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AN INTERPOLATION PROBLEM FOR MULTIVARIATE STATIONARY SEQUENCES

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Let X and Y be stationarily cross-correlated multivariate stationary sequences. Assume that all values of Y and all but one values of X are known. We determine the best linear interpolation of the unknown value on the basis of the known values and derive a formula for the interpolation error matrix. Our assertions generalize a result of Budinský [1].

1. INTRODUCTION

In [1] Budinský studied the following problem. Let X and Y be two univariate stationarily cross-correlated stationary sequences. Assume that all values of Y and all but one values of X are known. Find the linear interpolation error of the unknown value of X on the basis of all known values. In the present paper we generalize Budinský's result to multivariate sequences X and Y. The main tool of our investigations is the Hellinger-spectral domain of a stationary sequence. H. Salehi first used Hellinger integrals in the interpolation of multivariate stationary sequences, see [6] and [7]. His method was developed and completed by Makagon and Weron, cf. [2, 3], and [8]. Some results of these authors, on which we heavily lean, are summarized in Section 2. Section 3 is devoted to the solution of the interpolation problem mentioned above. We obtain a formula for the interpolation error matrix as well as a recipe for determining the best linear interpolation of the unknown value. Since our formulas are rather difficult to apply in the general situation, in Section 4 we study some special cases and, using some facts on the Moore–Penrose inverse of a non-negative Hermitian block matrix, derive more tractable formulas for the interpolation error matrix.

2. PRELIMINARIES AND NOTATIONS

Let $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{C}$ be the sets of positive integers, integers, and complex numbers, resp. For $r \in \mathbb{N}$, the symbol $\mathcal{M}_r$, stands for the space of $r \times r$-matrices with complex entries. If $A \in \mathcal{M}_r$, then A^*, $\mathcal{R}(A)$, $\operatorname{Ker} A$, and $\rho(A)$ denote its adjoint, range, kernel, and rank, resp. Furthermore, A^+ is the Moore–Penrose inverse of A, cf. formulas (1.2) in [4]. If A is regular, its inverse A^{-1} coincides with A^+.

The symbol I stands for a unit matrix, where its size should become clear from the context.

Let $\mathcal{H}$ be a Hilbert space over $\boldsymbol{C}$ and $\mathcal{H}^r$ the Cartesian product of r copies of $\mathcal{H}$. We will consider $\mathcal{H}^r$ as a left $\mathcal{M}_r$-module, i.e., the generic element $\boldsymbol{u}$ of $\mathcal{H}^r$ is written as a column vector so that for each $A \in \mathcal{M}_r$ the product $A\boldsymbol{u}$ is defined in a natural way and belongs to $\mathcal{H}^r$. The zero element of $\mathcal{H}^r$ is denoted by $\boldsymbol{O}_r$, whereas the symbol 0 stands for $\boldsymbol{O}_1$ as well as for several zero matrices. For two vectors $\boldsymbol{u}$, $\boldsymbol{v}$ of $\mathcal{H}^r$ let $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ be their Grammian matrix. Finally, $\boldsymbol{e}_k$ denotes the kth unit vector of $\boldsymbol{C}^r$, i.e. the vector whose kth entry is 1 and all its other elements are $0, k \in \{1, \ldots, r\}$.

An r-variate stationary sequence is a map $\boldsymbol{S} : \boldsymbol{Z} \ni n \to \boldsymbol{s}_n \in \mathcal{H}^r$ such that $\langle \boldsymbol{s}_m, \boldsymbol{s}_n \rangle$ depends only on $m - n, m,\ n \in \boldsymbol{Z}$. By $\widetilde{\mathcal{M}}$ we denote the time domain of $\boldsymbol{S}$, i.e. the closed subspace of $\mathcal{H}^r$ spanned by all $\boldsymbol{s}_n,\ n \in \boldsymbol{Z}$, with coefficients from $\mathcal{M}_r$. Recall that $\widetilde{\mathcal{M}} = \mathcal{M}^r$, where $\mathcal{M}$ is the closed linear subspace of $\mathcal{H}$, spanned by the entries of all $\boldsymbol{s}_n,\ n \in \boldsymbol{Z}$.

Let us assume that the spectral measure F of $\boldsymbol{S}$ is absolutely continuous with respect to the Lebesgue measure σ on the σ-algebra $\mathcal{B}$ of Borel sets of $[-\pi, \pi)$. Let f be the spectral density and $L^2(F)$ the spectral domain of $\boldsymbol{S}$, i.e. the left Hilbert $\mathcal{M}_r$-module of (equivalence classes of) $\mathcal{B}$-measurable $\mathcal{M}_r$-valued functions Φ such that $\int_{-\pi}^{\pi} \Phi(\lambda) f(\lambda) \Phi(\lambda)^* \sigma(\mathrm{d}\lambda) = \int \Phi f \Phi^* \mathrm{d}\sigma$ exists.

In the following we will omit the integration variable and the domain of integration $[-\pi, \pi)$ in the notation. Furthermore, relations between $\mathcal{B}$-measurable functions are to be understood as relations that hold σ-a.e., although we will not emphasize this each time.

Let U be Kolmogorov's isomorphism between the time domain and the spectral domain of $\boldsymbol{S}$, i.e., U is an isometric $\mathcal{M}_r$-linear isomorphism of $\widetilde{\mathcal{M}}$ onto $L^2(F)$ such that

$$U\boldsymbol{s}_n = e^{in\cdot} I, \quad n \in \boldsymbol{Z}.$$

Let us consider the Hilbert-$\mathcal{M}_r$-module $H^2(F)$ of (equivalence classes of) $\mathcal{B}$-measurable $\mathcal{M}_r$-valued functions M such that $\operatorname{Ker} M \supseteq \operatorname{Ker} f$ and $\int M f^+ M^* \mathrm{d}\sigma$ exists. The mapping

$$V : \Phi \to \Phi f$$

establishes an isometric $\mathcal{M}_r$-linear isomorphism of $L^2(F)$ onto $H^2(F)$, cf. [6, Theorem 1] and [3, Theorem 3.3 (b)].

It is not hard to see that

$$V^{-1} M = M f^+, \quad M \in H^2(F). \tag{1}$$

In [3, Theorem 3.4 and Lemma 3.7] and [8, Lemma 4.5 (b)] it was proved the following result.

Lemma 1. A vector $\boldsymbol{u}$ of $\widetilde{\mathcal{M}}$ is orthogonal to all $\boldsymbol{s}_n$, $n \in \boldsymbol{Z} \setminus \{0\}$, if and only if $VU\boldsymbol{u}$ is equal to a constant $\mathcal{M}_r$-valued function, where its value A has the following properties:

$$\mathcal{R}(A) \subseteq \mathcal{R}(f) \tag{2}$$

and

$$\int Af^+A^* \mathrm{d}\sigma \tag{3}$$

exists. The matrix A can be computed by

$$A = \langle \boldsymbol{u}, \boldsymbol{s}_0 \rangle. \tag{4}$$

Conversely, if $A \in \mathcal{M}_r$ has properties (2) and (3), then there exists a vector $\boldsymbol{u} \in \widetilde{\mathcal{M}}$, which is orthogonal to all $\boldsymbol{s}_n$, $n \in \boldsymbol{Z} \setminus \{0\}$, such that $VU\boldsymbol{u} = A$ σ-a. e.

3. AN INTERPOLATION PROBLEM

Let $p, q \in \mathbb{N}$ and let $\boldsymbol{X}$ be a p-variate and $\boldsymbol{Y}$ a q-variate stationary sequence such that $\boldsymbol{S} : \boldsymbol{s}_n := \begin{pmatrix} \boldsymbol{x}_n \\ \boldsymbol{y}_n \end{pmatrix}$, $n \in \boldsymbol{Z}$, is a $(p+q)$-variate stationary sequence. Let $\widetilde{\mathcal{M}}_0$ be the closed $\mathcal{M}_{p+q}$-linear hull of all $\boldsymbol{s}_n$, $n \in \boldsymbol{Z} \setminus \{0\}$, and $\begin{pmatrix} \boldsymbol{O}_p \\ \boldsymbol{y}_0 \end{pmatrix}$. Denote the vector $\begin{pmatrix} \boldsymbol{x}_0 \\ \boldsymbol{O}_q \end{pmatrix}$ by $\boldsymbol{x}'_0$. Motivated by a paper of Budinský [1] we study the following interpolation problem:
Find the orthogonal projection $\widehat{\boldsymbol{x}}_0$ of $\boldsymbol{x}'_0$ onto $\widetilde{\mathcal{M}}_0$ and the interpolation error matrix

$$\triangle := \langle \boldsymbol{x}'_0 - \widehat{\boldsymbol{x}}_0, \boldsymbol{x}'_0 - \widehat{\boldsymbol{x}}_0 \rangle .$$

Since $\widetilde{\mathcal{M}}_0$ is of the form $\widetilde{\mathcal{M}}_0 = \mathcal{M}_0^{p+q}$, where $\mathcal{M}_0$ is the closed subspace of $\mathcal{H}$ spanned by the entries of all $\boldsymbol{s}_n$, $n \in \boldsymbol{Z} \setminus \{0\}$, and the entries of $\boldsymbol{y}_0$, the problem is equivalent to determining the orthogonal projections of the entries of $\boldsymbol{x}_0$ onto $\mathcal{M}_0$. However, we find it convenient to study the larger space $\widetilde{\mathcal{M}}_0$ since this allows us to use the isomorphisms U and V.

First note that the singular part of the spectral measure F of $\boldsymbol{S}$ does not affect on the interpolation error. So we assume that $\boldsymbol{S}$ has a spectral density f.
Let

$$\widetilde{\boldsymbol{x}}_0 := \boldsymbol{x}'_0 - \widehat{\boldsymbol{x}}_0.$$

In the following we have to consider block partitions $A := \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ of matrices A from $\mathcal{M}_{p+q}$. In all these cases the left upper block A_{11} is assumed to belong to $\mathcal{M}_p$. In particular, the block partition $f = \begin{pmatrix} f_{11} & f_{12} \\ f_{12}^* & f_{22} \end{pmatrix}$ of f corresponds to the partition of $\boldsymbol{S}$ into $\boldsymbol{X}$ and $\boldsymbol{Y}$ and the interpolation error matrix $\triangle$ has the form

$$\triangle = \begin{pmatrix} \triangle_{11} & 0 \\ 0 & 0 \end{pmatrix}, \tag{5}$$

where $\triangle_{11}$ is non-negative Hermitian and belongs to $\mathcal{M}_p$.

Consider the subset

$L := \left\{ \begin{pmatrix} c \\ O_q \end{pmatrix} \in \boldsymbol{C}^{p+q} : \begin{pmatrix} c \\ O_q \end{pmatrix} \in \mathcal{R}(f) \text{ and } \int \begin{pmatrix} c \\ O_q \end{pmatrix}^* f^+ \begin{pmatrix} c \\ O_q \end{pmatrix} \mathrm{d}\sigma \text{ exists} \right\}$ of $\boldsymbol{C}^{p+q}$. Since $\begin{pmatrix} c \\ O_q \end{pmatrix} \in L$ if and only if $\begin{pmatrix} c \\ O_q \end{pmatrix} \in \mathcal{R}(f)$ and the $\boldsymbol{C}^p$-valued function $((f^+)_{11})^{\frac{1}{2}} c$ is square-integrable, the set L is a subspace of $\boldsymbol{C}^{p+q}$. Denote by P the orthogonal projection in $\boldsymbol{C}^{p+q}$ onto L.

Let E be the $\mathcal{M}_{p+q}$-valued function

$$E(\lambda) := \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad \lambda \in [-\pi, \pi).$$

Theorem 2. The interpolation error matrix Δ can be calculated by

$$\Delta = \left(\int P f^+ P \, \mathrm{d}\sigma \right)^+ . \tag{6}$$

The orthogonal projection of $\boldsymbol{x}_0'$ onto $\widetilde{\mathcal{M}}_0$ is equal to

$$\widehat{\boldsymbol{x}}_0 = U^{-1}(E - \Delta f^+). \tag{7}$$

Proof. Note that $\widetilde{\boldsymbol{x}}_0$ is of the form $\begin{pmatrix} \boldsymbol{u} \\ O_q \end{pmatrix}$ for some $\boldsymbol{u} \in \mathcal{M}_0^p$, which implies

$$\langle \widetilde{\boldsymbol{x}}_0, \boldsymbol{s}_0 \rangle = \langle \widetilde{\boldsymbol{x}}_0, \widetilde{\boldsymbol{x}}_0 \rangle = \Delta. \tag{8}$$

Since $\widetilde{\boldsymbol{x}}_0$ is orthogonal to all $\boldsymbol{s}_n$, $n \in \boldsymbol{Z} \setminus \{0\}$, from Lemma 1 and (8) we obtain that $VU\widetilde{\boldsymbol{x}}_0$ is a constant function whose value is equal to Δ. Since VU is an isometry of $\widetilde{\mathcal{M}}$ onto $H^2(F)$, it follows

$$\Delta = \langle \widetilde{\boldsymbol{x}}_0, \widetilde{\boldsymbol{x}}_0 \rangle = \int \Delta f^+ \Delta \, \mathrm{d}\sigma. \tag{9}$$

Relations (2) and (5) yield $\mathcal{R}(\Delta) \subseteq \mathcal{R}(f)$. Thus $\int \Delta f^+ \Delta \, \mathrm{d}\sigma = \int \Delta P f^+ P \Delta \, \mathrm{d}\sigma = \Delta \int P f^+ P \mathrm{d}\sigma \Delta$. Comparing this with (9), we get

$$\Delta = \Delta \int P f^+ P \mathrm{d}\sigma \, \Delta \, . \tag{10}$$

If we can show that the range of the matrix $B := \int P f^+ P \, \mathrm{d}\sigma$ is included in $\mathcal{R}(\Delta)$, the result immediately follows from (10). But $\mathcal{R}(B) \subseteq \mathcal{R}(P) \subseteq \mathcal{R}(f)$ and the integral $\int B f^+ B \, \mathrm{d}\sigma = B \int P f^+ P \, \mathrm{d}\sigma B$ exists. According to Lemma 1 there exists a vector $\boldsymbol{u}$ of $\widetilde{\mathcal{M}}$, which is orthogonal to all $\boldsymbol{s}_n$, $n \in \boldsymbol{Z} \setminus \{0\}$, such that $VU\boldsymbol{u} = B = \langle \boldsymbol{u}, \boldsymbol{s}_0 \rangle \sigma$-a. e. Moreover, since $\int VU\boldsymbol{u} f^+ VU \begin{pmatrix} O_p \\ \boldsymbol{y}_0 \end{pmatrix} d\sigma = \int B f^+ f \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \mathrm{d}\sigma = \int B \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \mathrm{d}\sigma = 0$, the vector $\boldsymbol{u}$ even belongs to the orthogonal complement of $\widetilde{\mathcal{M}}_0$. This means that it has the form $\boldsymbol{u} = D\widetilde{\boldsymbol{x}}_0$, for some $D \in \mathcal{M}_{p+q}$. Then $B = \langle \boldsymbol{u}, \boldsymbol{s}_0 \rangle = \langle D\widetilde{\boldsymbol{x}}_0, \boldsymbol{s}_0 \rangle = D\langle \widetilde{\boldsymbol{x}}_0, \widetilde{\boldsymbol{x}}_0 \rangle = D\Delta$, which implies $\operatorname{Ker} \Delta \subseteq \operatorname{Ker} B$ and, hence, $\mathcal{R}(B) \subseteq \mathcal{R}(\Delta)$.
To prove (7) note that $U\widehat{\boldsymbol{x}}_0 = U\boldsymbol{x}_0' - U\widetilde{\boldsymbol{x}}_0, U\boldsymbol{x}_0' = E$, and $VU\widetilde{\boldsymbol{x}}_0 = \Delta$ σ-a.e., thus $U\widetilde{\boldsymbol{x}}_0 = V^{-1}\Delta = \Delta f^+$ by (1). □

Corollary 3. The range of Δ is equal to the range of P.

Proof. It was shown in the proof of Theorem 2 that $\mathcal{R}(\Delta) \subseteq \mathcal{R}(P)$. Thus, if P_Δ denotes the orthogonal projector onto $\mathcal{R}(\Delta)$, we get $\int Pf^+Pd\sigma = \Delta^+ = P_\Delta\,\Delta^+\,P_\Delta = P_\Delta \int Pf^+P\,\mathrm{d}\sigma P_\Delta = \int P_\Delta Pf^+PP_\Delta \mathrm{d}\sigma = \int P_\Delta f^+ P_\Delta \mathrm{d}\sigma$. From this equality it is easy to conclude that $\mathcal{R}(P_\Delta) = \mathcal{R}(P)$. □

4. SPECIAL CASES

Under additional assumptions formula (6) can be brought into a more explicit form. Because of (5) it is enough to give expressions for Δ_{11}.

Corollary 4. If the values of f are regular matrices and

$$\int (f_{11} - f_{12}f_{22}^{-1}f_{12}^*)^{-1}\mathrm{d}\sigma \tag{11}$$

exists, then

$$\Delta_{11} = \left(\int (f_{11} - f_{12}f_{22}^{-1}f_{12}^*)^{-1}\mathrm{d}\sigma\right)^{-1}. \tag{12}$$

Proof. If the matrix $f(\lambda)$ is regular, then the left upper block of $f(\lambda)^{-1}$ is equal to $(f_{11}(\lambda) - f_{12}(\lambda)f_{22}(\lambda)^{-1}f_{12}(\lambda)^*)^{-1}$ by the well-known Frobenius formula, $\lambda \in [-\pi, \pi)$. Now the result immediately follows from (6). □

The following corollary generalizes Theorem 1 of [1].

Corollary 5. Let $p = 1$ and the values of f be regular matrices. Then Δ_{11} can be computed by (12), where the right-hand side of (12) is to be interpreted as 0, if the integral (11) does not exist.

Proof. If (11) exists, the result is a special case of Corollary 4. If (11) does not exist, the projection P is equal to 0. □

In the statement and the proof of our next corollary we make use of the following result on matrices, which can be easily obtained from formula (3.24) in [4]. If $A \in \mathcal{M}_{p+q}$ and A is non-negative Hermitian, then $\rho(A) = \rho(A_{22}) + \rho(A_{11} - A_{12}A_{22}^+A_{21})$. In particular, $\rho(A) = \rho(A_{22})$ if and only if $A_{11} - A_{12}A_{22}^+A_{21} = 0$.

Corollary 6. Let $p = 1$. Then $\Delta_{11} = 0$ if one of the following conditions hold:

(i) $\rho(f) = \rho(f_{22})$ or, equivalently, $f_{11} - f_{12}f_{22}^+f_{12}^* = 0$ on a set of positive measure σ.

(ii) $\rho(f) > \rho(f_{22})$ σ-a. e. and the integral

$$\int (f_{11} - f_{12}f_{22}^+f_{12}^*)^{-1}\mathrm{d}\sigma \tag{13}$$

does not exist.

If $\rho(f) > \rho(f_{22})$ σ-a.e. and (13) exists, then $\triangle_{11}$ is equal to

$$\triangle_{11} = \left(\int (f_{11} - f_{12} f_{22}^{+} f_{12}^{*})^{-1} \mathrm{d}\sigma \right)^{-1}. \tag{14}$$

Proof. It is not hard to see that the condition $\rho(f(\lambda)) = \rho(f_{22}(\lambda))$ is equivalent to the fact that e_1 does not belong to $\mathcal{R}(f(\lambda))$, $\lambda \in [-\pi, \pi)$. So, (i) yields $P = 0$ and, hence, $\triangle_{11} = 0$. If $\rho(f(\lambda)) > \rho(f_{22}(\lambda))$, we have $\rho(f_{11}(\lambda) - f_{12}(\lambda) f_{22}(\lambda)^{+} f_{12}(\lambda)^{*}) = 1 = \rho(f_{11}(\lambda))$ and therefore $\rho(f(\lambda)) = \rho(f_{11}(\lambda)) + \rho(f_{22}(\lambda))$. Under this condition the left upper block of $f(\lambda)^{+}$ is equal to $(f_{11}(\lambda) - f_{12}(\lambda) f_{22}(\lambda)^{+} f_{12}(\lambda)^{*})^{-1}$, cf. formula (3.32) in [4]. Thus, from the non-existence of (13) we again conclude $P = 0$ and the existence of (13) yields (14) because of (6). □

Corollary 7. Let $p = 1$. Then $\triangle_{11} = 0$ if and only if e_1 belongs to $\mathcal{R}(f)$ σ-a.e. and the integral (13) exists.

Proof. In the proof of Corollary 6 it was mentioned that e_1 belongs to $\mathcal{R}(f(\lambda))$ if and only if $\rho(f(\lambda)) > \rho(f_{22}(\lambda))$, $\lambda \in [-\pi, \pi)$. Hence, Corollary 7 is a consequence of Corollary 6. □

Now let us use our results to derive a minimality condition for r-variate stationary sequences due to Rozanov [5, Theorem 10.2 of Ch. 2].

An r-variate stationary sequence $\boldsymbol{S}$ is called minimal in the sense of Rozanov if for each $k \in \{1, \ldots, r\}$ the kth entry $s_0^{(k)}$ of $\boldsymbol{s}_0$ does not belong to the space $\mathcal{H}_k$ spanned by the entries of all $\boldsymbol{s}_n$, $n \in \boldsymbol{Z} \setminus \{0\}$, and the elements $s_0^{(j)}, j \neq k$.

Corollary 8. An r-variate stationary process $\boldsymbol{S}$ is minimal in the sense of Rozanov if and only if the values of f are regular matrices and all functions on the principal diagonal of f^{-1} are integrable.

Proof. From Corollary 7 it follows that $s_0^{(k)}$ does not belong to $\mathcal{H}_k$ if and only if e_k belongs to $\mathcal{R}(f)$ and the kth function on the principal diagonal of f^{+} is integrable. But $e_k \in \mathcal{R}(f)$ for all $k \in \{1, \ldots, r\}$ if and only if f^{-1} exists. □

Remark 9. We conclude with the remark that all results of the present paper can be extended to a multivariate stationary process on a discrete Abelian group in an obvious way.

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