementation were discussed. For instance, it has been shown that, even though a saturation at 1200 N of the active suspensions' actuators is present, a considerable disturbance decoupling action is obtained. Work is in progress on the synthesis of the disturbance decoupling control law taking into account different actuators' dynamics other than the saturation level.

Finally, let us remark that the aim of this paper consists in enlightening some structural properties of vehicles with active suspensions more than synthesizing different algorithms taking into account various kinds of available actuators' dynamics.

APPENDIX

This Appendix analyzes the roll/height dynamics of the chassis of the vehicle. Notation refers to Figure 1–a). The system dynamics in the state space and its properties are analyzed.

For the roll/height dynamics of the chassis, the controlled output vector reduces to

$$\mathbf{e} = (\theta_r,\ z)^T. \tag{26}$$

The 8–dimensional state vector, the 2–dimensional input and disturbance vectors are

$$\begin{aligned}
\mathbf{x} &= (\mathbf{x}_r^T\ \mathbf{x}_v^T)^T; && (27)\\
\mathbf{x}_r &= (\theta_r\ \theta_{a1}\ \dot{\theta}_r\ \dot{\theta}_{a1})^T; &&\\
\mathbf{x}_v &= (z\ z_1\ \dot{z}\ \dot{z}_1)^T; &&\\
\mathbf{u} &= (u_1\ u_2)^T; && (28)\\
\mathbf{d} &= (d_1\ d_2)^T && (29)
\end{aligned}$$

and the state space linearized dynamics around the equilibrium configuration is given by

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{D}\mathbf{d}; \\ \mathbf{e} = \mathbf{E}\mathbf{x}, \end{cases} \tag{30}$$

where the state matrix is

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{0}_4 \\ \mathbf{0}_4 & \mathbf{A}_{22} \end{bmatrix},$$

with

$$\mathbf{A}_{11} = \begin{bmatrix} \mathbf{0}_2 & \mathbf{I}_2 \\ \mathbf{M}_{1k} & \mathbf{M}_{1\beta} \end{bmatrix}; \quad \mathbf{A}_{22} = \begin{bmatrix} \mathbf{0}_2 & \mathbf{I}_2 \\ \mathbf{M}_{2k} & \mathbf{M}_{2\beta} \end{bmatrix}$$

$$\mathbf{M}_{1k} = \begin{bmatrix} \frac{-2kl^2}{I_r} & \frac{2kl^2}{I_r} \\ \frac{2kl^2}{I_{a1}} & \frac{-2(k_t+k)l^2}{I_{a1}} \end{bmatrix}; \quad \mathbf{M}_{1\beta} = \begin{bmatrix} \frac{-2\beta l^2}{I_r} & \frac{2\beta l^2}{I_r} \\ \frac{2\beta l^2}{I_{a1}} & \frac{-2(\beta_t+\beta)l^2}{I_{a1}} \end{bmatrix};$$

$$\mathbf{M}_{2k} = \begin{bmatrix} \frac{-2k}{M_b} & \frac{2k}{M_b} \\ \frac{2k}{M_{a1}} & \frac{-2(k_t+k)}{M_{a1}} \end{bmatrix}; \quad \mathbf{M}_{2\beta} = \begin{bmatrix} \frac{-2\beta}{M_b} & \frac{2\beta}{M_b} \\ \frac{2\beta}{M_{a1}} & \frac{-2(\beta_t+\beta)}{M_{a1}} \end{bmatrix},$$

the input matrix is

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix};$$

with

$$\mathbf{B}_1 = \begin{bmatrix} \mathbf{0}_2 \\ \mathbf{B}_{1L} \end{bmatrix}; \quad \mathbf{B}_2 = \begin{bmatrix} \mathbf{0}_2 \\ \mathbf{B}_{2L} \end{bmatrix};$$

$$\mathbf{B}_{1L} = \begin{bmatrix} \frac{-l}{I_r} & \frac{l}{I_r} \\ \frac{l}{I_{a1}} & -\frac{l}{I_{a1}} \end{bmatrix}; \quad \mathbf{B}_{2L} = \begin{bmatrix} \frac{1}{M_b} & \frac{1}{M_b} \\ \frac{-1}{M_{a1}} & \frac{-1}{M_{a1}} \end{bmatrix},$$

the disturbance matrix is

$$\mathbf{D} = \begin{bmatrix} \mathbf{D}_1 \\ \mathbf{D}_2 \end{bmatrix};$$

with

$$\mathbf{D}_1 = \begin{bmatrix} \mathbf{0}_2 \\ \mathbf{D}_{1L} \end{bmatrix}; \quad \mathbf{D}_2 = \begin{bmatrix} \mathbf{0}_2 \\ \mathbf{D}_{2L} \end{bmatrix};$$

$$\mathbf{D}_{1L} = \begin{bmatrix} 0 & 0 \\ \frac{-l}{I_{a1}} & \frac{l}{I_{a1}} \end{bmatrix}; \quad \mathbf{D}_{2L} = \begin{bmatrix} 0 & 0 \\ \frac{1}{M_{a1}} & \frac{1}{M_{a1}} \end{bmatrix},$$

and finally the output matrix of $\mathbf{e}$ in (26) is

$$\mathbf{E} = \left[\begin{array}{c|c|c|c} \begin{matrix} 1 \\ 0 \end{matrix} & \mathbf{0}_{(2\times 3)} & \begin{matrix} 0 \\ 1 \end{matrix} & \mathbf{0}_{(2\times 3)} \end{array} \right]. \tag{31}$$

For this simplified model, the controlled invariant $\mathcal{J}$ solving Problem 1 is given by the column space of

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2 \end{bmatrix}; \quad \mathbf{J}_1 = \mathbf{J}_2 = \left[\begin{array}{c|c} \begin{matrix} 0 \\ 1 \end{matrix} & \mathbf{0}_{(2\times 1)} \\ \hline \mathbf{0}_{(2\times 1)} & \begin{matrix} 0 \\ 1 \end{matrix} \end{array} \right]. \tag{32}$$

It can be shown that $\mathcal{J} = \max \mathcal{V}(\mathbf{A}, \mathbf{B}, \ker \mathbf{E})$ and that all the unassignable eigenvalues of $\mathcal{J}$ are in the strict left half plane. Moreover system $(\mathbf{A}, \mathbf{B}, \mathbf{E})$ is controllable and observable almost everywhere and therefore, $\mathcal{J}$ is the resolvent subspace also for the disturbance decoupling problem with stability (Proposition 4).

As regard the algebraic output feedback, measured outputs are for the simplified model

$$\mathbf{y} = \left[\mathbf{y}_h^T, \dot{\mathbf{y}}_h^T\right]^T \quad \text{with} \quad \mathbf{y}_h = \begin{bmatrix} (z - \theta_r l) - (z_1 - \theta_{a1} l) \\ (z + \theta_r l) - (z_1 + \theta_{a1} l) \end{bmatrix}. \tag{33}$$

The next proposition proves the left invertibility of triple $(\mathbf{A}, \mathbf{B}, \mathbf{E})$ and allows to apply Theorem 2 to the simplified vehicle dynamics.

Proposition 6. The roll/height dynamics of a vehicle with active suspensions described in (30) is *left invertible.*

Proof. Recall that a necessary and sufficient condition for a system to be left invertible with respect to input $\mathbf{u}$ is that

$$\max \mathcal{V}(\mathbf{A}, \mathbf{B}, \ker \mathbf{E}) \cap \operatorname{im} \mathbf{B} = \emptyset. \tag{34}$$

Equation (34) is easy to prove:

$$\left[\begin{array}{cc|cc|c} 0 & & & & \\ 1 & & \multicolumn{2}{c|}{\mathbf{0}_{(2\times 1)}} & \mathbf{0}_{(4\times 2)} \\ \hline \multicolumn{2}{c|}{\mathbf{0}_{(2\times 1)}} & 0 & & \\ & & 1 & & \\ \hline \multicolumn{4}{c|}{} & \begin{array}{c|c} \begin{array}{c} 0 \\ 1 \end{array} & \mathbf{0}_{(2\times 1)} \\ \hline \mathbf{0}_{(2\times 1)} & \begin{array}{c} 0 \\ 1 \end{array} \end{array} \\ \multicolumn{4}{c|}{\mathbf{0}_{(4\times 2)}} & \end{array}\right] \cap \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{-l}{I_r} & \frac{l}{I_r} \\ \frac{l}{I_{a1}} & \frac{-l}{I_{a1}} \\ 0 & 0 \\ 0 & 0 \\ \frac{l}{M_b} & \frac{l}{M_b} \\ \frac{-1}{M_{a1}} & \frac{-1}{M_{a1}} \end{bmatrix} = \emptyset.$$

□

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