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# ON CALCULATION OF STATIONARY DENSITY OF AUTOREGRESSIVE PROCESSES

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An iterative procedure for computation of stationary density of autoregressive processes is proposed. On an example with exponentially distributed white noise it is demonstrated that the procedure converges geometrically fast. The AR(1) and AR(2) models are analyzed in detail.

## 1. INTRODUCTION AND PRELIMINARIES

Let $\{X_t\}$ be a stationary autoregressive process of the first order defined by

$$X_t = bX_{t-1} + e_t, \quad 0 \neq b \in (-1,1) \tag{1.1}$$

where $\{e_t\}$ are i.i.d. random variables with a finite second moment. From (1.1) we have

$$X_t = e_t + be_{t-1} + b^2 e_{t-2} + \dots \tag{1.2}$$

and the series converges in the quadratic mean. It is clear from (1.2) that $\{X_t\}$ is strictly stationary. We are interested in the stationary distribution of the process $\{X_t\}$. The following assertions show that this distribution is continuous.

**Theorem 1.1.** Let $\{\xi_t\}$ be i.i.d. random variables. If the series $X = \sum_{t=-\infty}^{\infty} k_t \xi_t$ converges almost surely and infinite many $k_t$ are different from 0 then $X$ has a continuous distribution.

Proof. See [10]. □

**Theorem 1.2.** Let $\{\eta_t\}$ be independent random variables such that $E\eta_t^2 < \infty$ and $\sum var\, \eta_t < \infty$. Then the series $\sum(\eta_t - E\eta_t)$ converges almost surely.

Proof. See [15], § 16.2 and § 17.2, or [18], Theorem IV.1.4, p. 241. □

**Theorem 1.3.** The stationary distribution of the process $\{X_t\}$ defined by (1.1) is continuous.

Proof. The assertion is a direct consequence of Theorems 1.1 and 1.2. □

For example, if $b = 0.5$ and $e_t$ is a discrete random variable such that $P(e_t = 0.5) = P(e_t = -0.5) = 0.5$ then $X_t$ has the continuous rectangular distribution $R(-1, 1)$. A review of some results of this kind can be found in [2].

However, in many cases stronger assumptions about $e_t$ are made.

**Theorem 1.4.** If $e_t$ has a density, then $X_t$ has also a density.

Proof. Using (1.2) we can write $X_t = e_t + Z_t$ where $Z_t = be_{t-1} + b^2 e_{t-2} + \ldots$ Since $e_t$ and $Z_t$ are independent and $e_t$ has a density, their sum $e_t + Z_t = X_t$ has an absolutely continuous distribution (see [16], p. 196). □

Now, we introduce a known equation for the stationary distribution of $X_t$ and a formula for the characteristic function of $X_t$.

**Theorem 1.5.** Let $e_t$ have a density $f$. Then the density $h$ of $X_t$ satisfies the equation

$$h(x) = \int f(x - bu)\, h(u)\, \mathrm{d}u. \tag{1.3}$$

Proof. The equation follows from (1.1), since $X_{t-1}$ has also the density $h$ and $X_{t-1}$ and $e_t$ are independent. □

**Theorem 1.6.** Let $\psi(t)$ be the characteristic function of $e_t$ and let $\rho(t)$ be the characteristic function of $X_t$. Then

$$\rho(t) = \prod_{n=0}^{\infty} \psi(b^n t).$$

Proof. Define

$$Y_{t,n} = e_t + be_{t-1} + \ldots + b^n e_{t-n}. \tag{1.4}$$

Then $Y_{t,n} \to X_t$ in the quadratic mean as $n \to \infty$. Thus $Y_{t,n} \to X_t$ also in the distribution. It implies that the characteristic functions of the variables $Y_{t,n}$ converge pointwise to $\rho(t)$. But the characteristic function of $Y_{t,n}$ is $\psi(t)\,\psi(bt)\ldots\psi(b^n t)$. □

If $e_t$ has normal distribution then it is well known that $X_t$ is also normally distributed. One of the first attempts to find a connection between the distributions of $X_t$ and $e_t$ in a non-normal case was published in [9]. For some special non-normal distributions of $e_t$ (continuous and discrete rectangular distributions, Laplace distribution) the stationary distribution of $X_t$ is calculated in [2]. It is also known that if $e_t$ has a stable distribution of exponent $\theta$, $(0 < \theta \leq 2)$ then $X_t$ also has a stable distribution of the same exponent (e.g., see [19], p. 208, Ex. 11).

By the way, the opposite problem was more popular, viz. to find a distribution of $e_t$ for a given stationary distribution of $X_t$. The famous paper [12] contains the cases when $X_t$ has exponential or gamma distributions. A review of such results can be found in [1] and [2].

The methods mentioned above were applicable only to some special distributions of $X_t$ and $e_t$. Another approach was proposed in [3]. The goal of this paper was to find a distribution of $e_t$ such that the stationary distribution of $X_t$ has given moments. This method was derived for general linear processes so that autoregressive models form only a special class. Detailed results for AR(1) models are published in [4]. A different method based on Hermite polynomials can be found in [17].

The problem how to calculate stationary distribution of a process from the distribution of a white noise seems to be even more popular in non-linear models. The method of moments was applied in [6]. Exact explicit results are quite rare. One of them can be found in [7] but formulas for stationary density of the absolute autoregression published in [8] and [5] became more familiar (cf. [19], pp. 140–142 and p. 205). Numerical procedures suggested for computation of stationary density in non-linear models (see [19], p. 152) can be, of course, also used in the model (1.1). It corresponds to numerical solution of the equation (1.3).

Quite recently (see [13]) some theoretic results for the stationary density of $X_t$ in the model (1.1) were derived in the case that $e_t$ has the rectangular distribution on $(0, 1)$. It was proved that the stationary density belongs to the class $C^\infty$ and some bounds for tails of this stationary distribution were derived. The derivation was based on investigation of asymptotic properties of $e_t + be_{t-1} + \cdots + b^n e_{t-n}$ as $n \to \infty$.

## 2. AN ITERATIVE METHOD FOR AR(1) PROCESSES

An explicit formula for $h$ satisfying (1.3) given $f$ is known only in a few cases. Here we propose an iterative method for its computation. Let $h_0$ be an arbitrary density. For $n \geq 1$ define

$$h_n(x) = \int f(x - bu)\, h_{n-1}(u)\, du. \tag{2.1}$$

It is obvious that every function $h_n$ defined by (2.1) is a density.

**Theorem 2.1.** Let $h_0$ be a density. Define $h_n$ by (2.1). Assume that there exists an integer $m \geq 0$ such that

$$\int_{-\infty}^{\infty} |\psi(t)\, \psi(bt) \ldots \psi(b^m t)|\, dt < \infty. \tag{2.2}$$

Then $h_n(x) \to h(x)$ for all $x$ as $n \to \infty$.

Proof. Let $\lambda_n$ be the characteristic function corresponding to $h_n$. Using (2.1) we obtain

$$\begin{aligned}\lambda_n(t) &= \int e^{itx}h_n(x)\,\mathrm{d}x = \int e^{itx}\left[\int f(x-bu)\,h_{n-1}(u)\,\mathrm{d}u\right]\mathrm{d}x\\ &= \int h_{n-1}(u)\left[\int e^{itx}f(x-bu)\,\mathrm{d}x\right]\mathrm{d}u = \int h_{n-1}(u)\left[\int e^{itbu+ity}f(y)\,\mathrm{d}y\right]\mathrm{d}u\\ &= \psi(t)\int e^{itbu}h_{n-1}(u)\,\mathrm{d}u = \psi(t)\,\lambda_{n-1}(bt).\end{aligned}$$

Thus

$$\lambda_n(t) = \psi(t)\,\psi(bt)\,\psi(b^2t)\dots\psi(b^{n-1}t)\,\lambda_0(b^nt).$$

From the continuity of the characteristic function we have

$$\lambda_0(b^nt) \to \lambda_0(0) = 1 \quad \text{as} \quad n\to\infty.$$

Using Theorem 1.6 we get $\lambda_n(t)\to\rho(t)$ as $n\to\infty$. Assume that $n>m$. Because

$$h_n(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\lambda_n(t)\,\mathrm{d}t \tag{2.3}$$

and $|e^{-itx}\lambda_n(t)| \le |\psi(t)\,\psi(bt)\dots\psi(b^mt)|$, Lebesgue theorem gives

$$\lim_{n\to\infty} h_n(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\left[\lim_{n\to\infty}\lambda_n(t)\right]\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\rho(t)\,\mathrm{d}t = h(x). \qquad \square$$

Of course, speed of the convergence and complexity of formulas for $h_n$ depend heavily on the choice of $h_0$.

## 3. AN EXAMPLE

Sometimes it is easy to derive speed of convergence $h_n(x)\to h(x)$. From (2.3) we have

$$|h_{n+1}(x)-h_n(x)| \le \frac{1}{2\pi}\int_{-\infty}^{\infty}|\lambda_{n+1}(t)-\lambda_n(t)|\,\mathrm{d}t.$$

Define

$$\Delta_{n+1} = \sup_x |h_{n+1}(x)-h_n(x)|.$$

Then we get

$$\Delta_{n+1} \le \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\psi(t)\,\psi(bt)\dots\psi(b^{n-1}t)[\psi(b^nt)\,\lambda_0(b^{n+1}t)-\lambda_0(b^nt)]\right|\mathrm{d}t. \tag{3.1}$$

Consider the process $\{X_t\}$ given by (1.1) such that $b\in(0,1)$ and $e_t\sim Ex(1)$, i.e., $f(x)=e^{-x}$ for $x>0$. For simplicity, choose $h_0(x)=f(x)$. Then

$$\psi(t) = \lambda_0(t) = \frac{1}{1-it}.$$

Since

$$|\psi(t)| = \frac{1}{\sqrt{1+t^2}}, \tag{3.2}$$

the assumption (2.2) is fulfilled for $m = 1$. The inequality (3.1) can be written in the form

$$\Delta_{n+1} \leq \int_{-\infty}^{\infty} \left| \left[ \prod_{s=0}^{n} \psi(b^s t) \right] [\psi(b^{n+1}t) - 1] \right| \mathrm{d}t. \tag{3.3}$$

Assume that $n \geq 3$. Then (3.3) and (3.2) yield

$$\begin{aligned}
\Delta_{n+1} &\leq \frac{1}{\pi} \int_{-\infty}^{\infty} |\psi(t)\, \psi(bt)| . |\psi(b^2 t) \psi(b^3 t)| . |\psi(b^{n+1}t) - 1| \, \mathrm{d}t \\
&\leq \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1+b^2t^2} \frac{1}{1+b^6t^2} \frac{b^{n+1}t}{\sqrt{1+b^{2n+2}t^2}} \, \mathrm{d}t \\
&\leq b^{n+1} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1+b^2t^2} \frac{1}{1+b^6t^2} \, \mathrm{d}t.
\end{aligned}$$

Since

$$\frac{t}{1+b^6t^2} \leq \frac{1}{2b^3}, \qquad \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{1+b^2t^2} = \frac{1}{2b},$$

we have

$$\Delta_{n+1} \leq \frac{1}{4} b^{n-3}.$$

In this case the iterative procedure converges to the limit density geometrically fast.

For small values of $n$ we write (3.3) in the form

$$\Delta_{n+1} \leq \frac{b^{n+1}}{\pi} \int_0^{\infty} t \prod_{s=0}^{n+1} (1 + b^{2s} t^2)^{-1/2} \, \mathrm{d}t.$$

Especially,

$$\begin{aligned}
\Delta_2 &\leq \frac{b^2}{\pi} \int_0^{\infty} t \frac{1}{\sqrt{1+t^2}} \frac{1}{\sqrt{1+b^2t^2}} \frac{1}{\sqrt{1+b^4t^2}} \, \mathrm{d}t \\
&\leq \frac{b^2}{\pi} \int_0^{\infty} \frac{t}{(1+b^4t^2)\sqrt{1+t^2}} \, \mathrm{d}t = \frac{b^2}{2\pi} \int_0^{\infty} \frac{\mathrm{d}x}{(1+b^4x)\sqrt{1+x}} \\
&= \frac{\pi - 2 \operatorname{arctg} \frac{b^2}{\sqrt{1-b^4}}}{2\pi\sqrt{1-b^4}} = D_2(b).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\Delta_3 &\leq \frac{b^3}{\pi} \int_0^{\infty} \frac{t}{\sqrt{1+t^2}} \frac{1}{\sqrt{1+b^2t^2}} \frac{1}{\sqrt{1+b^4t^2}} \frac{1}{\sqrt{1+b^6t^2}} \, \mathrm{d}t \\
&\leq \frac{b^3}{\pi} \int_0^{\infty} \frac{1}{1+b^2t^2} \frac{1}{1+b^6t^2} \, \mathrm{d}t = -\frac{2b \ln b}{\pi(1-b^4)} = D_3(b).
\end{aligned}$$

Some values of $D_2(b)$ and $D_3(b)$ are introduced in Table 3.1.

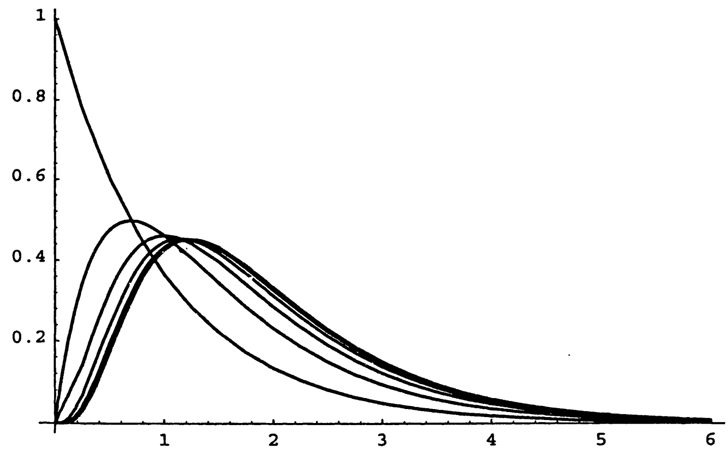


Fig. 3.1.

Table 3.1.

| $b$ | 0.01 | 0.1 | 0.5 | 0.9 | 0.99 |
|---|---|---|---|---|---|
| $D_2(b)$ | 0.50 | 0.50 | 0.43 | 0.34 | 0.32 |
| $D_3(b)$ | 0.03 | 0.15 | 0.24 | 0.18 | 0.16 |

Explicit formulas for $h_n(x)$ can be written down if $n$ is small. For example,

$$h_1(x) = \frac{e^{-x}}{1-b}\left(1 - e^{x-x/b}\right),$$

$$h_2(x) = \frac{e^{-x}}{(1-b)^2}\left[1 - e^{x-x/b} - \frac{b - e^{x(b^2+b-1)/b-1/b}}{1+b}\right]$$

and so on. However, it is easier to use a program package for calculating and processing the functions $h_n(x)$. We used Mathematica. Figure 3.1 shows functions $h_0(x), \ldots, h_5(x)$ in the case $b = 0.5$.

## 4. A GENERALIZATION TO AR PROCESSES OF HIGHER ORDER

The iterative method for calculating stationary density can be generalized to autoregressive processes of higher order. Here we present a derivation for AR(2) model.

Let $X_t$ be a stationary AR(2) process defined by

$$X_t = b_1 X_{t-1} + b_2 X_{t-2} + e_t \tag{4.1}$$

where $e_t$ is a strict white noise with a density $f$ and a finite second moment. Let $\psi$ be the characteristic function of $e_t$ and

$$\boldsymbol{F} = \begin{pmatrix} b_1 & b_2 \\ 1 & 0 \end{pmatrix}.$$

It is known that

$$X_t = \sum_{j=0}^{\infty} a_j e_{t-j}$$

where $a_j$ is the (1,1)-element of the matrix $\boldsymbol{F}^j$ (see [14], p. 57). It follows from the assumption of stationarity that all roots of $\boldsymbol{F}$ lie inside the unit circle and thus the series (4.1) converges in the quadratic mean. If we define $\boldsymbol{c} = (1,0)'$ then $a_j = \boldsymbol{c}'\boldsymbol{F}^j\boldsymbol{c} = \boldsymbol{c}'\boldsymbol{F}'^j\boldsymbol{c}$ and the characteristic function $\lambda$ of $X_t$ is given by

$$\lambda(v) = \prod_{j=0}^{\infty} \psi(va_j) = \prod_{j=0}^{\infty} \psi(v\boldsymbol{c}'\boldsymbol{F}'^j\boldsymbol{c}). \tag{4.2}$$

Since we assume that $e_t$ has a density, it follows from (4.1) that the random vector $(X_t, X_{t-1})'$ has a joint density, say $q(x,y)$. The stationary density of $\{X_t\}$ is $\int q(x,y)\,\mathrm{d}y$. Because $\{X_t\}$ is stationary, the vector $(X_{t-1}, X_{t-2})'$ has also the density $q$. The joint density of $(X_t, X_{t-1}, X_{t-2})'$ is $q(x_{t-1}, x_{t-2})\, f(x_t - b_1 x_{t-1} - b_2 x_{t-2})$ and so we have an integral equation

$$q(x_t, x_{t-1}) = \int q(x_{t-1}, x_{t-2})\, f(x_t - b_1 x_{t-1} - b_2 x_{t-2})\,\mathrm{d}x_{t-2}. \tag{4.3}$$

Let $q_0(y,z)$ be a density. Formula (4.3) suggests that a method for calculating $q$ can be based on the recurrent relation

$$q_n(x,y) = \int q_{n-1}(y,z) f(x - b_1 y - b_2 z)\,\mathrm{d}z. \tag{4.4}$$

We prove that under some conditions concerning $q_0$ the functions $q_n$ converge pointwise to $q$.

**Theorem 4.1.** Let $\lambda_n$ be the characteristic function corresponding to $q_n$. Then for arbitrary $\boldsymbol{t} = (t_1, t_2)'$ we have $\lambda_n(\boldsymbol{t}) \to \lambda(\boldsymbol{t})$.

Proof. Using (4.4) we get

$$\begin{aligned} \lambda_n(t_1, t_2) &= \iint e^{it_1x + it_2y} q_n(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \iiint e^{it_1x+it_2y} q_{n-1}(y,z) f(x - b_1 y - b_2 z)\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y \end{aligned}$$

$$\begin{aligned} &= \iiint e^{it_1(w+b_1y+b_2z)+it_2y} q_{n-1}(y,z) f(w)\,\mathrm{d}z\,\mathrm{d}w\,\mathrm{d}y \\ &= \iint e^{i(t_1b_1+t_2)y+it_1b_2z} q_{n-1}(y,z)\,\mathrm{d}y\,\mathrm{d}z \int e^{it_1w} f(w)\,\mathrm{d}w \\ &= \lambda_{n-1}(t_1b_1+t_2, t_1b_2)\,\psi(t_1) \\ &= \lambda_{n-1}(\boldsymbol{F}'\boldsymbol{t})\,\psi(\boldsymbol{c}'\boldsymbol{t}). \end{aligned}$$

This gives

$$\lambda_n(\boldsymbol{t}) = \psi(\boldsymbol{c}'\boldsymbol{t})\,\psi(\boldsymbol{c}'\boldsymbol{F}'\boldsymbol{t})\ldots\psi(\boldsymbol{c}'\boldsymbol{F}'^{n-1}\boldsymbol{t})\,\lambda_0(\boldsymbol{F}'^{n}\boldsymbol{t}).$$

Since $\boldsymbol{F}^n \to \boldsymbol{0}$ as $n \to \infty$, we have $\lambda_0(\boldsymbol{F}'^n\boldsymbol{t}) \to 1$ and in view of (4.2) it follows that $\lambda_n(\boldsymbol{t}) \to \lambda(\boldsymbol{t})$. □

**Theorem 4.2.** Let $q_0$ be a density. Assume that there exists an integer $m \geq 0$ such that

$$\iint |\psi(\boldsymbol{c}'\boldsymbol{t})\,\psi(\boldsymbol{c}'\boldsymbol{F}'\boldsymbol{t})\ldots\psi(\boldsymbol{c}'\boldsymbol{F}'^m\boldsymbol{t})|\,\mathrm{d}t_1\,\mathrm{d}t_2 < \infty.$$

Then $q_n(x,y) \to q(x,y)$ for all $(x,y)$ as $n \to \infty$.

Proof. For $n \geq m$ we have

$$|\lambda_n(t_1,t_2)| \leq |\psi(\boldsymbol{c}'\boldsymbol{t})\psi(\boldsymbol{c}'\boldsymbol{F}'\boldsymbol{t})\ldots\psi(\boldsymbol{c}'\boldsymbol{F}'^m\boldsymbol{t})|$$

and thus $\iint |\lambda_n(t_1,t_2)|\,\mathrm{d}t_1\,\mathrm{d}t_2 < \infty$. Then $q_n(x,y)$ is bounded, continuous, and

$$q_n(x,y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(xt_1+yt_2)}\lambda_n(t_1,t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2$$

(see [11], formula 7.12). Theorem 4.1 and Lebesgue theorem imply

$$\lim_{n\to\infty}\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(xt_1+yt_2)}\lambda_n(t_1,t_2)\,\mathrm{d}t_1\,\mathrm{d}t_2 = q(x,y). \qquad \square$$

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