KYBERNETIKA — VOLUME 36 (2000), NUMBER 5, PAGES 531–538

ON THE STATE OBSERVATION AND STABILITY FOR UNCERTAIN NONLINEAR SYSTEMS

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In this paper, we treat the class of nonlinear uncertain dynamic systems that was considered in [1, 5, 9, 10]. We consider continuous-time dynamical systems whose nominal part is linear and whose uncertain part is norm-bounded. We study the problems of state observation and obtaining stabilizing controller for uncertain nonlinear systems, where the uncertainties are characterized by known bounds.

1. INTRODUCTION

Consider the following class of nonlinear systems

$$\begin{cases} \dot{x} = Ax + Bu + f(t, x, u) \\ y = Cx \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, A is a known constant $(n \times n)$ matrix, B is a known constant $(n \times m)$ matrix and C is a known constant $(p \times n)$ matrix. The $(n \times 1)$ vector $f(t, x, u)$ assumed continuous in x represents the nonlinear uncertainties in the plant.

For such systems, the authors in [3, 4, 7], obtained results concerning the construction of observers relied on exact knowledge of the plant. Many authors [2, 3, 8] solved the problem of stabilization by an estimated state feedback law given by an observer design for nonlinear system of the form (1) where $f(t, x, u)$ is a Lipschitz nonlinearity.

In this paper, we start by presenting a nonlinear observer, in the presence of plant, that guarantees the observation error is globally exponentially stable. This observer design incorporates only the bound of the nonlinearities (uncertainties), and does not require exact knowledge concerning the structure of the plant nonlinearities $f(t, x, u)$. We show that the result of [9] subsist using a Kalman like observer. Next, a continuous feedback control is proposed to exponentially stabilize nonlinear dynamical systems (1) using the Lyapunov approach, based on the stabilizability of the nominal system $\dot{x} = Ax + Bu$. Furthermore, for more general systems and under some conditions on the uncertainties we show that the system can be uniformly exponentially stable.

2. OBSERVER DESIGN

Given a nonlinear system (1), one can estimate the states by using an observer, whose structure is as follows

$$\dot{\hat{x}}(t) = G(\hat{x}(t), u(t), y(t))$$

where $\hat{x}(t)$ is the state of the observer. It is needed that the estimation error,

$$e(t) = \hat{x}(t) - x(t)$$

has to converge as fast as possible to zero. Most current methods lead to the design of an exponential observer, exponential stability is the most wanted. With the model given in (1), the problem is to design a continuous observer with input $y(t)$ such that the estimates denoted by $\hat{x}(t)$ converge to $x(t)$ exponentially fast. We shall assume the following assumptions.

($\mathcal{H}_1$) The pair (A, C) is observable.

($\mathcal{H}_2$) There exist $\theta > 0$ large enough and a function h_θ, where $h_\theta(\cdot,\cdot,\cdot) : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^p$ such that

$$f(t, x, u) = S_\theta^{-1} C^T h_\theta(t, x, u)$$

where S_θ satisfies the following stationary equation

$$0 = -\theta S_\theta - A^T S_\theta - S_\theta A + C^T C, \quad \theta > 0, \tag{2}$$

and

$$S_\theta = \lim_{t \to +\infty} S_t$$

with $S_t \in \mathcal{S}^+$ the cone of symmetric positive definite matrices on $\mathbb{R}^n$, which satisfies

$$\dot{S}_t = -\theta S_t - A^T S_t - S_t A + C^T C$$

($\mathcal{H}_3$) There exists a positive scalar valued function ρ_θ ([1]) such that

$$||h_\theta(t, x, u)|| \leq \rho_\theta(t, u), \quad \text{for all } t \in \mathbb{R}_+,\ x \in \mathbb{R}^n \text{ and } u \in \mathbb{R}^m,$$

where $||.||$ denotes the usual Euclidean norm on $\mathbb{R}^n$.

Let $e(t) = \hat{x}(t) - x(t)$, the error difference between the estimated $\hat{x}(t)$ given by the following nonlinear observer dynamical equation, and the true state $x(t)$ of the system (1),

$$\dot{\hat{x}} = A\hat{x} + Bu - S_\theta^{-1} C^T (C\hat{x} - y) + \varphi_\theta(\hat{x}, y, \rho_\theta) \tag{3}$$

where

$$\varphi_\theta(\hat{x}, y, \rho_\theta) = \begin{cases} \sigma_\theta(\hat{x}, y, \rho_\theta) & \text{for all} \quad e \notin \text{Ker } C \\ 0 & \text{for all} \quad e \in \text{Ker } C \end{cases}$$

with

$$\sigma_\theta(\hat{x}, y, \rho_\theta) = -\rho_\theta(t, u)\frac{S_\theta^{-1} C^T C e}{||Ce||}.$$

We now state the following stability theorem for the observation error system given by (3) and the auxiliary function $\varphi_\theta(\hat{x}, y, \rho_\theta)$.

Theorem 1. If the assumptions $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ hold, then the system (3) is an observer of the system (1).

Proof. Using $(\mathcal{H}_1)$ one can consider, as in [3], the Lyapunov function

$$V(e) = e^T S_\theta e.$$

Then, one can verify that, the matrix S_θ solution of (3) is positive definite. Indeed, it suffices to change the equation (2) into a Lyapunov equation

$$\left(-A^T - \frac{\theta}{2} I\right) S_\theta + S_\theta \left(-\frac{\theta}{2} I - A\right) = -C^T C \leq 0$$

and for θ large enough, the eigenvalues of the matrix $\left(-\frac{\theta}{2} I - A\right)$ are in the open-left-half plane.

Taking into account $(\mathcal{H}_3)$, the error equation is given by

$$\dot{e}(t) = Ae(t) - S_\theta^{-1} C^T C e(t) + \varphi_\theta(\hat{x}, y, \rho_\theta) - f(t, x, u).$$

The derivative of V is given by

$$\dot{V}(e) = e^T S_\theta \dot{e} + \dot{e}^T S_\theta e$$

$$\dot{V}(e) = e^T S_\theta A e + e^T A^T S_\theta e - 2e^T C^T C e + 2e^T S_\theta(\varphi_\theta(\hat{x}, y, \rho_\theta) - f(t, x, u)).$$

Using (2) and $(\mathcal{H}_2)$,

$$\dot{V}(e) \leq -\theta e^T S_\theta e - e^T C^T C e + 2||Ce||\rho_\theta(t, u) - 2\rho_\theta(t, u)\frac{e^T S_\theta S_\theta^{-1} C^T C e}{||Ce||}.$$

Thus,

$$\dot{V}(e) \leq -\theta e^T S_\theta e - ||Ce||^2 + 2||Ce||\rho_\theta(t, u) - 2||Ce||\rho_\theta(t, u).$$

Hence,

$$\dot{V}(e) \leq -\theta e^T S_\theta e.$$

This estimation holds for θ large enough. Therefore,

$$\dot{V}(e) \leq -\theta V(e).$$

Using the fact that V satisfies

$$\lambda_{\min}(S_\theta)\|e\|^2 \leq V(e)$$

one obtains the following estimation for θ large enough

$$\|e\|^2 \leq K\exp - \theta t, \quad K > 0.$$

It follows that the origin of the error equation is globally exponentially stable.

3. GLOBAL STABILITY

We shall construct a suitable control Lyapunov function at zero, which according to [6], guarantees feedback asymptotic stabilization. Suppose that the following assumptions hold.

$(\mathcal{H}_4)$ The pair (A, B) is stabilizable, therefore, there exists an $(m \times n)$ matrix K such that the eigenvalues of the matrix A_K, defined by $A_K = A + BK$ are in the open-left-half plane.

$(\mathcal{H}_5)$ There exist a $(n \times q)$ constant matrix D and a function $h(\cdot,\cdot,\cdot) : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^q$, such that

$$f(t, x, u) = Dh(t, x, u) \tag{4}$$

and

$$\text{Ker } B^T P \subset \text{Ker } D^T P. \tag{5}$$

$(\mathcal{H}_6)$ There exists a positive scalar valued function ρ ([2]) such that

$$\|h(t, x, u)\| \leq \rho(x), \quad \text{for all } t \in \mathbb{R}_+,\ x \in \mathbb{R}^n \text{ and } u \in \mathbb{R}^m. \tag{6}$$

The following theorem shows asymptotic stability of system (1).

Theorem 2. If the assumptions $(\mathcal{H}_4)$, $(\mathcal{H}_5)$ and $(\mathcal{H}_6)$ hold, then there exists a smooth function v such that the feedback law

$$u(x) = Kx + v(x)$$

stabilizes globally and asymptotically the system (1).

Proof. Consider the Lyapunov function

$$V(x) = x^T P x$$

where $P = P^T > 0$ is such that $P(A + BK) + (A + BK)^T P = -Q$, $Q > 0$.

The derivative of V along the trajectory of the system (1) in closed loop with $u = Kx + v$ where v is an other control is given by

$$\dot{V}(x) = -x^T Qx - 2x^T PBv + 2x^T Pf(t, x, u).$$

Thus, using (4)

$$\dot{V}(x) = -x^T Qx - 2x^T PBv + 2x^T PDh(t, x, u).$$

Taking into account (6), we obtain

$$\dot{V}(x) \leq -x^T Qx - 2x^T PBv + 2||D^T Px||\rho(x)$$

Let

$$\varrho(x) = -x^T Qx + 2||D^T Px \ \rho(x)$$

and

$$\varsigma(x) = 2x^T PB.$$

Therefore by (5), one has

$$B^T Px = 0 \Rightarrow D^T Px = 0, \quad \forall x \in \mathbb{R}^n.$$

It follows that

$$\varsigma(x) = 0 \ \Rightarrow \ \varrho(x) < 0$$

One can deduce that V is a control Lyapunov function for the system

$$\dot{x} = (A + BK)x + Bv + Dh(t, x, u) \qquad (7).$$

Let

$$\nu(x) = ||\varsigma(x)||^2 \quad \text{with} \quad \varsigma_i(x) = 2x^T P\nu_i, \ i = 1, \ldots, m.$$

According to Sontag's theorem [6], the following C^∞ state feedback for $x \neq 0$,

$$v(x) = \Big(v_1(x, y), \ldots, v_m(x, y)\Big)^T$$

defined by

$$v_i(x, y) = \begin{cases} 0 & \text{if } x = 0 \\ \sigma_i(x) & \text{if not} \end{cases}$$

where

$$\sigma_i(x) = \begin{cases} 0 & \text{if } \nu = 0 \text{ and } \varrho < 0 \\ -\varsigma_i \frac{\varrho + \sqrt{\varrho^2 + \nu^4}}{\nu} & \text{if not} \end{cases}$$

stabilizes globally and asymptotically the system (7). Hence, the feedback law

$$u(x) = Kx + v(x)$$

stabilizes the system (1).

Remark 1. Notice that this class of nonlinear systems includes the case treated in [10], it means that if $D = B$ with $q = m$, one can refine the result of [10].

Remark 2. Suppose that $(\mathcal{H}_4)$ holds and the uncertainties $f(t, x, u)$ satisfies for all (t, x, u),

$$f(t, x, u) \leq \rho(x) + \alpha \|u\|, \quad \alpha > 0.$$

Then there exists a smooth function v such that the feedback $u(x) = Kx + v(x)$ stabilizes globally and asymptotically the system (1) provided that $\alpha < \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}$. Indeed, it suffices to take the Lyapunov function $V(x) = x^T P x$. The derivative of V along the trajectories of (1) with $u = Kx + v$ satisfies

$$\dot{V}(x) \leq -(-\lambda_{\min(Q)} + 2\alpha\lambda_{\max(P)})\|x\|^2 - 2x^T PBv + 2\|D^T Px\|\rho(x).$$

Therefore, one can reach conclusion about the sign definiteness argument of $\dot{V}$ by using the same argument of the proof of the above theorem.

Finally, we consider the following class of nonlinear systems

$$\dot{x} = F(t, x, u) + f(t, x, u) \tag{8}$$

where F, f are continuous in t and locally Lipschitz in x such that

$$F(t, 0, 0) = f(t, 0, 0) = 0$$

$\forall t \geq 0$, $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. The corresponding system without uncertainties $f(t, x, u)$, called nominal system is described by

$$\dot{x} = F(t, x, u) \qquad (9).$$

Suppose that the system (8) satisfies the following assumptions required for stability purpose.

$(\mathcal{H}_7)$ The function F as well as the uncertainties function f are continuous uniformly bounded with respect to time and locally uniformly bounded with respect to the state x.

$(\mathcal{H}_8)$ There exists a controller $u = u(t, x)$ which makes the origin $x = 0$ of (9) uniformly exponentially stable equilibrium point. In particular, there exists a C^1-function

$$\mathcal{V} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^+$$

which satisfies

$$\lambda_1 \|x\|^2 \leq \mathcal{V}(t, x) \leq \lambda_2 \|x\|^2,$$

$$\frac{\partial \mathcal{V}}{\partial t}(t, x) + \nabla \mathcal{V}(t, x) F(t, x, u(t)) \leq -\lambda_3 \mathcal{V}(t, x)$$

for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, $\lambda_1, \lambda_2, \lambda_3 > 0$, and

$$\left\| \frac{\partial \mathcal{V}}{\partial x} \right\| \leq \lambda_4 \|x\|.$$

Suppose that, the uncertainties part of the system (8) satisfies,

$(\mathcal{H}_9)$ There exists a positive function ρ, such that $\forall t \in \mathbb{R}$, $\forall x \in \mathbb{R}^n$,

$$\|f(t,x,u)\| \leq \rho(t,x)\|x\|.$$

It is worth noting that assumption $(\mathcal{H}_7)$ is made to guarantee the existence of a classical solution for system (8) under any controller that is continuous and locally uniformly bounded.

Then, one has

Theorem 3. Suppose that the assumptions $(\mathcal{H}_7)$, $(\mathcal{H}_8)$, $(\mathcal{H}_9)$ hold and the non-negative function ρ satisfies for all $(t,x) \in \mathbb{R} \times \mathbb{R}^n$,

$$\rho(t,x) < \frac{\lambda_3\lambda_1}{\lambda_4}.$$

Then the system (8) with the controller $u = u(t,x)$ given in $(\mathcal{H}_8)$ is globally exponentially stable.

Proof. Consider the Lyapunov function $\mathcal{V}(t,x)$ given in $(\mathcal{H}_8)$. The derivative of $\mathcal{V}$ along the trajectories of the system (8) with the controller $u = u(t,x)$ is given by

$$\dot{\mathcal{V}}(t,x) = \frac{\partial \mathcal{V}}{\partial t}(t,x) + \frac{\partial \mathcal{V}}{\partial x}(F(t,x,u(t)) + f(t,x,u)).$$

Thus, by $(\mathcal{H}_8)$ we obtain

$$\dot{\mathcal{V}}(t,x) \leq -\lambda_3\mathcal{V}(t,x) + \frac{\partial \mathcal{V}}{\partial x}(f(t,x,u)).$$

$$\dot{\mathcal{V}}(t,x) \leq -\lambda_3\mathcal{V}(t,x) + \|\frac{\partial \mathcal{V}}{\partial x}\| \, \|f(t,x,u)\|.$$

Therefore,

$$\dot{\mathcal{V}}(t,x) \leq -\lambda_3\lambda_1\|x\|^2 + \lambda_4\|x\| \, \|f(t,x,u)\|.$$

It follows by $(\mathcal{H}_9)$, that

$$\dot{\mathcal{V}}(t,x) \leq -\lambda_3\lambda_1\|x\|^2 + \lambda_4\rho(t,x)\|x\|^2.$$

Hence,

$$\dot{\mathcal{V}}(t,x) \leq (-\lambda_3\lambda_1 + \lambda_4\rho(t,x))\|x\|^2.$$

Therefore, using the fact that $\rho(t,x) < \frac{\lambda_3\lambda_1}{\lambda_4}$, one can obtain the following estimation

$$\dot{\mathcal{V}}(t,x) \leq -l\|x\|^2, \; l > 0$$

which implies, using the properties of $\mathcal{V}$ given in $(\mathcal{H}_8)$, that the system (8) with $u = u(t,x)$ is globally exponentially stable.

4. CONCLUSION

It is shown in this paper that the use of the degenerated Kalman observer solve the problem in observer design for a class of uncertain systems. Furthermore, we construct a continuous nonlinear state controller which used to produce an exponential stability of the whole system in the presence of nonlinearity (uncertainties $f(t, x, u)$).

(Received November 11, 1999.)

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