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SUPERADDITIVITY IN GENERAL FUZZY COALITION GAMES[1]

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The superadditivity and related concepts belong to the fundamental ones in the coalition game theory. Their definition in general coalition games (games without side-payments) is based on the set theoretical approaches. It means that in the case of fuzzy coalition games the set theoretical model can be modified into the fuzzy set theoretical one. In this paper, the coalition games without side-payments and with fuzzy expectations of the pay-offs of players are considered and it is shown that for such games the properties of superadditivity, subadditivity and additivity turn into fuzzy properties. Their relations to their deterministic counterparts are shown and some results regarding their formal structure are derived.

1. INTRODUCTION

The classical deterministic coalition game theory is based on a hidden presumption that each player and each coalition know, at the very beginning of negotiations, the exact values of payoffs. This assumption does not reflect the reality of practical cooperative games and decision-making. In real situations, the expectations of profit are always more or less vague, and they can be represented by fuzzy set theoretical tools as shown, e.g., in [8]. This approach to the fuzzy coalition games is focused to the fuzzification of the characteristic function of the game and differs from the one used in [1].

Obviously, the fuzzification of the input data of the bargaining leads to the fuzzification of other properties and objects like the superadditivity, core, etc., as discussed in [8] and some other papers.

The superadditivity and related properties (subadditivity, additivity) in fuzzy coalition games were briefly viewed in [7] and for the case of coalition games with side-payments they were in a more detailed way investigated in [6]. This paper, in certain sense, continues [6] and develops the investigation of superadditivity and related concepts started in [7] for coalition games without side-payments or (in

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another terminology) general coalition games. The main attention is paid to the relations between the validity of superadditivity, subadditivity and additivity in deterministic coalition games and their fuzzy extensions.

As the side-payments are not considered, the fuzzification of the game regards the sets of admissible pay-off vectors (imputations) and the superadditivity is defined analogously to the deterministic case [3] by means of set theoretical relations between these sets. This set-theoretical approach demands another introduction of the fuzziness into the superadditivity concept, different from the one used in [6] for the games with side-payments. Both concepts of fuzzy superadditivity are compared in the conclusive section.

2. GENERAL COALITION GAME AND ITS SUPERADDITIVITY

Let us briefly remember the deterministic game theoretical concepts which are to be fuzzified in the following sections.

A *general coalition game* (or a *coalition game without side-payments*) is the pair $(I, \mathcal{V})$ where I is the (non-empty and finite) *set of players*, every subset $K \subset I$ is called a *coalition*. $\mathcal{V}$ is a mapping of the class of coalitions in the class of subsets of the multidimensional real space R^I, such that for any $K \subset I$

— $\mathcal{V}(K)$ is closed,

— if $\boldsymbol{x} = (x_i)_{i \in I} \in \mathcal{V}(K)$, $\boldsymbol{y} = (y_i)_{i \in I} \in R^I$ and $x_i \geq y_i$ for all $i \in K$, then $\boldsymbol{y} \in \mathcal{V}(K)$,

— $\mathcal{V}(K) \neq \emptyset$,

— $\mathcal{V}(K) = R^I$ if and only if $K = \emptyset$.

The mapping $\mathcal{V}$ is called *characteristic function* of the game $(I, \mathcal{V})$ and every vector $\boldsymbol{x} \in R^I$ is an *imputation*.

If $\boldsymbol{x}, \boldsymbol{y} \in R^I$ and $K \subset I$ then we say that $\boldsymbol{x}$ *dominates* $\boldsymbol{y}$ *via* K and write $\boldsymbol{x} \operatorname{dom}_K \boldsymbol{y}$ iff

— $x_i \geq y_i$ for all $i \in K$,

— $x_j > y_j$ for at least one $j \in K$.

Then we can define for every coalition $K \subset I$ its *superoptimum* set $\mathcal{V}^*(K)$ by

$$\begin{aligned} \mathcal{V}^*(K) &= \left\{ \boldsymbol{y} \in R^I : \text{there does not exist } \boldsymbol{x} \in \mathcal{V}(K) \text{ such that } \boldsymbol{x} \operatorname{dom}_K \boldsymbol{y} \right\} \\ &= \left\{ \boldsymbol{y} \in R^I : \text{for any } \boldsymbol{x} \in \mathcal{V}(K) \text{ either } x_i < y_i \right. \qquad (1) \\ &\qquad \left. \text{for some } i \in K \text{ or } x_j = y_j \text{ for all } j \in K \right\}. \end{aligned}$$

Introducing the concept of superoptimum we have gathered all tools necessary for the definition of superadditivity and related concepts.

We say that the coalition game $(I, \mathcal{V})$ is *superadditive* if for any pair of disjoint coalitions $K, L \subset I$, $K \cap L = \emptyset$, the inclusion

$$\mathcal{V}(K \cup L) \supset \mathcal{V}(K) \cap \mathcal{V}(L) \qquad (2)$$

is fulfilled. Analogously, we say that the coalition game $(I, \mathcal{V})$ is *subadditive* iff for any pair of disjoint coalitions $K,\ L \subset I$

$$\mathcal{V}^*(K \cup L) \supset \mathcal{V}^*(K) \cap \mathcal{V}^*(L). \tag{3}$$

We say that $(I, \mathcal{V})$ is *additive* iff it is superadditive and subadditive. It is not difficult to verify (see [3]) that if for any disjoint $K,\ L \subset I$

$$\mathcal{V}^*(K \cup L) \subset \mathcal{V}^*(K) \cap \mathcal{V}^*(L)$$

then the game $(I, \mathcal{V})$ is superadditive and, on the other hand, if for any disjoint $K, L \subset I$

$$\mathcal{V}(K \cup L) \subset \mathcal{V}(K) \cap \mathcal{V}(L)$$

then the game is subadditive.

3. FUZZY EXTENSION OF A GAME

If $(I, \mathcal{V})$ is a general coalition game then the sets $\mathcal{V}(K)$, for $K \subset I$, describe the possible vectors of individual pay-offs of members of coalition K. In the classical coalition games model these sets are deterministic and this determinism reflects the hidden assumption that the information about the possible outcomes is perfectly known at the very beginning of the negotiation. The real bargaining situations are different. Some of the imputations are only possible but not certainly achievable results of the cooperative behaviour. If we assume that the sets $\mathcal{V}(K)$ contain certainly achievable imputations of coalition K then the set $\mathcal{V}(K)$ can be extended into a fuzzy subset $\mathcal{W}(K)$ of R^I by adding only possible but not guaranteed outcomes.

We say that $(I, \mathcal{W})$ is a *fuzzy extension of* $(I, \mathcal{V})$ iff I is the set of players and $\mathcal{W}$ is a mapping connecting every coalition K with a fuzzy subset $\mathcal{W}(K)$ of R^I with membership function μ_K such that (cf. [7])

if $\boldsymbol{x} \in \mathcal{V}(K)$ then $\mu_K(\boldsymbol{x}) = 1$,

the set $\{\boldsymbol{x} \in R^I : \mu_K(\boldsymbol{x}) = 1\}$ is closed,

if $\boldsymbol{x},\ \boldsymbol{y} \in R^I,\ x_i \geq y_i$ for all $i \in K$, then $\mu_K(\boldsymbol{x}) \leq \mu_K(\boldsymbol{y})$,

if $K \neq \emptyset$ then there exists $\boldsymbol{x} \in R^I$ such that

$$\mu_K(\boldsymbol{x}) = 0,$$

if $K = \emptyset$ then $\mu_\emptyset(\boldsymbol{x}) = 1$ for all $\boldsymbol{x} \in R^I$.

The mapping $\mathcal{W}$ is called *fuzzy characteristic function* of $(I, \mathcal{W})$.

It is rational to consider the domination relation via a coalition to be rather dependent on the possibilities of achievability of the input vectors. Namely, if the domination relation is to be considered then we demand the dominating imputation to be at least as possible (for the relevant coalition) as the dominated one. Then the possibility of the validity of domination relation is equal to the possibility of the dominating vector, i.e., the possibility that a pair of imputations fulfilling the relation is achievable for the given coalition. In symbols, if $\boldsymbol{x}, \boldsymbol{y} \in R^I$, $K \subset I$, then we say that $\boldsymbol{x}$ *significantly dominates* $\boldsymbol{y}$ *via* K iff $\boldsymbol{x} \,\mathrm{dom}_K\, \boldsymbol{y}$ in the sense defined above and $\mu_K(\boldsymbol{x}) \geq \mu_K(\boldsymbol{y})$. If $\boldsymbol{x} \,\mathrm{dom}_K\, \boldsymbol{y}$ then the possibility that the domination is significant is equal to $\mu_K(\boldsymbol{x})$ iff $\mu_K(\boldsymbol{x}) \geq \mu_K(\boldsymbol{y})$ and it is equal to 0 iff the opposite inequality $\mu_K(\boldsymbol{x}) < \mu_K(\boldsymbol{y})$ is fulfilled (cf. [8]). We denote this possibility by

$$\begin{aligned} \nu_K(\boldsymbol{x}, \boldsymbol{y}) &= \mu_K(\boldsymbol{x}) \quad \text{iff} \quad \mu_K(\boldsymbol{x}) \geq \mu_K(\boldsymbol{y}) \quad \text{and} \quad \boldsymbol{x} \,\mathrm{dom}_K\, \boldsymbol{y}, \\ &= 0 \quad \text{in other cases.} \end{aligned} \tag{4}$$

Remark 1. The definition of $\mathcal{W}(K)$ implies that if $\boldsymbol{x} \,\mathrm{dom}_K\, \boldsymbol{y}$ then $\mu_K(\boldsymbol{y}) \geq \mu_K(\boldsymbol{x})$ and, consequently, if the domination is significant then $\mu_K(\boldsymbol{x}) = \mu_K(\boldsymbol{y})$.

Having introduced the concept of significant domination, we can define the *fuzzy superoptimum* of the coalition K as a fuzzy subset $\mathcal{W}^*(K)$ of R^I with membership function $\mu_K^* : R^I \to [0, 1]$. Intuitively, $\mathcal{W}^*(K)$ is to be a set of imputations from R^I which are not significantly dominated by any imputation from $\mathcal{W}(K)$, and, in the terminology of fuzzy sets and possibilities, the possibility that $\boldsymbol{y} \in R^I$ is an element of $\mathcal{W}^*(K)$ is complementary to the possibility that some $\boldsymbol{x} \in R^I$ significantly dominating $\boldsymbol{y}$ via K belongs to $\mathcal{W}(K)$. It means, in formal expression, that

$$\mu_K^*(\boldsymbol{y}) = 1 - \sup_{\boldsymbol{x} \in R^I} \nu_K(\boldsymbol{x}, \boldsymbol{y}) = \inf_{\boldsymbol{x} \in R^I} (1 - \nu_K(\boldsymbol{x}, \boldsymbol{y})) . \tag{5}$$

Remark 2. If $\mu_K(\boldsymbol{x})$ strictly decreases in some coordinates $i \in K$ for $\boldsymbol{x} \notin \mathcal{V}(K)$ and for $\mu_K(\boldsymbol{x}) \neq 0$, where $(I, \mathcal{W})$ is a fuzzy extension of $(I, \mathcal{V})$, then $\mathcal{W}^*(K)$ is a crisp subset of R^I and $\mathcal{W}^*(K) = \mathcal{V}^*(K)$.

Remark 3. If $(I, \mathcal{W})$ is a fuzzy extension of $(I, \mathcal{V})$ then for every $K \subset I$, the inclusion $\mathcal{V}^*(K) \supset \mathcal{W}^*(K)$ in the fuzzy set theoretical sense, i.e., $\mu_K^*(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \notin \mathcal{V}^*(K)$, holds.

4. SUPERADDITIVITY OF FUZZY EXTENSIONS

The concept of superadditivity in fuzzy coalition games can be approached in qualitatively different ways. The simplest one is a direct analogy of the deterministic model (2).

If $(I, \mathcal{W})$ is a fuzzy extension of a coalition game $(I, \mathcal{V})$ then we say that $(I, \mathcal{W})$ is *superadditive* iff for every pair of disjoint coalitions $K, L \subset I$, $K \cap L = \emptyset$,

$$\mathcal{W}(K \cap L) \supset \mathcal{W}(K) \cap \mathcal{W}(L), \tag{6}$$

it means, for any $\boldsymbol{x} \in R^I$

$$\mu_{K \cup L}(\boldsymbol{x}) \geq \min (\mu_K(\boldsymbol{x}), \mu_L(\boldsymbol{x})) . \tag{7}$$

Due to this definition, the superadditivity in fuzzy coalition game is presented as a deterministic property – relations (6) or (7) are either true or false. This fact rather contradicts our presumption that characteristic properties of fuzzy coalition game, like its superadditivity, would be also fuzzy. Anyhow, even the simple model introduced above gains some interesting properties. Some of them are already mentioned in [7].

Theorem 1. Let $(I, \mathcal{V})$ be a coalition game and $(I, \mathcal{W})$ its fuzzy extension such that $\mu_K(\boldsymbol{x}) = 1$ if and only if $\boldsymbol{x} \in \mathcal{V}(K)$ for all $K \subset I$. If $(I, \mathcal{W})$ is superadditive in the sense of (6) then $(I, \mathcal{V})$ is superadditive in the sense of (2).

Proof. The theorem follows from (2), (6) and from the condition that $\mu_K(\boldsymbol{x}) = 1$ exactly for $\boldsymbol{x} \in \mathcal{V}(K)$. If $(I, \mathcal{W})$ is superadditive then for all $\boldsymbol{x} \in \mathcal{V}(K) \cap \mathcal{V}(L)$, where $K, L \subset I,\ K \cap L = \emptyset$,

$$\mu_{K \cup L}(\boldsymbol{x}) = \min\left(\mu_K(\boldsymbol{x}),\, \mu_L(\boldsymbol{x})\right) = 1$$

and, consequently, $\mathcal{V}(K \cup L) \supset \mathcal{V}(K) \cap \mathcal{V}(L)$. □

Theorem 2. If $(I, \mathcal{V})$ is a coalition game then there exists its fuzzy extension $(I, \mathcal{W})$ which is superadditive.

Proof. If $(I, \mathcal{V})$ is given then we can define for every one-element coalition $\{i\},\ i \in I,\ \ \mathcal{W}(\{i\})$ and $\mu_{\{i\}}$ arbitrarily, respecting $\mu_{\{i\}}(\boldsymbol{x}) = 1$ for $\boldsymbol{x} \in \mathcal{V}(\{i\})$. Then for every two-element coalition $\{i, j\}$ we define

$$\mu_{\{i,j\}}(\boldsymbol{x}) = \max\left(\chi_{\{i,j\}}(\boldsymbol{x}), \min(\mu_{\{i\}}(\boldsymbol{x}),\, \mu_{\{j\}}(\boldsymbol{x}))\right)$$

where $\chi_{\{i,j\}}$ is the characteristic function of $\mathcal{V}(\{i, j\})$, i.e. $\chi_{\{i,j\}}(\boldsymbol{x}) = 1$ for $\boldsymbol{x} \in \mathcal{V}(\{i, j\})$, $\chi_{\{i,j\}}(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \notin \mathcal{V}(\{i, j\})$. □

We can continue in the recursive procedure and for any coalition $K \subset I$ of at least 3 players define

$$\mu_K(\boldsymbol{x}) = \max\left(\chi_K(\boldsymbol{x}),\, \psi_K(\boldsymbol{x})\right)$$

where for any $\boldsymbol{x} \in R^I$

$$\psi_K(\boldsymbol{x}) = \max\left[\min(\mu_L(\boldsymbol{x}),\, \mu_{K-L}(\boldsymbol{x})) : L \subset K,\, \emptyset \neq L \neq K\right]$$

and χ_K is the $0 - 1$ characteristic function of $\mathcal{V}(K)$. The fuzzy game $(I, \mathcal{W})$ with membership functions μ_K constructed in this way is evidently a fuzzy extension of $(I, \mathcal{V})$ as for any $K \subset I,\ \boldsymbol{x} \in \mathcal{V}(K)$ means $\mu_K(\boldsymbol{x}) = 1$ and it is superadditive as for any pair of disjoint coalitions K, L

$$\mu_{K \cup L}(\boldsymbol{x}) \geq \min\left(\mu_K(\boldsymbol{x}),\, \mu_L(\boldsymbol{x})\right), \quad \boldsymbol{x} \in R^I.$$

Corollary. If $(I, \mathcal{V})$ is a superadditive coalition game then there exists its fuzzy extension $(I, \mathcal{W})$ such that for any $K \subset I$, $\mu_K(\boldsymbol{x}) = 1$ if and only if $\boldsymbol{x} \in \mathcal{V}(K)$, which is superadditive. The statement is evidently fulfilled by any $(I, \mathcal{W})$ such that for each $\alpha \in (0, 1]$ the functions $\mu_K^{(\alpha)}(\boldsymbol{x}) = \min(\alpha, \mu_K(\boldsymbol{x}))$ fulfil the superadditivity condition $\mu_{K \cup L}^{(\alpha)}(\boldsymbol{x}) \geq \min(\mu_K^{(\alpha)}(\boldsymbol{x}),\, \mu_L^{(\alpha)}(\boldsymbol{x}))$ for any disjoint $K, L \subset I$.

Lemma 1. Let $(I, \mathcal{V})$ be a coalition game and let $(I, \mathcal{W})$ and $(I, \mathcal{W}')$ be its fuzzy extensions. Then also the pairs $(I, [\mathcal{W} \cup \mathcal{W}'])$ and $(I, [\mathcal{W} \cap \mathcal{W}'])$, where for any $K \subset I$ $[\mathcal{W} \cup \mathcal{W}'](K) = \mathcal{W}(K) \cup \mathcal{W}'(K)$ and $[\mathcal{W} \cap \mathcal{W}'](K) = \mathcal{W}(K) \cap \mathcal{W}'(K)$, are fuzzy extensions of $(I, \mathcal{V})$.

Proof. It is not difficult to verify the properties of fuzzy extension for $(I, [\mathcal{W} \cup \mathcal{W}'])$ and $(I, [\mathcal{W} \cap \mathcal{W}'])$. Let us denote the membership functions of $[\mathcal{W} \cup \mathcal{W}']$ and $[\mathcal{W} \cap \mathcal{W}']$,

$$\mu_K^+(\boldsymbol{x}) = \max(\mu_K(\boldsymbol{x}), \mu'_K(\boldsymbol{x})), \quad \mu_K^{++}(\boldsymbol{x}) = \min(\mu_K(\boldsymbol{x}), \mu'_K(\boldsymbol{x})),$$

respectively, for $K \subset I$, $\boldsymbol{x} \in R^I$, and μ_K, μ'_K being membership functions of $\mathcal{W}(K)$, $\mathcal{W}'(K)$, respectively. Then for $\boldsymbol{x} \in \mathcal{V}(K)$, $K \subset I$, $\mu_K^+(\boldsymbol{x}) = \mu_K^{++}(\boldsymbol{x}) = 1$, the sets $\{\boldsymbol{x} \in R^I : \mu_K^+(\boldsymbol{x}) = 1\} = \{\boldsymbol{x} \in R^I : \mu_K(\boldsymbol{x}) = 1\} \cup \{\boldsymbol{x} \in R^I : \mu'_K(\boldsymbol{x}) = 1\}$, $\{\boldsymbol{x} \in R^I : \mu_K^{++}(\boldsymbol{x}) = 1\} = \{\boldsymbol{x} \in R^I : \mu_K(\boldsymbol{x}) = 1\} \cap \{\boldsymbol{x} \in R^I : \mu'_K(\boldsymbol{x}) = 1\}$ are closed. If $\boldsymbol{x} \in R^I$, $\boldsymbol{y} \in R^I$, $x_i \geq y_i$ for all $i \in K$, then $\mu_K^+(\boldsymbol{x}) \leq \mu_K^+(\boldsymbol{y})$ and $\mu_K^{++}(\boldsymbol{x}) \leq \mu_K^{++}(\boldsymbol{y})$. If we denote by $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ the vectors for which $\mu_K(\boldsymbol{x}^{(1)}) = 0$ and $\mu'_K(\boldsymbol{x}^{(2)}) = 0$ then $\mu_K(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in R^I$ such that $x_i \geq x_i^{(1)}$ for all $i \in K$, and $\mu'_K(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in R^I$ such that $x_i \geq x_i^{(2)}$ for all $i \in K$. It means that there exists $\boldsymbol{y} \in R^I$ such that $y_i \geq \max(x_i^{(1)}, x_i^{(2)})$, $i \in K$, and for this $\boldsymbol{y}$ both, $\mu_K^+(\boldsymbol{y}) = 0$ and $\mu_K^{++}(\boldsymbol{y}) = 0$.

Theorem 3. Let $(I, \mathcal{V})$ be a coalition game, $(I, \mathcal{W})$, $(I, \mathcal{W}')$ be its fuzzy extensions and $(I, [\mathcal{W} \cap \mathcal{W}'])$ fulfils the assumptions of the previous Lemma. If $(I, \mathcal{W})$ and $(I, \mathcal{W}')$ are superadditive then also $(I, [\mathcal{W} \cap \mathcal{W}'])$ is superadditive.

Proof. Let us preserve the notations used in the proof of Lemma 1. If for all $\boldsymbol{x} \in R^I$ and $K, L \subset I$, $K \cap L = \emptyset$,

$$\mu_{K \cup L}(\boldsymbol{x}) \geq \min(\mu_K(\boldsymbol{x}), \mu_L(\boldsymbol{x})), \quad \mu'_{K \cup L}(\boldsymbol{x}) \geq \min(\mu'_K(\boldsymbol{x}), \mu'_L(\boldsymbol{x})) \tag{8}$$

then

$$\min(\mu_{K \cup L}(\boldsymbol{x}), \mu'_{K \cup L}(\boldsymbol{x})) \geq \min[\min(\mu_K(\boldsymbol{x}), \mu'_K(\boldsymbol{x})), \min(\mu_L(\boldsymbol{x}), \mu'_L(\boldsymbol{x}))]$$

and, consequently, $(I, [\mathcal{W} \cap \mathcal{W}'])$ is superadditive. □

5. FUZZY SUPERADDITIVITY

The concept of superadditivity presented in the previous section and based on relations (6) and (7) does not fully reflect the general paradigm of the fuzzy coalition games, namely, the presumption that the properties of fuzzy games are to be also fuzzy. The superadditivity defined by (6) is a completely deterministic concept – a fuzzy extension $(I, \mathcal{W})$ of some game $(I, \mathcal{V})$ either is superadditive or it is not. If we want to implement vagueness also into the superadditivity of fuzzy extensions of coalition games, we can apply the approach suggested already in [7], namely, we can use the concept of α-cuts of fuzzy sets to structurize the possibilities of the superadditivity.

Let $(I, \mathcal{V})$ be a coalition game and $(I, \mathcal{W})$ be its fuzzy extension. Let $\alpha \in (0,1]$ be real number, and let us define for every coalition $K \subset I$ a fuzzy subset $\mathcal{W}_\alpha(K)$ of R^I with membership function $\mu_K^{(\alpha)} : R^I \to [0, \alpha]$ such that

$$\mu_K^{(\alpha)}(\boldsymbol{x}) = \min(\alpha, \mu_K(\boldsymbol{x})), \quad \boldsymbol{x} \in R^I.$$

Then we say that the pair $(I, \mathcal{W}_\alpha)$ is an *α-reduction* of $(I, \mathcal{W})$, and we say that $(I, \mathcal{W})$ is *α-fuzzy superadditive* iff for any $K,\ L \subset I,\ K \cap L = \emptyset$,

$$\mathcal{W}_\alpha(K \cup L) \supset \mathcal{W}_\alpha(K) \cap \mathcal{W}_\alpha(L) \tag{9}$$

it means

$$\mu_{K \cup L}^{(\alpha)}(\boldsymbol{x}) \geq \min\left(\mu_K^{(\alpha)}(\boldsymbol{x}),\, \mu_L^{(\alpha)}(\boldsymbol{x})\right), \quad \boldsymbol{x} \in R. \tag{10}$$

Remark 4. If $(I, \mathcal{W})$ is a fuzzy extension of $(I, \mathcal{V})$, if $0 < \alpha \leq \beta \leq 1$ and if $(I, \mathcal{W})$ is β-fuzzy superadditive then it is α-fuzzy superadditive. Consequently, if $(I, \mathcal{W})$ is fuzzy superadditive in the sense of (6) then it is α-fuzzy superadditive for any $\alpha \in (0,1]$.

The concepts introduced above allow us to define the possibility with which $(I, \mathcal{W})$ is superadditive as a number

$$\sigma(I, \mathcal{W}) = \sup\left(\alpha \in (0,1] : (I, \mathcal{W}) \text{ is } \alpha\text{-fuzzy superadditive}\right). \tag{11}$$

Obviously, $\sigma(I, \mathcal{W}) = 0$ if $(I, \mathcal{W})$ is not α-fuzzy superadditive for any $\alpha > 0$.

Of course, fuzzy superadditivity (6) is identical with 1–fuzzy superadditivity, $\sigma(I, \mathcal{W}) = 1$. The previous notes and Theorem 1 mean that equality $\sigma(I, \mathcal{W}) = 1$ implies the superadditivity of the deterministic game $(I, \mathcal{V})$ if $\mu_K(\boldsymbol{x}) = 1$ exactly for $\boldsymbol{x} \in \mathcal{V}(K)$, for all $K \subset I$.

Lemma 2. If $(I, \mathcal{W})$ is a fuzzy extension of $(I, \mathcal{V})$ and if $\beta \in (0,1]$, then $\sigma(I, \mathcal{W}) = \beta$ if and only if

$$\beta = \sup\left(\alpha \in (0,1] : (I, \mathcal{W}) \text{ is } \alpha\text{-superadditive}\right).$$

P r o o f. If $(I, \mathcal{W})$ is α-superadditive then, due to Remark 4, it is γ-superadditive for all $0 < \gamma \leq \alpha$ and (11) can be reduced in correspondence with the statement of Lemma. □

The previous result about the existence of superadditive fuzzy extension (see Theorem 2) can be extended to the case of α-fuzzy superadditivity and to arbitrary value of the possibility $\sigma(I, \mathcal{W})$.

Theorem 4. Let $(I, \mathcal{V})$ be a coalition game and $\beta \in (0,1]$. Then there exists a fuzzy extension $(I, \mathcal{W})$ of $(I, \mathcal{V})$ which is fuzzy superadditive with possibility $\sigma(I, \mathcal{W}) = \beta$.

Proof. Let us suppose that there already exists a fuzzy extension $(I, \mathcal{W}')$ of $(I, \mathcal{V})$ with membership functions μ'_K, $K \subset I$. Then we can construct, for every player $i \in I$, the membership functions $\mu_{\{i\}}$ such that $\mu_{\{i\}}(\boldsymbol{x}) \geq \mu'_{\{i\}}(\boldsymbol{x})$ for all $\boldsymbol{x} \in R^I$ and, moreover, there exists at least one $\boldsymbol{x} \in R^I$ such that $\mu'_I(\boldsymbol{x}) < \beta$ and $\mu_{\{i\}}(\boldsymbol{x}) = 1$ for all $i \in I$. The fuzzy extension $(I, \mathcal{W})$ with these membership functions, certainly is not superadditive with possibility β. Let us put $\mu_K(\boldsymbol{x}) = \mu'_K(\boldsymbol{x})$ for all $K \subset I$, where K contains at least two players. then we construct the β-cuts

$$\mu_K^{(\beta)}(\boldsymbol{x}) = \min(\beta, \mu_K(\boldsymbol{x})), \quad \boldsymbol{x} \in R^I, \ K \subset I$$

and use the recursive procedure used in the proof of Theorem 2. Applying it, we obtain membership functions μ^*_K, $K \subset I$, where K contains at least two players. Applying it, we obtain membership functions μ^*_K, $K \subset I$, such that for all $i \in I$

$$\begin{aligned} \mu''_{\{i\}}(\boldsymbol{x}) &= \beta & \text{if } & 0 < \mu_{\{i\}}(\boldsymbol{x}) \leq \beta, \\ &= \mu_{\{i\}}(\boldsymbol{x}) & \text{if } & \mu_{\{i\}}(\boldsymbol{x}) > \beta, \\ &= 0 & \text{if } & \mu_{\{i\}} = 0 \end{aligned}$$

for all $i, j \in I, \ i < j$,

$$\mu''_{\{i\}}(\boldsymbol{x}) = \max\left(\mu_{\{i\}}(\boldsymbol{x}), \min(\mu''_{\{i\}}(\boldsymbol{x}), \mu''_{\{i\}}(\boldsymbol{x}))\right)$$

and, generally, for $K \subset I$, K containing more than 3 players

$$\mu''_K(\boldsymbol{x}) = \max(\mu_K(\boldsymbol{x}), \psi_K(\boldsymbol{x}))$$

where

$$\psi_K(\boldsymbol{x}) = \max\left(\min(\mu''_L(\boldsymbol{x}), \mu''_{K-L}(\boldsymbol{x})) : L \subset K, \ \emptyset \neq L \neq K\right).$$

Then the pair $(I, \mathcal{W}'')$ where for any $K \subset I$, $W''(K)$ are fuzzy subsets of R^I with membership functions μ''_K, is a fuzzy extension of $(I, \mathcal{V})$ and it is fuzzy superadditive with possibility $\sigma(I, \mathcal{W}'') = \beta$. □

6. SUBADDITIVITY AND ADDITIVITY OF FUZZY EXTENSIONS

If $(I, \mathcal{V})$ is a deterministic coalition game and $(I, \mathcal{W})$ its fuzzy extension then it appears natural to define the subadditivity of $(I, \mathcal{W})$ by means of the (fuzzy) sets $\mathcal{W}^*(K)$, $K \subset I$, with membership functions μ^*_K defined by (5). Remarks 2 and 3 illustrate that for any $K \subset I$ the set $\mathcal{W}^*(K)$ is not a fuzzy extension of $\mathcal{V}^*(K)$ but, generally, its reduction. It significantly changes the relation between fuzzy and deterministic subadditivity (and, consequently, additivity) in coalition games. Remarks 2 and 3 can be completed by the following statement.

Lemma 3. Let $(I, \mathcal{W})$ be a fuzzy extension of $(I, \mathcal{V})$ such that for any coalition $K \subset I$ and any $\boldsymbol{x} \in R^I$, $\mu_K(\boldsymbol{x}) = 1$ if and only if $\boldsymbol{x} \in \mathcal{V}(K)$. Then $\mu_K^*(\boldsymbol{x}) = 1$ for all $\boldsymbol{x} \in \mathcal{V}(K) \cap \mathcal{V}^*(K)$, where μ_K^* is the membership function of $\mathcal{W}^*(K)$.

Proof. If $\boldsymbol{x} \in \mathcal{V}(K) \cap \mathcal{V}^*(K)$ and $\mu(\boldsymbol{y}) < 1$ for all $\boldsymbol{y} \in R^I$ such that $\boldsymbol{y} \operatorname{dom}_K \boldsymbol{x}$ then, due to (5) and due to the definition of significant domination, $\boldsymbol{x}$ cannot be significantly dominated by any $\boldsymbol{y} \in R^I$ via K and $\mu_K^*(\boldsymbol{x}) = 1$. □

If $(I, \mathcal{W})$ is a fuzzy extension of $(I, \mathcal{V})$ then we say that $(I, \mathcal{W})$ is *subadditive* iff for any $K, L \subset I$, $K \cap L = \emptyset$,

$$\mathcal{W}^*(K \cup L) \supset \mathcal{W}^*(K) \cap \mathcal{W}^*(L) \tag{12}$$

it means

$$\mu_{K\cup L}^*(\boldsymbol{x}) \geq \min(\mu_K^*(\boldsymbol{x}), \mu_L^*(\boldsymbol{x})), \quad \text{for all } \boldsymbol{x} \in R^I. \tag{13}$$

Remark 5. Obviously, Remark 2 implies that if $(I, \mathcal{V})$ is subadditive then there exists its subadditive extension and, for all $K \subset I$, if $\mu_K(\boldsymbol{x})$ strictly decreases in some coordinates for $\boldsymbol{x} \notin \mathcal{V}(K)$ then $(I, \mathcal{W})$ is subadditive if and only if $(I, \mathcal{V})$ is subadditive.

Analogously to the case of superadditivity, it is possible to construct a subadditive extension of any deterministic coalition game $(I, \mathcal{V})$ but, as follows from Remark 5, it does not fulfil the condition $\mu_K(\boldsymbol{x}) = 1$ iff $\boldsymbol{x} \in \mathcal{V}(K)$ for some $K \subset I$.

Theorem 5. If $(I, \mathcal{V})$ is coalition game then there exists its fuzzy extension $(I, \mathcal{W})$ which is subadditive.

Proof. The statement is quite obvious if we remember Remark 5. It is sufficient to construct new deterministic game $(I, \mathcal{V}_0)$ which is subadditive and such that for any $K \subset I$, $\mathcal{V}_0(K) \supset \mathcal{V}(K)$. Such construction is always possible. Let us put, for example, $\mathcal{V}_0(I) = \mathcal{V}(I)$ and for any $K \subset I$, $K \neq I$, $\mathcal{V}_0(K)$ such that for any $L \supset I$

$$\mathcal{V}_0^*(K) \cap \mathcal{V}_0(L) = \emptyset.$$

Then any fuzzy extension $(I, \mathcal{W})$ of $(I, \mathcal{V}_0)$ is also a fuzzy extension of $(I, \mathcal{V})$. If $\mathcal{W}(K)$ are constructed so that $\mu_K(\boldsymbol{x})$ is strictly decreasing in some coordinates and for those $\boldsymbol{x}$ for which $\mu_K(\boldsymbol{x}) \in (0, 1)$ from K then $\mathcal{W}(K)$ is fuzzy subadditive. □

The *fuzzy additivity* can be defined, analogously to the deterministic case, as a conjunction of fuzzy superadditivity and fuzzy subadditivity.

If $(I, \mathcal{W})$ is a fuzzy extension of $(I, \mathcal{V})$ then we say that it is *additive* iff for any pair of disjoint coalitions $K, L \subset I$, $K \cap L = \emptyset$,

$$\mathcal{W}(K \cup L) \supset \mathcal{W}(K) \cap \mathcal{W}(L) \quad \text{and} \quad \mathcal{W}^*(K \cup L) \supset \mathcal{W}^*(K) \cap \mathcal{W}^*(L) \tag{14}$$

which means for all $\boldsymbol{x} \in R^I$

$$\mu_{K\cup L}(\boldsymbol{x}) \geq \min(\mu_K(\boldsymbol{x}), \mu_L(\boldsymbol{x})) \quad \text{and} \quad \mu_{K\cup L}^*(\boldsymbol{x}) \geq \min(\mu_K^*(\boldsymbol{x}), \mu_L^*(\boldsymbol{x})). \tag{15}$$

Certain potential complexity of, namely, the fuzzy superoptimum set causes simplicity of achievable results which are similar to those derived for the subadditivity.

Theorem 6. Let $(I, \mathcal{W})$ be a fuzzy extension of $(I, \mathcal{V})$ such that for any $K \subset I$ the membership function $\mu_K(\boldsymbol{x})$ strictly decreases for some coordinates from K. If $(I, \mathcal{W})$ is additive then $(I, \mathcal{V})$ is additive.

Proof. The statement follows from Theorem 1 and Remark 5. □

Theorem 7. If $(I, \mathcal{V})$ is a coalition game then there always exists its additive fuzzy extension $(I, \mathcal{W})$.

Proof. The proof is analogous to those ones of Theorem 2 and Theorem 5. It is sufficient to construct a new deterministic coalition game $(I, \mathcal{V}_0)$ which is additive and $\mathcal{V}_0(K) \supset \mathcal{V}(K)$ for all $K \subset I$. Such game always exists and it has to fulfil for every $K \subset I$ the conditions

$$\mathcal{V}_0(K) \supset \bigcap_{i \in K} \mathcal{V}_0(\{i\}), \quad \mathcal{V}_0^*(K) \supset \bigcap_{i \in K} \mathcal{V}_0^*(\{i\})$$

and

$$\boldsymbol{y} \in \mathcal{V}_0(K) \cap \mathcal{V}_0^*(K) \quad \text{for} \quad \boldsymbol{y} \in R^I,\ y_i = \max(x_i : \boldsymbol{x} \in \mathcal{V}_0(\{i\})), \quad \text{for all } i \in I.$$

These conditions are necessary (but not sufficient) for the additivity. Fulfilling these conditions it is possible to construct the sets $\mathcal{V}_0(K)$ recursively, where for $i, j \in I$, $i < j$,

$$\mathcal{V}_0(\{i,j\}) \supset \mathcal{V}_0(\{i\}) \cap \mathcal{V}_0(\{j\}) \quad \text{and} \quad \mathcal{V}_0^*(\{i,j\}) \supset \mathcal{V}_0^*(\{i\}) \cap \mathcal{V}_0^*(\{j\})$$

and generally

$$\mathcal{V}_0(K) \supset \bigcup_{\substack{L \subset K \\ \emptyset \neq L \neq K}} (\mathcal{V}_0(L) \cap \mathcal{V}_0(K-L)), \quad \mathcal{V}_0^*(K) \supset \bigcup_{\substack{L \subset K \\ \emptyset \neq L \neq K}} (\mathcal{V}_0^*(L) \cap \mathcal{V}_0^*(K-L)).$$

The desired fuzzy extension is, for example, the game $(I, \mathcal{V}_0)$ itself but there exist also other fuzzy extensions $(I, \mathcal{W})$ of $(I, \mathcal{V}_0)$ such that for any $K \subset I$, $\mu_K(\boldsymbol{x}) = 1$ iff $\boldsymbol{x} \in \mathcal{V}_0(K)$, $\mu_K(\boldsymbol{x})$ is strictly decreasing for some coordinates from K and for $\boldsymbol{x}$ for which $1 > \mu_K(\boldsymbol{x}) > 0$, and the inclusion

$$\mathcal{W}^*(K \cup L) \supset \mathcal{W}(K) \cap \mathcal{W}(L)$$

is fulfilled for any pair of disjoint coalitions. □

7. FUZZY SUPERADDITIVITY IN LINEAR GAMES

The general coalition games may in some cases gain a special form called *coalition games with side-payments* (cf., for example, [7, 10]). Such games are defined as pairs (I, v) where I is the set of players and $v : 2^I \to R$ is a mapping connecting each coalition $K \subset I$ with a real number $v(K) \in R$, $v(\emptyset) = 0$. If we construct for every coalition $K \subset I$ a set

$$\mathcal{V}(K) = \left\{ \boldsymbol{x} \in R^I : \sum_K x_i \leq v(K) \right\} \tag{16}$$

then it is easy to verify that the pair $(I, \mathcal{V})$ is a deterministic general coalition game in the sense considered in the previous sections. It is also easy to prove (cf. [3]) that $(I, \mathcal{V})$ is superadditive if and only if for any pair of disjoint coalitions $K, L \subset I,\ K \cap L = \emptyset$,

$$v(K \cup L) \geq v(K) + v(L) \tag{17}$$

and, analogously, it is subadditive and additive iff for any $K,\ L \subset I,\ K \cap L = \emptyset$,

$$v(K \cup L) \leq v(K) + v(L) \quad \text{and} \quad v(K \cup L) = v(K) + v(L), \tag{18}$$

respectively.

As the coalition games with side-payments are defined by means of real-valued characteristic function v, it is natural to fuzzify them by means of using fuzzy numbers instead of the deterministic values $v(K)$. It was done in [7] and [6] and in this section we briefly investigate the relation between both types of results – those derived for games with side payments and those presented above for the general games. We will see that in the fuzzy case both the approaches are not equivalent.

Before remembering the results summarized in [6] it is necessary to introduce several elementary concepts of the theory of fuzzy quantities [4, 5]. *Fuzzy quantity* is any fuzzy subset a of the real line R with membership function $\mu_a : R \to [0, 1]$ such that

— there exists $x_a \in R$ for which $\mu_a(x_a) = 1$,

— there exist $x_1,\ x_2 \in R,\ x_1 < x_2$, where $\mu_a(x) = 0$ for all $x \notin [x_1, x_2]$.

The algebraic operations with fuzzy quantities can be defined by means of so called extension principle. For our purpose, we need the concept of *sum* of fuzzy quantities a and b defined as a fuzzy quantity $a \oplus b$ with membership function

$$\mu_{a\oplus b}(x) = \sup_{y\in R} \left[\min(\mu_a(y),\ \mu_b(x - y))\right], \quad x \in R. \tag{19}$$

Moreover, the investigation of superadditivity demands also the ordering relation over fuzzy quantities. There exist many different definitions of such relation. In this paper we use, in accordance with [6, 7, 8] the one based on the idea that *ordering relation* over vague fuzzy quantities is also a vague fuzzy relation valid with some possibility. If a, b are fuzzy quantities then we say that $a \succsim b$ with possibility $\nu_{\succsim}(a, b)$ iff

$$\nu_{\succsim}(a, b) = \sup_{\substack{x,y\in R \\ x\geq y}} \left[\min(\mu_a(x),\ \mu_b(y))\right]. \tag{20}$$

The basic properties of this ordering relation are discussed, e. g., in [5].

Having mentioned some concepts of the fuzzy quantities theory we can remember the fuzzification of coalition game with side-payments (I, v) with characteristic function $v : 2^I \to R$, as suggested in [6, 7].

We say that a pair (I, w) is a *fuzzy extension of a coalition game with side-payments* (I, v) if for every coalition $K \subset I$, $w(K)$ is a fuzzy quantity with membership function π_K such that $\pi_K(v(K)) = 1$. Analogously to (17) we say that (I, w)

is *fuzzy superadditive* iff for any $K,\ L \subset I,\ K \cap L = \emptyset$,

$$w(K \cup L) \succsim w(K) \oplus w(L). \tag{21}$$

If we abbreviate the notation and introduce the symbol π_{K+L} for the membership function of the fuzzy quantity $w(K) \oplus w(L)$, i.e.

$$\pi_{K+L}(x) = \sup_{y \in R} \left[\min(\mu_K(y),\ \mu_L(x-y))\right], \quad x \in R,$$

then the possibility of relation (21) is, due to (20),

$$\nu_{\succsim}(w(K \cup L),\ w(K) \oplus w(L)) = \sup_{\substack{x,y \in R \\ x \geq y}} \left[\min(\mu_{K \cup L}(x),\ \mu_{K+L}(y))\right] \tag{22}$$

and the possibility that (I, w) is fuzzy superadditive will be denoted

$$\lambda(I, w) = \min(\nu_{\succsim}(w(K \cup L),\ w(K) \oplus w(L)) : K,\ L \subset I,\ K \cap L = \emptyset). \tag{23}$$

Analogously to the deterministic procedure (16), it is easy to derive also from a fuzzy extension (I, w) of coalition game with side-payment (I, v) for every coalition $K \subset I$ the fuzzy subset $\mathcal{W}(K)$ of R^I with membership function $\mu_K : R^I \to [0, 1]$ such that for every $\boldsymbol{x} \in R^I$

$$\mu_K(\boldsymbol{x}) = \sup \left(\pi_K(\boldsymbol{y}) : \boldsymbol{y} \in R^I, \sum_K y_i \geq \sum_K x_i \right). \tag{24}$$

It is easy to see that the pair $(I, \mathcal{W})$ is a fuzzy extension of the general coalition game $(I, \mathcal{V})$ where the sets $\mathcal{V}(K)$ were derived by (16). Moreover, if for a coalition $K \subset I$, $\pi_K(x)$ is strictly decreasing for all $x > v(K)$ then also $\mu_K(\boldsymbol{x})$ is strictly decreasing for all $\boldsymbol{x} \notin \mathcal{V}(K)$ and all coordinates from K.

The results derived in the previous sections imply the following relation between the fuzzy superadditivity of both extensions, (I, w) and $(I, \mathcal{W})$.

Theorem 8. Let (I, v) be a coalition game with side-payments, let (I, w) be its fuzzy extension such that for all coalitions $K \subset I$ the membership function $\pi_K(x)$ is strictly decreasing for all $x > v(K)$, and let $(I, \mathcal{V})$ and $(I, \mathcal{W})$ be derived from (I, v) and (I, w) by means of (16) and (24), respectively. Let $\lambda(I, w)$ be the possibility that (I, w) is fuzzy superadditive in the sense of (21). If $(I, \mathcal{W})$ is superadditive in the sense of (6) and (7) then $(I, \mathcal{V})$ is superadditive in the sense of (2), (I, v) is superadditive in the sense of (17) and $\lambda(I, w) = 1$. Moreover, under the assumptions of this theorem the following statements are equivalent

— (I, v) is superadditive due to (17),

— $(I, \mathcal{V})$ is superadditive due to (2),

— $\lambda(I, w) = 1$.

Proof. The statement follows from Theorem 1, from the obvious equivalence between (17) and (2) (cf. [3]) and from [6], namely its Theorem 3. □

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