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CONTINUOUS–TIME PERIODIC SYSTEMS IN H_2 AND H_∞ Part I: Theoretical Aspects

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The paper is divided in two parts. In the first part a deep investigation is made on some system theoretical aspects of periodic systems and control, including the notions of H_2 and H_∞ norms, the parametrization of stabilizing controllers, and the existence of periodic solutions to Riccati differential equations and/or inequalities. All these aspects are useful in the second part, where some parametrization and control problems in H_2 and H_∞ are introduced and solved.

1. INTRODUCTION AND PROBLEM POSITION

The analysis and design problems for periodic systems have a long history in the scientific literature, although only recently various issues concerning theoretical aspects have been succesfully clarified: see the survey paper [1] for an overview on the structural properties of periodic systems, [2] for the properties of periodic solutions to periodic Riccati equations and [3] for the study of the periodic Lyapunov equations.

The paper benefits from the development of the theory of H_∞ control for shift-invariant systems. In this regard, specially important is the celebrated paper [4], the additional parametrization results given in [5], the parametrization of memoryless state-feedback controllers via LMI and the mixed H_2/H_∞ control results in [6].
The application of the above theory to periodic systems is far from being trivial, since it requires, besides non standard results on the differential periodic Riccati equations, an appropriate extension of the mathematical machinery concerning system theoretical aspects such as spectral properties, Youla–Kučera parametrization, small gain results, H_2 and H_∞ norm, BIBO stability of feedback systems and so on so forth. Presenting new results concerning the theory of periodic systems is one of the scopes of the paper. Part of the work is inspired by the recent definition, for continuous-time periodic systems, of the so called lifted shift-invariant reformulation (well known in the discrete-time case), see [7, 8]. All the above arguments are the object of the first part of the paper.

In the second part of the paper we consider the continuous-time T-periodic linear system $\mathcal{P}$, described by the differential equations

$$\dot{x} = A(t)\,x + B_1(t)\,w + B_2(t)\,u \tag{1}$$
$$z_1 = C_1(t)\,x + D_1(t)\,u \tag{2}$$
$$z_2 = C_2(t)\,x + D_2(t)\,u \tag{3}$$

where
$$A(\cdot),\ B_1(\cdot),\ B_2(\cdot),\ C_1(\cdot),\ D_1(\cdot),\ C_2(\cdot),\ D_2(\cdot)$$
are T-periodic piecewise continuous function matrices. The signal $u(t)$ is the control input, $w(t)$ is an input disturbance and $z_1(t), z_2(t)$ are controlled output variables.

The following state-feedback problems are dealt with.

(1) Find a necessary and sufficient condition for the existence of a T-periodic causal controller fed by (x, w) and yielding u such that the H_∞ norm (to be properly defined) from w to z_1 is less than a prescribed positive attenuation level γ

(2) Parametrize all stabilizing T-periodic controllers fed by (x, w) and yielding u such that the H_∞ norm from w to z_1 is less than a prescribed positive attenuation level γ

(3) Parametrize all memoryless T-periodic controllers $(u(t) = K(t)\,x(t))$ such that the H_∞ norm from w to z_1 is less than (or equal to) a prescribed positive attenuation level γ

(4) Find a memoryless T-periodic controller $(u(t) = K(t)\,x(t))$ which minimizes the H_2 norm (to be properly defined) between w and z_2

(5) Find a memoryless T-periodic controller of the kind $u(t) = K(t)\,x(t)$ which minimizes the H_2 norm between w and z_2 while keeping the H_∞ norm from w to z_1 less than or equal to a prescribed positive attenuation level γ.

The paper aims at providing a rather complete picture of the theory underlying the above mentioned issues. As such, the reader could easily found overlapping material with respect to past contributions. Indeed, preliminary results on the H_∞ type periodic Riccati equation and on the full information H_∞ periodic control problem are contained in [9, 10] and [11]. The sensitivity minimization problem for periodic systems was approached in [12, 13, 14] and [15, 16]. More in detail, the sufficient part of Theorem 3.1 of [17] is proven in [13] and [16] and the parametrization of H_∞ performant controllers in Theorem 3.2 of [17] can be deduced along the lines traced in [12]. Concerning this last point, the proof presented here exploits only the properties of Hamiltonian periodic systems and does not require any frequency domain considerations. The analysis result stated in Lemma 2.6 was proven in [10], by exploiting the results in [13]. Here a different and self-contained proof is provided. As for the concept of exponentially modulated signal and spectral properties of periodic systems, they were introduced, in a different way, in [7]. Finally, Theorem 2.2 of [17] is standard in the literature of optimal control of time-varying systems, and can be found in many papers.

This first part of the paper is organized into four Sections. All the mathematical material on periodic systems is concentrated in Section 2, which includes 13 lemmas. The proofs of them are gathered in Section 3. Part II contains 7 theorems concerning the parametrization of stabilizing memoryless state-feedback controllers (Theorem 2.1 of [17]), the optimal H_2 control problem (Theorem 2.2 of [17]), the H_∞ full-information control problem (Theorem 3.1 of [17]), the parametrization of H_∞ performant controllers (Theorem 3.2 of [17]), the parametrization of memoryless state-feedback controllers via differential LMI (Theorem 4.1 of [17]), and the so-called convex and post-optimization procedures for the mixed H_2/H_∞ control problem (Theorems 5.1 and 5.2 of [17]).

2. THEORETICAL ASPECTS OF PERIODIC SYSTEMS

In this section reference is made to a T-periodic system $\mathcal{G} = (F, G, H, E)$ described by

$$\dot{\theta} = F(t)\,\theta + G(t)\,w \tag{4}$$

$$z = H(t)\,\theta + E(t)\,w. \tag{5}$$

Matrices $F(\cdot)$, $G(\cdot)$, $H(\cdot)$ and $E(\cdot)$ are T-periodic piecewise continuous matrix functions of period T.

A number of theoretical results concerning system $\mathcal{G}$ are provided. They can be considered as non trivial generalization to periodic systems of concepts taken from the realm of shift-invariant systems. In particular, Lemmas 2.1 and 2.2. relate the solution of a periodic differential Lyapunov equation to some time-domain specifications of the system (minimum energy input, maximum output overshoot). The H_∞ norm of a periodic system is then defined and shown to be independent of time (Lemma 2.3) and equal to the classical L_2-induced norm of the input-output operator (Lemma 2.4). Then, Lemmas 2.6 – 2.8 extend to periodic systems the necessary and sufficient conditions for the H_∞ norm to be bounded by a prescribed scalar γ. This is done in terms of differential periodic Riccati equations (Lemma 2.6), periodic differential inequalities (Lemmas 2.6, 2.7), periodic Hamiltonial properties (Lemma 2.5) and differential periodic game theory approach (Lemma 2.8). An Hankel–Toeplitz operator is defined and characterized in Lemma 2.9. It will be useful in Section 4. Lemmas 2.10, 2.11 extend to periodic systems the double coprime factorization result and the Youla–Kučera parametrization, respectively. A small gain result is then provided in Lemma 2.12, and, finally, the relation between internal and external stability of feedback periodic systems is pointed out in Lemma 2.13.

2.1. Stability

Internal stability of $\mathcal{G}$ refers to the free state motion and as such depends only on matrix $F(\cdot)$. Associated with $F(\cdot)$ is the transition matrix $\Phi_F(t,\tau)$, which is nonsingular, for each t and τ, thanks to the Jacobi formula

$$\det[\Phi_F(t,\tau)] = e^{\int_t^\tau \mathrm{trace}[F(\sigma)]\,\mathrm{d}\sigma}.$$

The above expression puts into light the fact that a continuous-time periodic system is reversible.
In continuous-time it is always possible to find a T-periodic state space transformation $S(\cdot)$ which solves the so-called Floquet problem, i. e. such that, in the new coordinates, the dynamic matrix, say $\widehat{F}$, is constant. Indeed $\widehat{F}$ can be obtained by solving $e^{\widehat{F}T} = \Phi_F(\tau + T, \tau)$, where τ is any given time point. The appropriate transfomation $S(\cdot)$ is simply given by

$$S(t) = e^{\widehat{F}(t-\tau)}\Phi_F(\tau, t).$$

Such a matrix is indeed pèriodic of period T and solves the linear differential equation

$$\dot{S}(t) = \widehat{F}S(t) - S(t)\,F(t)$$

with initial condition $S(\tau) = I$. The eigenvaues of $\widehat{F}$ are called the *characteristic exponents* of $F(\cdot)$, whereas the eigenvalues of the *monodromy matrix* $\Phi_F(\tau+T, \tau)$ are called the *characteristic multipliers* of $F(\cdot)$. The relation between the characteristic exponents λ and the characteristic multipliers z of $F(\cdot)$ is given by the simple formula $z = e^{\lambda T}$.

The characteristic multipliers z are different from zero (recall the Jacobi formula or the formula $z = e^{\lambda T}$) and, most important, they do not depend on τ. Indeed the equivalence relation holds

$$\Phi_F(\tau_1 + T, \tau_1) = \Phi_F(\tau_1 + T, \tau_2)\,\Phi_F(\tau_2 + T, \tau_2)\,\Phi_F(\tau_1 + T, \tau_2)^{-1}$$

which shows that, for each τ_1 and τ_2 the matrices $\Phi_F(\tau_1 + T, \tau_1)$ and $\Phi_F(\tau_2 + T, \tau_2)$ are similar.
The system is said to be (internally) stable if the free motion $\theta_f(\cdot)$ satisfying

$$\theta_f(t) = \Phi_F(t, \tau)\,\theta_f(\tau)$$

converges to zero for any initial state $x(\tau)$ and any initial time instant τ. Notice that it is always possible to write $\Phi_F(t, \tau) = \Phi_F(\tau + \tau_1, \tau)\,\Phi_(\tau + T, \tau)^k$, where k is a nonnegative integer and $\tau_1 \in [0, T)$. Hence stability holds iff $\Phi_(\tau + T, \tau)^k$ goes to zero as k goes to infinity. This occurs iff the characteristic multipliers of $F(\cdot))$ are inside the open unit disc (equivalently, iff the characteristic exponents are in the open left hand side of the complex plane).

2.2. Structural properties and invariant zeros

Here we are concerned with the concepts of reachability, observability, stabilizability and detectability of $\mathcal{G}$. As is well known, differently from the discrete-time case, the dimensions of the reachable (observable) subspace does not depend on the particular time-instant so that it is possible to find a periodic state-space transformation which brings the system to the Kalman canonical form in four parts. Among the different yet equivalent characterizations of reachability and observability, we will here refer to the extension of the so-called PBH (modal) test. System $\mathcal{G}$ descibed by equation (4) is reachable (the periodic pair $(F(\cdot), G(\cdot))$ is reachable) iff for any complex λ there does not exist nonzero periodic solutions $\theta(\cdot)$ of

$$\begin{bmatrix} \lambda I - F(t)' \\ -G(t)' \end{bmatrix} \theta = \begin{bmatrix} \dot{\theta} \\ 0 \end{bmatrix}, \quad t \geq \tau. \tag{6}$$

Analogously, system $\mathcal{G}$ described by equations (4), (5) is observable (in other words, the pair $(F(\cdot), G(\cdot)))$ is observable), iff for any complex λ there does not exist periodic solutions $\theta(\cdot)$ of

$$\begin{bmatrix} -\lambda I + F(t) \\ H(t) \end{bmatrix} \theta = \begin{bmatrix} \dot{\theta} \\ 0 \end{bmatrix}, \quad t \geq \tau. \tag{7}$$

Notice that a number λ satisfying (6) or (7) is a characteristic exponent of $F(\cdot)$, i.e. $e^{\lambda T}$ is a characterisic multiplier of $F(\cdot)$. Indeed, take for example equation (7). It follows that $\theta(\cdot)$ is a T-periodic solution of $\dot{\theta} = (F(t) - \lambda I)\,\theta$. Hence $\theta(\tau) = \theta(\tau + T) = \Phi_F(\tau + T, \tau)\, e^{-\lambda T}\theta(\tau)$ so that $\Phi_F(\tau + T, \tau)\,\theta(\tau) = e^{\lambda T}\theta(\tau)$.
The same modal tests can be given for stabilizability and detectability as well, by simply restricting equations (6), (7) to unstable system's modes, i.e. $Re(\lambda) \geq 0$. In the sequel we will also make use of the so-called Wonham characterizations: namely, system $\mathcal{G}$ is stabilizable iff there exists a T-periodic feedback gain $K(\cdot)$ such that $F(\cdot) + G(\cdot)K(\cdot)$ is stable. Analogously, system $\mathcal{G}$ is detectable iff there exists a stable state reconstructor, i.e. a T-periodic matrix $L(\cdot)$ such that $F(\cdot) + L(\cdot)\,H(\cdot)$ is stable. The notion of system zero can be extended following the same line of reasoning. In particular, an *invariant zero* of a "tall" (i.e. the number of outputs greater than that of the inputs) T-periodic system is defined as any complex number $z = e^{\lambda T}$ such that there exists two T-periodic function $w(\cdot)$ and $\theta(\cdot)$, with $(\theta, w) \neq (0,0)$, such that

$$\begin{bmatrix} -\lambda I + F(t) & G(t) \\ H(t) & (t) \end{bmatrix} \begin{bmatrix} \theta \\ w \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ 0 \end{bmatrix}, \quad t \geq \tau. \tag{8}$$

An analoguous definition holds for "fat" systems provided that the system matrix is transposed.

2.3. The input output–operator

The input-output operator associated with $\mathcal{G}$, with zero initial condition at $t = \tau$, will be denoted with $\mathcal{G}_{\rm op}(\tau)$. Hence,

$$z(t) = [\mathcal{G}_{\rm op}(\tau)\, w]\,(t), \quad t \geq \tau$$

denotes the output of system (4),(5) with $\theta(\tau) = 0$. The *adjoint* system of $\mathcal{G}$ is denoted by $\mathcal{G}^\sim$. It is readily seen that the adjoint $\mathcal{G}_{\rm op}(\tau)^\sim$ of the operator $\mathcal{G}_{\rm op}(\tau)$ is realized by $\mathcal{G}^\sim = (-F', -H', G', E')$. System (4)–(5) is said to be *inner at* $t = \tau$ if $\mathcal{G}_{\rm op}(\tau)^\sim \mathcal{G}_{\rm op}(\tau) = I$. If E is square and $\det[E] \neq 0$, the *inverse* of $\mathcal{G}$ is $\mathcal{G}^{-1} = (F - GE^{-1}H, GE^{-1}, -E^{-1}H, E^{-1})$. Of course, the input-output operator

$\mathcal{G}_{\mathrm{op}}^{-1}(\tau)$ of $\mathcal{G}^{-1}$ coincides with the inverse operator $\mathcal{G}_{\mathrm{op}}(\tau)^{-1}$, i.e. $\mathcal{G}_{\mathrm{op}}(\tau)^{-1} = \mathcal{G}_{\mathrm{op}}^{-1}(\tau)$. Similar definitions can be given for the right/left inverses.

Assume again that $\theta(\tau) = 0$ and that system (4)–(5) is stable. Then, we can define the L_2 induced norm

$$||\mathcal{G}_{\mathrm{op}}(\tau)|| = \sup_{w \neq 0, w \in L_2[\tau,\infty)} \frac{||\mathcal{G}_{\mathrm{op}}(\tau)\, w||_2}{||w||_2}. \tag{9}$$

Notice that, thanks to periodicity, the above norm does not depend upon τ. Actually, if $\tau - T \leq \tau' < \tau$, then $||\mathcal{G}_{\mathrm{op}}(\tau)|| \leq ||\mathcal{G}_{\mathrm{op}}(\tau')|| \leq ||\mathcal{G}_{\mathrm{op}}(\tau - T)|| = ||\mathcal{G}_{\mathrm{op}}(\tau)||$. Hence, we define

$$||\mathcal{G}_{\mathrm{op}}|| = ||\mathcal{G}_{\mathrm{op}}(\tau)||. \tag{10}$$

2.4. BIBO stability

System $\mathcal{G}$ is said to be BIBO stable (at τ) if the forced output response $z(\cdot)$ (with zero initial state at τ) is bouded for any bounded input. This occurs iff the reachable and observable part of the system is (internally) stable. In this case we simply say that the input output operator $\mathcal{G}_{\mathrm{op}}(\tau)$ is stable. As apparent, this stability concept does not depend on τ. Of course, internal stability implies external (BIBO) stability. The converse is true iff the system is stabilizable and detectable. It is readily seen that both internal and BIBO stability are preserved in cascade and parallel block configurations.

2.5. The H_2 norm

We now define and characterize in time-domain the H_2 norm of the periodic system from w to z. It is here assumed that the system is stable, i.e. $F(\cdot)$ has all characteristic multipliers inside the open unit disk, and that the system is strictly proper. This last assumption is necessary for continuous-time system to ensure the boundeness of the impulse response. Now, define as $\delta(t)$ the impulse (Dirac) function and e_i the ith column of the identity matrix (whose dimension will be clear from the context). Hence, the H_2 norm of $\mathcal{G}$ at time τ is defined as follows:

$$||\mathcal{G}(\tau)||_2 = \left[\sum_{i=1}^{m} \int_0^T ||\mathcal{G}_{\mathrm{op}}(\tau)\delta(t - \tau - \sigma)\, e_i||_2^2 \mathrm{d}\sigma \right]^{1/2}.$$

Notice that

$$[\mathcal{G}_{\mathrm{op}}(\tau)\delta(t - \tau - \sigma)\, e_i]\,(t) = C(t)\,\Phi_F(t, \tau + \sigma)G(\sigma)\, e_i, \quad t \geq \tau + \sigma \tag{11}$$

is the response to the input $w(t) = \delta(t - \tau - \sigma)\, e_i$ and initial state $\theta(\tau) = 0$. Thanks to periodicity with respect to τ, this norm does not depend on τ, and hence the H_2 norm of the system can be defined as

$$||\mathcal{G}||_2 = ||\mathcal{G}(\tau)||_2, \ \forall\, \tau.$$

It is easy to verify that this norm can be computed by solving one of the two differential Lyapunov equations with periodic coefficients

$$\dot{P}_2(t) = P_2(t)\,F(t)' + F(t)\,P_2(t) + G(t)\,G(t)' \tag{12}$$

$$-\dot{Q}_2(t) = F(t)'Q_2(t) + Q_2(t)\,F(t) + H(t)'H(t). \tag{13}$$

Actually, recall that, since $F(\cdot)$ is stable, eqs. (12), (13) admit a unique T-periodic solutions $Q_2(\cdot)$ and $P_2(\cdot)$, see [3]. From (11) it readily follows that

$$\|\mathcal{G}\|_2^2 = \text{trace}\left[\int_0^T H(\sigma)\,P_2(\sigma)\,H(\sigma)'\mathrm{d}\sigma\right] = \text{trace}\left[\int_0^T G(\sigma)'Q_2(\sigma)\,G(t)\,\mathrm{d}\sigma\right]. \tag{14}$$

The periodic Lyapunov equation is useful to characterize time-domain properties of the periodic system. The first result, stated below, considers the properties of the reachability Grammian $P_2(t)$.

Lemma 2.1. Consider system $\mathcal{G}$ given by equations (4), (5) and assume that
(i) $E(t) = 0, \forall t$
(ii) $F(\cdot)$ is stable
(iii) $\theta(-\infty) = 0$
(iv) the periodic pair $(F(\cdot), G(\cdot))$ is reachable.
If $\theta(\xi)$ is a final state of (4), (5), then

$$\inf_{w\neq 0, w\in L_2(-\infty\ \xi)} \|w\|_2^2 = \theta(\xi)'P_2(\xi)^{-1}\theta(\xi),$$

where the T-periodic positive definite matrix $P_2(t)$ is the unique T-periodic solution of the T-periodic Lyapunov equation (12).

A second result concerns a time-domain specification. Precisely, we want to link the solution of the Lyapunov equation (12) with the maximum overshoot of the output signal z when the input signal w is bounded in the unit ball of L_2. To this aim, define

$$\|z\|_\infty = \sup_{t\geq\tau}(z(t)'z(t))^{1/2}.$$

The following result holds.

Lemma 2.2. Consider system $\mathcal{G}$ given by equations (4),(5) and assume that
(i) $E(t) = 0, \forall t$
(ii) $F(\cdot)$ is stable
(iii) $\theta(\tau) = 0$
(iv) the periodic pair $(F(\cdot), G(\cdot))$ is reachable.
Then,

$$\sup_{w\in L_2[\tau,\ \infty],\ \|w\|_2\leq 1} \|z\|_\infty = \sup_{t\in[0,\ \tau]} \lambda_{\max}[H(t)\,P_2(t)\,H(t)']^{1/2}$$

where the T-periodic positive definite matrix $P_2(t)$ is the unique T-periodic solution of the T-periodic Lyapunov equation (12) and $\lambda_{\max}$ denotes the maximum eigenvalue.

2.6. EP signals, transfer function operator and spectral properties

The analysis of periodic systems can be addressed by making reference to its spectral properties. Define in system $\mathcal{G}$ given by equations (4) – (5) the input signal $w(t)$ as an *exponentially periodic signal* (EPS) in the symbol λ, i. e.

$$w(t+kT) = w(t)\, e^{\lambda kT}, \quad \forall t.$$

Now, chosen any tag point τ, it follows that the initial state

$$\theta_\lambda(\tau) = (e^{\lambda T} I - \Phi_F(\tau+T,\tau))^{-1} \int_\tau^{\tau+T} \Phi_F(\tau+T,\sigma)\, G(\sigma) w(\sigma)\, \mathrm{d}\sigma \qquad (15)$$

is such that both the state and the corresponding output are still EPS with symbol λ, i. e.

$$\theta(t+kT) = \theta(t)\, e^{\lambda kT}, \quad \forall t$$
$$z(t+kT) = z(t)\, e^{\lambda kT}, \quad \forall t.$$

The corresponding input/output operator mapping $w(\sigma)$, $\sigma \in [t, t+T)$ to $z(t)$ will be denoted by $\mathcal{G}(t,\lambda)$. After some computations, it follows that

$$\begin{aligned} z(t) &= [\mathcal{G}(t,\lambda)\, w]\,(t) \qquad (16)\\ &= H(t)\,\Phi_F(t,\tau)\,\theta_\lambda(\tau) + H(t)\int_\tau^t \Phi_F(t,\sigma) G(\sigma)\, w(\sigma)\, \mathrm{d}\sigma + Ew(t)\\ &= H(t)\,(e^{\lambda T} I - \Phi_F(t+T,t))^{-1} \int_t^{t+T} \Phi_F(t+T,\sigma)\, G(\sigma)\, w(\sigma)\, \mathrm{d}\sigma + E(t)\, w(t). \end{aligned}$$

This operator satisfies (the simple check is left to the reader)

$$[\mathcal{G}(t,\lambda)\, w]\,(t+kT) = e^{\lambda kT}[\mathcal{G}(t,\lambda)\, w]\,(t).$$

Its norm is defined as

$$\|\mathcal{G}(t,\lambda)\| = \sup_{w\neq 0, w\in L_2[t\ t+T)} \frac{\|[\mathcal{G}(t,\lambda)w]\,(t)\|}{\|w\|_{\tau,T}} \qquad (17)$$

where the norm of a (complex) signal $v(\cdot)$ in $L_2[t, t+T)$ is taken as

$$\|v\|_{t,T}^2 = \int_t^{t+T} v(\sigma)^* v(\sigma)\, \mathrm{d}\sigma$$

and * denotes here the complex conjugate. It should be noted (see e. g. [7, 8]) that $\mathcal{G}(t,\lambda)$ coincides with the "transfer function" of the lifted state-sampled reformulation of the periodic system between the sampled input function $u(kT+\sigma)$, $\sigma \in [t, t+T]$ and the sampled output signal $y(kT+t+T)$, both seen as discrete-time function of the integer k. The adjoint $\mathcal{G}(t,\lambda)^\sim$ of $\mathcal{G}(t,\lambda)$ is easily shown to be related with the transfer function $\mathcal{G}^\sim(t,\lambda)$ of the adjoint system $\mathcal{G}^\sim = (-F', -H', G', E')$ in the following way

$$\mathcal{G}(t,\lambda)^\sim = \mathcal{G}^\sim(t, -\lambda^*).$$

Analogously, as for the inverse system $\mathcal{G}^{-1}$, it is

$$\mathcal{G}(t,\lambda)^{-1} = \mathcal{G}^{-1}(t,\lambda).$$

The Fourier analysis for periodic systems can be properly extended as follows. Consider, for each t, the input and output discrete-time signals at time k

$$\{w(t+kT)\},\ \{z(t+kT)\}$$

and define the formal series

$$w^{(\lambda)}(t) = \sum_{k=-\infty}^{\infty} w(t+kT)\, e^{-\lambda kT}$$

$$z^{(\lambda)}(t) = \sum_{k=-\infty}^{\infty} z(t+kT)\, e^{-\lambda kT}.$$

Then it results that $w^{(\lambda)}(t)$ and $z^{(\lambda)}(t)$ are EPS signals, i. e. for example $z^{(\lambda)}(t + kT) = z^{(\lambda)}(t)\, e^{\lambda kT}$, $\forall t$. Hence, it follows that $z^{(\lambda)}(t)$ is given by $z^{(\lambda)}(t) = [\mathcal{G}(t,\lambda)\, w^{(\lambda)}]\,(t)$, so that

$$[\mathcal{G}_{\mathrm{op}}(\tau)\, w]^{(\lambda)}(t) = [\mathcal{G}(t,\lambda)\, w^{(\lambda)}]\,(t), \quad t \geq \tau \tag{18}$$

The above equation serves as a proper generalization of the concept of transfer function for periodic system. As a matter of fact, the operator $\mathcal{G}(\tau,\lambda)$ acts as a transfer function since it transforms the input Fourier trasform $w^{(j\omega)}(\sigma)$, $\sigma \in [t, t+T)$, into the output Fourier transform $z^{(j\omega)}(t)$. Recalling now the inverse Fourier transformation formulas,

$$w(t+kT) = \frac{T}{2\pi}\int_{-\pi}^{\pi} w^{(j\omega)}(t)\, e^{j\omega kT}\mathrm{d}\omega$$

$$z(t+kT) = \frac{T}{2\pi}\int_{-\pi}^{\pi} z^{(j\omega)}(t)\, e^{j\omega kT}\mathrm{d}\omega$$

it is easy to work out the so-called Parseval rule. Actually, since,

$$\int_{-\infty}^{\infty} z(t)^* z(t)\,\mathrm{d}t = \frac{T}{2\pi}\int_{-\pi}^{\pi}\left[\int_{\tau}^{\tau+T} z^{(j\omega)}(t)^* z^{(j\omega)}(t)\,\mathrm{d}t\right]\mathrm{d}\omega$$

it follows

$$\|z\|_2^2 = \frac{T}{2\pi}\int_{-\pi}^{\pi} \|z^{(j\omega)}\|_{\tau,T}^2\,\mathrm{d}\omega = \frac{T}{2\pi}\int_{-\pi}^{\pi} \|\mathcal{G}(t,j\omega)\, w^{(j\omega)}\|_{\tau,T}^2\mathrm{d}\omega. \tag{19}$$

Analogously

$$\|w\|_2^2 = \frac{T}{2\pi}\int_{-\pi}^{\pi} \|w^{(j\omega)}\|_{\tau,T}^2\mathrm{d}\omega. \tag{20}$$

Remark 2.1. It is important to stress further the relations between the two operators associated with the periodic system $\mathcal{G}$, namely $\mathcal{G}_{\text{op}}(\tau)$ and $\mathcal{G}(\tau,\lambda)$. The causal operator $\mathcal{G}_{\text{op}}(\tau)$ acts on signals defined for $t \geq \tau$ and yields the output *forced* response of the system (with initial zero conditions at $t = \tau$). On the contrary, the noncausal operator $\mathcal{G}(\tau,\lambda)$ acts on signals defined in the interval $[t, t+T)$ and yields the value of a signal at time t. Concerning causality, notice however that, since the input signal is EPS, the equation (16) can be equivalently rewritten as

$$z(t) = H(t)\,(I - \Phi_F(t+T,t)\lambda^{-T})^{-1} \int_{t-T}^{t} \Phi_F(t,\sigma)\,G(\sigma)\,w(\sigma)\,\mathrm{d}\sigma + E(t)\,w(t)$$

so that a causal operator mapping $w(\sigma)$, $\sigma \in (t-T, t]$ to $z(t)$ can be defined instead of $\mathcal{G}(t,\lambda)$. As in the time-invariant case, the two operators $\mathcal{G}(t,\lambda)$ and $\mathcal{G}_{\text{op}}(\tau)$ are related by equation (18), in which the signals on both sides are the z-transform (or Fourier transform) of the input and output signals uniformly sampled with period T at tag time t.

2.7. The H_∞ norm

The operator $\mathcal{G}(\tau,\lambda)$ is a consistent generalization of the transfer function of a shift-invariant system. It is then very natural to define the H_∞ *norm* of the periodic system $\mathcal{G}$ given by equations (4), (5) at τ as

$$||\mathcal{G}(\tau)||_\infty = \sup_{Re(\lambda)>0} ||\mathcal{G}(\tau,\lambda)||. \tag{21}$$

A first important result is that this norm does not depend on the initial time τ.

Lemma 2.3. $||\mathcal{G}(\tau)||_\infty$ is constant with respect to τ .

Based on this lemma, the H_∞ norm $||\mathcal{G}||_\infty$of a periodic system can be consistently defined as follows:

$$||\mathcal{G}||_\infty := \sup_{Re(\lambda)>0} ||\mathcal{G}(\tau,\lambda)||. \tag{22}$$

Interestingly, as in the shift invariant case, the L_2 induced norm given by (9), (10) coincides with the now defined H_∞ norm of the transfer function, given by (15)–(22).

Lemma 2.4. Consider system $\mathcal{G}$ given by equations (4), (5) and suppose that F is stable. Then

$$||\mathcal{G}||_\infty = ||\mathcal{G}_{\text{op}}||.$$

Differently from the H_2 norm, the characterization of the H_∞ norm in state-space only gives a necessary and sufficient condition for this norm to be less than or equal to a prescribed positive scalar γ. Let this scalar be such that $\bar{\sigma}(E(t)) < \gamma$, $\forall t$ (so that $\gamma^2 I - E(t)'E(t)$ is positive definite, $\forall\, t$) and consider the T-periodic Hamiltonian matrix $W(\cdot)$ associated with system (4), (5) :

$$W(t) = \begin{bmatrix} \widehat{F}(t) & \widehat{G}(t)\widehat{G}(t)' \\ -\widehat{H}(t)'\widehat{H}(t) & -\widehat{F}(t)' \end{bmatrix} \tag{23}$$

where

$$\widehat{F}(t) = F(t) + G(t)\,(\gamma^2 I - E(t)'E(t))^{-1}E(t)'H(t)$$
$$\widehat{G}(t) = G(t)\,(\gamma^2 I - E(t)'E(t))^{-1/2}$$
$$\widehat{H}(t) = (I - E(t)\,E(t)'\gamma^{-2})^{-1/2}H(t).$$

In the classical results of Riccati equation theory, the existence of Hermitian solutions is linked with properties of invariant subspace of the so-called Hamiltonian matrix. The lemma below extends this relation to the differential Riccati equation of H_∞ type

$$-\dot{P}(t) = F(t)'P(t) + P(t)\,F(t) + H(t)'H(t)$$
$$+(P(t)\,G(t) + H(t)'E(t))\,(\gamma^2 I - E(t)'E(t))^{-1}(G(t)'P(t) + E(t)'H(t)). \tag{24}$$

Lemma 2.5. Consider system $\mathcal{G}$ given by equations (4), (5) and assume that
(i) $\gamma^2 I - E(t)'E(t) > 0, \forall\, t$
(ii) $\forall\, w \in L_2[\tau\ \infty)$, $w \neq 0$, it results that $\|\mathcal{G}_{\mathrm{op}}(\tau)\,w\|_2 \neq \gamma\|w\|_2$
(iii) The Hamiltonian matrix $W(\cdot)$ in (23) does not have unit-modulus characteristic multipliers
(iv) The pair $(F(\cdot), G(\cdot))$ is stabilizable.
Then there exists a T-periodic stabilizing solution of the Riccati equation (24).

Lemma 2.6 below characterizes the H_∞ norm in terms of differential T-periodic Riccati equations and inequalities. Precisely it states equivalent conditions for such a norm to be less than a scalar γ.

Lemma 2.6. Now, consider system $\mathcal{G}$ given by equations (4),(5) and let $\gamma > 0$ be a given positive scalar. Then, the following statements are equivalent:
(i) $F(\cdot)$ is stable and $\|\mathcal{G}\|_\infty < \gamma$
(ii) $\gamma^2 I - E(t)'E(t) > 0, \forall\, t$ and there exists a T-periodic stabilizing positive semidefinite solution to

$$-\dot{P}(t) = F(t)'P(t) + P(t)\,F(t) + H(t)'H(t) +$$
$$+\,(P(t)\,G(t) + H(t)'E(t))\,(\gamma^2 I - E(t)'E(t))^{-1}(G(t)'P(t) + E(t)'H(t))$$

i.e. such that

$$F(\cdot) + G(\cdot)\,(\gamma^2 I - E(\cdot)'E(\cdot))^{-1}(G(\cdot)'P(\cdot) + E(\cdot)'H(\cdot))$$

is asymptotically stable.

(iii) $\gamma^2 I - E(t)\,E(t)' > 0, \forall t$ and there exists a T-periodic stabilizing positive semidefinite solution to

$$\dot{Q}(t) = F(t)Q(t) + Q(t)\,F(t)' + G(t)\,G(t)' + (Q(t)\,H(t)' + G(t)\,E(t)')(\gamma^2 I - E(t)\,E(t)')^{-1}(H(t)Q(t) + E(t)\,G(t)') \tag{25}$$

i. e. such that

$$F(\cdot) + (Q(\cdot)\,H(\cdot)' + G(\cdot)\,E(\cdot)')(\gamma^2 I - E(\cdot)\,E(\cdot)')^{-1}H(\cdot)$$

is asymptotically stable.

Notice that the equivalent conditions stated in Lemma 2.6 are concerned with differential Riccati equations. It is also easy to work out new equivalent conditions based on differential Riccati (strict) inequalities where the stabilizing property of the solution is no longer required. This is done, for example, in the lemma below, which, in addition, extends the results of Lemma 2.6 to cope with the case where also the equality sign in the H_∞ bound is considered.

Lemma 2.7. Consider system $\mathcal{G}$ given by equations (4)–(5). Let $\gamma > 0$ be a given positive scalar and assume that the pair $(F(\cdot), G(\cdot))$ is reachable. Then, the following statements are equivalent:

(i) $F(\cdot)$ is stable and $||\mathcal{G}||_\infty \leq \gamma$

(ii) $\gamma^2 I - E(t)'E(t) > 0, \forall t$ and there exists a positive definite T-periodic solution $Q(\cdot)$ of the differential Riccati matrix inequality

$$\dot{Q}(t) \geq F(t)Q(t) + Q(t)\,F(t)' + G(t)\,G(t)' + (G(t)\,E(t)' + Q(t)\,H(t)')\,(\gamma^2 I - E(t)\,E(t)')^{-1}(E(t)\,G(t)' + H(t)Q(t)). \tag{26}$$

(iii) $\gamma^2 I - E(t)'E(t) > 0$ and there exists a positive definite T-periodic solution $P(\cdot)$ of the differential Riccati matrix inequality

$$-\dot{P}(t) \geq F(t)'P(t) + P(t)\,F(t) + H(t)'H(t) + (P(t)\,G(t) + H(t)'E(t))\,(\gamma^2 I - E(t)\,E(t)')^{-1}(G(t)'P(t) + E(t)'H(t)).$$

(iv) $\gamma^2 I - E(t)'E(t) > 0$ and there exists a positive definite T-periodic solution $Q_s(\cdot)$ of the periodic Riccati equation

$$\dot{Q}_s(t) = F(t)Q_s(t) + Q_s(t)\,F(t)' + G(t)\,G(t)' + (G(t)\,E(t)' + Q_s(t)\,H(t)')\,(\gamma^2 I - E(t)\,E(t)')^{-1}(E(t)\,G(t)' + H(t)Q_s(t))$$

such that the characteristic multipliers of

$$F(\cdot) + (Q_s(\cdot)\,H(\cdot)' + G(\cdot)\,E(\cdot)')(\gamma^2 I - E(\cdot)\,E(\cdot)')^{-1}H(\cdot)$$

are all inside the *closed* unit disk (*strong solution*).

The theories of H_∞ control and game theory are strictly related to each other. The following analysis result puts the basis of such a relation by linking the maximization of a sign-indefinite functional with the periodic solution of a differential Riccati equation.

Lemma 2.8. Consider system $\mathcal{G}$ given by equations (4), (5) and assume that

(i) $E(t) = 0, \forall t$

(ii) $F(\cdot)$ is stable

(iii) $\|\mathcal{G}\|_\infty < \gamma$.

If $\theta(\tau)$ is an initial state of (4), (5) then

$$\sup_{w \neq 0, w \in L_2[\tau\ \infty)} \|z\|_2^2 - \gamma^2 \|w\|_2^2 = \theta(\tau)' P(\tau)\, \theta(\tau),$$

where $P(\tau)$ is the T-periodic stabilizing solution of (24) computed at $t = \tau$.

2.8. The mixed Hankel–Toeplitz operator

Suppose that the input vector w of system (4)–(5) is partitioned into two components, say w_1 and w_2. Accordingly, matrices $G(\cdot)$ and $E(\cdot)$ can be written as $G(t) = \left| \begin{array}{cc} G_1(t) & G_2(t) \end{array} \right|$ and $E(t) = \left| \begin{array}{cc} E_1(t) & E_2(t) \end{array} \right|$, respectively. Moreover, let

$$\mathcal{G}_1 = (F, G_1, H, E_1),\ \mathcal{G}_2 = (F, G_2, H, E_2).$$

Now, let the orthogonal projections Ω_+ and Ω_- be the operators mapping $L_2(-\infty\ ,\infty)$ to $L_2[\tau\ ,\infty)$ and $L_2(-\infty\ ,\tau]$, respectively, and let Ψ be the set

$$\Psi = \left\{ w : \left[\begin{array}{c} w_1 \\ w_2 \end{array} \right],\ w_1 \in L_2(-\infty\ \tau],\ w_2 \in L_2(-\infty\ +\infty) \right\}.$$

Finally, define the operator $\mathcal{M}_{\mathrm{op}} : \Psi \longrightarrow L_2[\tau\ \infty)$ as $\mathcal{M}_{\mathrm{op}} w = \Omega_+ z = \Omega_+ \mathcal{G}_{\mathrm{op}}(\tau)\, w$. It is readily seen that the adjoint operator $\mathcal{M}_{\mathrm{op}}^\sim : L_2[\tau\ \infty) \longrightarrow \Psi$ is defined by

$$\mathcal{M}_{\mathrm{op}}^\sim h = \left| \begin{array}{c} \Omega_- \mathcal{G}_{1\mathrm{op}}(\tau)^\sim \\ \mathcal{G}_{2\mathrm{op}}(\tau)^\sim \end{array} \right| h. \tag{27}$$

Lemma 2.9. Consider system $\mathcal{G}$ given by equations (4)–(5) and suppose that:

(i) $E(t) = 0, \forall t$

(ii) $w = [w_1'\ \ w_2']'$,

(iii) $G(\cdot) = [G_1(\cdot)\ \ G_2(\cdot)]$,

(iv) $F(\cdot)$ is stable

(v) the periodic pair $(F(\cdot), G(\cdot))$ is reachable.

If

$$\sup_{w\neq 0, w\in\Psi} \frac{\|\mathcal{M}_{\rm op}w\|_2}{\|w\|_2} < \gamma$$

then

(a) there exists the T-periodic positive semidefinite stabilizing solution $X(\cdot)$ of

$$-\dot{X}(t) = F(t)'X(t) + X(t)F(t) + \frac{1}{\gamma^2}X(t)\,G_2(t)\,G_2(t)'X(t) + H(t)'H(t).$$

(b) $X(t) < \gamma^2 P_2^{-1}(t)$, $\forall t$, where $P_2(\cdot)$ is the controllability Grammian of system $\mathcal{G}$ defined in equation (12).

2.9. Youla–Kučera parametrization

The periodic system $\mathcal{G}$ given by equations (4),(5) can be given a coprime fraction representation as indicated in the next result.

Lemma 2.10. Consider the periodic system $\mathcal{G}$ given by equations (4), (5) and assume that $(F(\cdot), G(\cdot))$ is stabilizable and $(F(\cdot), H(\cdot))$ is detectable. Then, there exist 8 stable periodic systems $\mathcal{S}$, $\mathcal{N}$, $\widehat{\mathcal{S}}$, $\widehat{\mathcal{N}}$, $\mathcal{X}$, $\mathcal{Y}$, $\widehat{\mathcal{X}}$ and $\widehat{\mathcal{Y}}$, such that

(i)

$$\mathcal{G}_{\rm op}(\tau) = \mathcal{N}_{\rm op}(\tau)\mathcal{S}_{\rm op}^{-1}(\tau) = \widehat{\mathcal{S}}_{\rm op}^{-1}(\tau)\widehat{\mathcal{N}}_{\rm op}(\tau).$$

(ii)

$$\begin{bmatrix} \widehat{\mathcal{X}}_{\rm op}(\tau) & -\widehat{\mathcal{Y}}_{\rm op}(\tau) \\ -\widehat{\mathcal{N}}_{\rm op}(\tau) & \widehat{\mathcal{S}}_{\rm op}(\tau) \end{bmatrix} \begin{bmatrix} \mathcal{S}_{\rm op}(\tau) & \mathcal{Y}_{\rm op}(\tau) \\ \mathcal{N}_{\rm op}(\tau) & \mathcal{X}_{\rm op}(\tau) \end{bmatrix} = I.$$

Moreover, a possible choice of the above systems is:

$$\begin{aligned}
&\mathcal{S} = (F+GK, G, K, I), \quad \mathcal{N} = (F+GK, G, H+EK, E) \\
&\mathcal{X} = (F+GK, L, -H-EK, I), \quad \mathcal{Y} = (F+GK, L, -K, 0) \\
&\widehat{\mathcal{S}} = (F+LH, L, H, I), \quad \widehat{\mathcal{N}} = (F+LH, G+LE, H, E) \\
&\widehat{\mathcal{X}} = (F+LH, G+LE, -K, I), \quad \widehat{\mathcal{Y}} = (F+LH, L, -K, 0)
\end{aligned}$$

where $K(\cdot)$ and $L(\cdot)$ are two T-periodic matrices such that $F(\cdot) + G(\cdot)K(\cdot)$ and $F(\cdot) + L(\cdot)\,H(\cdot)$ are stable.

Lemma 2.11. Consider the periodic system $\mathcal{G}$ given by equations (4),(5). All the periodic input-output operators $\mathcal{K}_{\rm op}(\tau)$ which internally stabilize $\mathcal{G}$ are given by

$$\begin{aligned}
\mathcal{K}_{\rm op}(\tau) &= [\mathcal{Y}_{\rm op}(\tau) + \mathcal{S}_{\rm op}(\tau)\mathcal{Q}_{\rm op}(\tau)][\mathcal{X}_{\rm op}(\tau) + \mathcal{N}_{\rm op}(\tau)\mathcal{Q}_{\rm op}(\tau)]^{-1} \\
&= [\widehat{\mathcal{X}}_{\rm op}(\tau) + \mathcal{Q}_{\rm op}(\tau)\widehat{\mathcal{N}}_{\rm op}(\tau)]^{-1}[\widehat{\mathcal{Y}}_{\rm op}(\tau) + \mathcal{Q}_{\rm op}(\tau)\widehat{\mathcal{S}}_{\rm op}(\tau)]
\end{aligned}$$

where $\mathcal{Q}$ is any stable periodic system such that $(I + \mathcal{G}_{\rm op}(\tau)\mathcal{Q}_{\rm op}(\tau))$ is invertible.

2.10. Stability of feedback systems

Consider a positive feedback connection between two periodic systems

$$\mathcal{H}_1 = (A_1, B_1, C_1, D_1), \quad \mathcal{H}_2 = (A_2, B_2, C_2, D_2).$$

For the well posedeness of the closed-loop system it is obviously required that $\det(I - D_1(t)\, D_2(t)) \neq 0, \forall\, t \in [\tau, \tau{+}T)$. We also assume that the two systems (see Figure 1) are given a right coprime factorization [1] as follows:

$$\mathcal{H}_{1\mathrm{op}}(\tau) = \mathcal{H}_{12\mathrm{op}}(\tau)\mathcal{H}_{11\mathrm{op}}(\tau)^{-1}, \quad \mathcal{H}_{2\mathrm{op}}(\tau) = \mathcal{H}_{21\mathrm{op}}(\tau)\mathcal{H}_{22\mathrm{op}}(\tau)^{-1},$$

Lemma 2.12 below provides (a simplified version) of the extension to periodic systems of the well known small gain theorem. Lemma 2.13 establishes an important link between internal and external stability of the feedback configuration in Figure 1.

Lemma 2.12. Assume that the two periodic systems $\mathcal{H}_1$ and $\mathcal{H}_2$ are both stable and that they have H_∞ norm less than 1, i. e. $||H_i||_\infty < 1, i = 1, 2$. Then, the closed loop system in Figure 1 is stable as well.

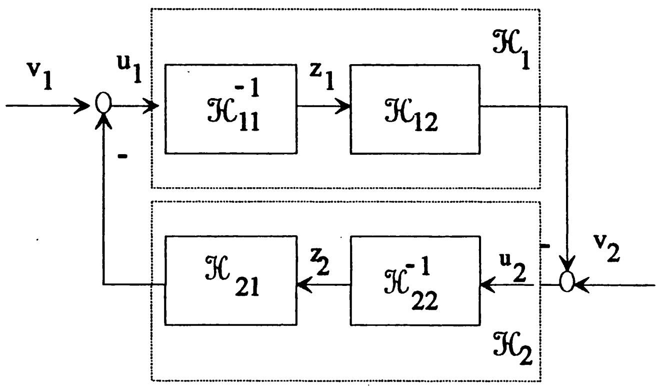

Fig. 1. Feedback configuration of two periodic systems.

Lemma 2.13. The system in Figure 1 is stable iff one of the two following equivalent conditions holds:

(i) The input-output operator $\bar{\mathcal{H}}_{\mathrm{op}}(\tau)$ from $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ to $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ is stable.

[1] Two T-periodic systems $\mathcal{A}$ and $\mathcal{B}$ with the same number of inputs are said to be right coprime if the relations

$$\mathcal{A}_{\mathrm{op}} = \bar{\mathcal{A}}_{\mathrm{op}}\mathcal{C}_{\mathrm{op}}$$
$$\mathcal{B}_{\mathrm{op}} = \bar{\mathcal{B}}_{\mathrm{op}}\mathcal{C}_{\mathrm{op}}$$

with $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ T-periodic systems, are verified only if $\mathcal{C}$ is a causal T-periodic system with causal inverse. It is possible to prove that an extended version of the Bezout identity holds for right coprime T-periodic systems as well.

(ii) The input-output operator $\widehat{\mathcal{H}}_{\rm op}(\tau)$ from $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ to $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ is stable.

3. PROOFS OF THE LEMMAS

3.1. Proof of Lemma 2.1

Since $F(\cdot)$ is stable, there exists only one periodic solution $P_2(\cdot)$ of equation (12). Moreover, such a solution is positive definite thanks to the reachability condition. Consider the function $v(\theta,t) = \theta' P_2(t)^{-1}\theta$ and compute its derivative along the trajectory of the system. It follows

$$\dot{v}(\theta,t) = w(t)'w(t) - p(t)'p(t)$$

where $p(t) = w(t) - G(t)'P_2(t)^{-1}\theta(t)$. Integration with $\theta(-\infty) = 0$ leads to

$$\theta'(\xi)\, P_2(\xi)^{-1}\theta(\xi) = -\|p\|_2^2 + \|w\|_2^2 \leq \|w\|_2^2.$$

The conclusion follows by noticing that $w(t) = G(t)'P_2(t)^{-1}\theta(t) \in L_2(-\infty\ \xi]$. Actually, with this choice we have that $\dot{\theta}(t) = (F(t) + G(t)G(t)'P_2(t)^{-1})\,\theta(t)$, and $F(t) + G(t)'G(t)'P_2(t)^{-1} {=} (F(t)\,P_2(t) + G(t)\,G(t)')\,P_2(t)^{-1} {=} (\dot{P}_2(t) - P_2((t)\,F(t)')\,P_2(t)^{-1}$. It is known that $\widehat{F}(t) = (\dot{P}_2(t) - P_2(t)\,F(t)')\,P_2(t)^{-1}$ and $-F(t)'$ are dynamic matrices of algebraically equivalent systems (if $\bar{\theta}(t) = P_2(t)^{-1}\theta(t)$ it follows that $\dot{\bar{\theta}}(t) = -\widehat{F}(t)^{-1}\bar{\theta}(t)$). Hence $\Phi_{\widehat{F}}(\tau+T,\tau) = P_2(\tau)\,\Phi_{-F'}(\tau,\tau+T)'P_2(\tau)^{-1}$. The conclusion follows from the stability of $F(\cdot)$

3.2. Proof of Lemma 2.2

Consider again the function $v(\theta,t) = \theta' P_2(t)^{-1}\theta$ and compute its derivative along the trajectory of the system. It follows

$$\dot{v}(\theta,t) = w(t)'w(t) - p(t)'p(t)$$

where $p(t) = w(t) - G(t)'P_2(t)^{-1}\theta(t)$. By integrating both members from τ to t and recalling that $\theta(\tau) = 0$ we have

$$\begin{aligned}\theta(t)'P_2(t)^{-1}\theta(t) &= -\int_\tau^t \|p(\sigma)\|^2 d\sigma + \int_\tau^t \|w(\sigma)\|^2 \mathrm{d}\sigma \\ &\leq \int_\tau^t \|w(\sigma)\|^2 \mathrm{d}\sigma \\ &\leq \int_\tau^\infty \|w(\sigma)\|^2 \mathrm{d}\sigma \leq 1.\end{aligned}$$

Hence the state trajectories $\theta(t)$, $t \geq \tau$ are in the set $\theta(t)'P_2(t)^{-1}\theta(t) \leq 1$. Now, letting $\tilde{\theta}(t) = P_2(t)^{-1/2}\theta(t)$ it follows

$$\sup_{w\in L_2[\tau,\ \infty],\ \|w\|_2\leq 1} \|z\|_\infty^2 \leq \sup_{t\geq\tau}\{z(t)'z(t),\ \theta(t)'P_2(t)^{-1}\theta(t) \leq 1\}$$

$$\leq \sup_{t\geq\tau}\{\tilde{\theta}(t)'P_2(t)^{1/2}H(t)'H(t)\,P_2(t)^{1/2}\tilde{\theta}(t),\ \tilde{\theta}(t)'\tilde{\theta}(t)\leq 1\}$$
$$\leq \sup_{t\geq\tau}\lambda_{\max}[H(t)\,P_2(t)\,H(t)']. \tag{28}$$

We only miss to show that there exists a feasible input $w(\cdot)$ which brings us arbitrarily close to the equality sign in equation (28). To this aim consider the solution $\Pi(t)$ of equation (12) with initial condition $\Pi(\tau)=0$, i.e.

$$\Pi(t)=\int_\tau^t \Phi_F(t,\sigma)G(\sigma)\,G(\sigma)'\Phi_F(t,\sigma)'\mathrm{d}\sigma.$$

Of course

$$\Pi(t)\leq P_2(t),\quad \lim_{t\to\infty}\Pi(t)-P_2(t)=0.$$

Now, consider a fixed but arbitrary time instant $L>\tau$ and the input function

$$w(t)=\begin{cases} G(t)'\Phi_F(L,t)'\Pi(L)^{-1/2}\psi, & \tau\leq t\leq L\\ 0, & t>L\end{cases}$$

where ψ is a unit-modulus eigenvector of $\Pi(L)^{1/2}H(L)'H(L)\Pi(L)^{1/2}$. Easy computations show that

$$||w||_2^2=1$$

and

$$z(L)=H(L)\Pi(L)^{1/2}\psi$$

so that

$$||z||_\infty^2\geq\psi'\Pi(L)^{1/2}H(L)'H(L)\Pi(L)^{1/2}\psi\geq\lambda_{\max}[H(L)\Pi(L)\,H(L)'].$$

Since $\Pi(t)$ tend to $P_2(t)$ as t tends to infinity, the thesis follows.

3.3. Proof of Lemma 2.3

Select $\tau_1\in(\tau,T+\tau)$. It is a matter of simple computation to show that

$$[\mathcal{G}(\tau_1,\lambda)\,w_1](t)=\begin{cases}[\mathcal{G}(\tau,\lambda)\,w](t) & \text{if } t\in[\tau_1,\tau+T)\\ e^{\lambda T}[\mathcal{G}(\tau,\lambda)\,w](t-T) & \text{if } t\in[\tau+T,\tau_1+T)\end{cases}.$$

where

$$w_1(t)=\begin{cases}w(t) & \text{if } t\in[\tau_1,\tau+T)\\ e^{\lambda T}w(t-T) & \text{if } t\in[\tau+T,\tau_1+T)\end{cases}.$$

Hence, if λ lies on the imaginary axis, i.e. $\lambda:=j\omega$, $[\mathcal{G}(\tau,j\omega)\,w](t)$ and w have the same norm as $[\mathcal{G}(\tau,j\theta)\,w_1](t)$ and w_1, respectively. Moreover, thanks to the maximum modulus theorem, $\forall\,\tau$

$$||\mathcal{G}(\tau)||_\infty=\sup_{Re(\lambda)>0}||\mathcal{G}(\tau,\lambda)||=\sup_\omega||\mathcal{G}(\tau,j\omega)||.$$

The proof is thus completed.

3.4. Proof of Lemma 2.4

Consider system $\mathcal{G}$ given by equations (4), (5) and let

$$\theta(-\infty) = 0$$

$$w(t) = \begin{cases} \bar{w}(t) & t < \tau + T \\ 0 & t \geq \tau + T \end{cases}$$

where

$$\bar{w}(t - kT) = \bar{w}(t)\, e^{-\lambda kT}, \quad Re(\lambda) \geq 0, \quad k \geq 0, \quad t \in [\tau, T + \tau).$$

It is easy to verify that

$$\begin{aligned} \theta(\tau) &= \int_{-\infty}^{\tau} \Phi_F(\tau, \sigma)\, G(\sigma)\bar{w}(\sigma)\, \mathrm{d}\sigma \\ &= (e^{\lambda T} I - \Phi_F(\tau + T, \tau))^{-1} \int_{\tau}^{\tau+T} \Phi_F(\tau + T, \sigma)\, G(\sigma)\bar{w}(\sigma)\, \mathrm{d}\sigma \end{aligned}$$

so that

$$[\mathcal{G}_{\mathrm{op}}(\tau)\, w]\,(t) = \begin{cases} \bar{z}(t) & t < \tau + T \\ H(t)\, \Phi_F(t, \tau + T)\, \theta(\tau + T) & t \geq \tau + T \end{cases}$$

where

$$\bar{z}(t - kT) = \bar{z}(t)\, e^{-\lambda kT}, \quad Re(\lambda) \geq 0, \quad k \geq 0, \quad t \in [\tau, T + \tau)$$

and

$$\bar{z}(t) = [\mathcal{G}(\tau, \lambda)\bar{w}]\,(t), \quad t \in [\tau, \tau + T).$$

Of course, due to the system stability and the fact that $Re(\lambda) \geq 0$, it is $\bar{z}(t) \in L_2(-\infty,\ \tau + T)$ and $s(t) = H(t)\, \Phi_F(t, \tau + T)\, \theta(\tau + T) \in L_2(\tau + T, \infty)$, so that

$$\begin{aligned} \|\mathcal{G}_{\mathrm{op}}(-\infty)\, w\|_2^2 &= \|\bar{z}\|_2^2 + \|s\|_2^2 \\ &\geq \|\bar{z}\|_2^2 \\ &= \sum_{k=0}^{\infty} \int_{\tau - kT}^{\tau - kT + T} \bar{z}^*(t)\bar{z}(t)\, \mathrm{d}t \\ &= \sum_{k=0}^{\infty} e^{-2Re(\lambda)kT} \int_{\tau}^{\tau+T} \bar{z}^*(t)\bar{z}(t)\, \mathrm{d}t. \end{aligned}$$

Analogously,

$$\|w\|_2^2 = \sum_{k=0}^{\infty} e^{-2Re(\lambda)kT} \int_{\tau}^{\tau+T} \bar{w}^*(t)\bar{w}(t)\, \mathrm{d}t.$$

Hence, recalling the fact that $\|\mathcal{G}_{\mathrm{op}}(\tau)\|$ does not depend on τ, it follows

$$\|\mathcal{G}_{\mathrm{op}}\|^2 = \sup_{w \neq 0, w \in L_2(-\infty\ \infty)} \frac{\|\mathcal{G}_{\mathrm{op}}(-\infty)\, w\|_2^2}{\|w\|_2^2}$$

$$
\begin{aligned}
&\geq \sup_{Re(\lambda)\geq 0,\ \bar{w}\in L_2[\tau\ \tau+T)} \frac{\int_\tau^{\tau+T} \|\bar{z}\|_2^2}{\int_\tau^{\tau+T} \|\bar{w}\|_2^2} \\
&= \sup_{Re(\lambda)\geq 0, \bar{w}\in L_2[\tau\ \tau+T)} \frac{\|\mathcal{G}(\tau,\lambda)\bar{w}\|_{\tau,T}^2}{\|\bar{w}\|_{\tau,T}^2} \\
&\geq \sup_{Re(\lambda)\geq 0} \|\mathcal{G}(\tau,\lambda)\|^2 \\
&= \|\mathcal{G}\|_\infty^2 .
\end{aligned}
$$

Viceversa, recalling the Parseval rule (equation (19), (20)),

$$
\begin{aligned}
\|\mathcal{G}_{\mathrm{op}}\|^2 &= \sup_{w\in L_2(-\infty\ \infty)} \frac{\|\mathcal{G}_{\mathrm{op}}(-\infty)\, w\|_2^2}{\|w\|_2^2} \\
&= \sup_{w^{(j\omega)}\in L_2[\tau\ \tau+T)} \frac{\int_{-\pi}^{\pi} [\int_\tau^{\tau+T} z^{(j\omega)}(t)^* z^{(j\omega)}(t)\, \mathrm{d}t]\, \mathrm{d}\omega}{\int_{-\pi}^{\pi} [\int_\tau^{\tau+T} w^{(j\omega)}(t)^* w^{(j\omega)}(t)\, \mathrm{d}t]\, \mathrm{d}\omega} \\
&= \sup_{w^{(j\omega)}\in L_2[\tau\ \tau+T)} \frac{\int_{-\pi}^{\pi} \|\mathcal{G}(\tau,j\omega)\, w^{(j\omega)}\|_{\tau,T}^2 \mathrm{d}\omega}{\int_{-\pi}^{\pi} \|w^{(j\omega)}\|_{\tau,T}^2 \mathrm{d}\omega} \\
&\leq \sup_{w^{(j\omega)}\in L_2[\tau\ \tau+T)} \frac{\int_{-\pi}^{\pi} \|w^{(j\omega)}\|_{\tau,T}^2 \mathrm{d}\omega}{\int_{-\pi}^{\pi} \|w^{(j\omega)}\|_{\tau,T}^2 \mathrm{d}\omega} \|\mathcal{G}\|_\infty^2 \\
&= \|\mathcal{G}\|_\infty^2
\end{aligned}
$$

3.5. Proof of Lemma 2.5

Without any loss of generality, let us assume that $\gamma = 1$. Moreover, let $\Phi_W(T+\tau,\tau)$ be the monodromy matrix associated with $W(\cdot)$ (see (23)). From the assumptions, we know (see e.g [9]), that there exist two matrices X_0 and Y_0 such that

(i) rank $\begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} = n$.

(ii) $X_0^* Y_0 = Y_0^* X_0$.

(iii) $\Phi_W(\tau+T,\tau) \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} = \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} Q$.

(iv) the eigenvalues of the constant matrix Q are inside the open unit disc.

Now let $\widehat{R}(t)$ be any (T-periodic) matrix such that $\Phi_{\widehat{R}}(\tau+T,\tau)=Q$ and consider the differential equation

$$\begin{bmatrix} \dot{X}(t) \\ \dot{Y}(t) \end{bmatrix} = W(t) \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} - \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} \widehat{R}(t)$$

with initial condition $\begin{bmatrix} X(\tau) \\ Y(\tau) \end{bmatrix} = \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$. It follows that the solutions $X(\cdot)$ and $Y(\cdot)$ of the associated differential equations are T-periodic matrices. Consider now $V(t)=X(t)^*Y(t)$. Simple computations show that

$$\begin{aligned} -\dot{V}(t) &= \widehat{R}(t)^* V(t) + V(t)\widehat{R}(t) \\ &\quad + X(t)^* H(t)'(I-E(t)E(t)')^{-1}H(t)X(t) - Y(t)^* G(t)\,(I-E(t)'E(t))^{-1}G(t)'Y(t) \end{aligned}$$

so that the periodic generator $V(\tau)$ satisfies

$$\begin{aligned} V(\tau) = &\int_\tau^\infty \Phi_{\widehat{R}}(t,\tau)' X(t)^* H(t)'(I-E(t)E(t)')^{-1}H(t)X(t)\,\Phi_{\widehat{R}}(t,\tau) + \\ &- \int_\tau^\infty \Phi_{\widehat{R}}(t,\tau)' Y(t)^* G(t)\,(I-E(t)'E(t))^{-1}G(t)'Y(t)\,\Phi_{\widehat{R}}(t,\tau)\,\mathrm{d}t. \end{aligned}$$

The thesis is proven ones it is shown that X_0 is invertible. Suppose by contradiction that there exists $y \neq 0$ such that $X_0 y = 0$. Hence $V(\tau)\,y = 0$. Letting

$$w(t) = (I-E(t)'E(t))^{-1}(G(t)'Y(t) + E(t)'H(t)X(t))\,\Phi_{\widehat{R}}(t,\tau)\,y$$

we have that $\theta(t) = X(t)\,\Phi_{\widehat{R}}(t,\tau)\,y$ is the state solution of system (4) and the output (5) is $z(t) = H(t)\,\theta(t) + E(t)\,w$. Moreover, it turns out that (the computation is left to the reader)

$$\int_\tau^\infty (z(t)'z(t) - w(t)'w(t))\,\mathrm{d}t = y'V(\tau)\,y = 0.$$

Hence, $\|z\|_2 = \|w\|_2$ with $w \in L_2[\tau\ \infty)$. Therefore, assumption (ii) entails that $w(t) = 0$, $\forall t$. Since $X_0 y = \theta(\tau) = 0$, it follows that $\theta(t) = 0$, $\forall t$, and $z(t) = 0$, $\forall t$, as well. Now, letting

$$h(t) = Y(t)\,\Phi_{\widehat{R}}(t,\tau)\,y.$$

It results that, for any $y \in \operatorname{Ker}(X_0)$,

$$G(t)'h(t) = 0,\ \forall t \tag{29}$$

$$\dot{h}(t) = -F(t)'h(t) \tag{30}$$

$$h(\tau) = Y_0 y \tag{31}$$

$$\Phi_F(\tau,\tau+T)'Y_0 y = Y_0 Q y \tag{32}$$

$$X_0 Q^k y = 0,\quad k = 0,1,\ldots \tag{33}$$

Let now $\mu(\lambda)$ be the monic minimum degree polynomial such that $\mu(Q)y = 0$ and write $\mu(Q) = (\lambda I - Q)\nu(Q)$. Hence, if $\widehat{y} = \nu(Q)y$, it is $Q\widehat{y} = \lambda\widehat{y}$ where $\widehat{y} \neq 0$ and $|\lambda| < 1$. Let d such that $e^{-dT} = \lambda$. Obviously, Real$(d) \geq 0$. Since $\widehat{y} \in Ker(X_0)$ (recall equation (33)), from equations (29)-(32) it follows that $\widehat{h}(t) = \Phi_F(\tau,t)'Y_0\widehat{y}e^{dt}$ is a T-periodic solution of

$$\begin{bmatrix} dI - F(t)' \\ G(t)' \end{bmatrix} \widehat{h}(t) = \begin{bmatrix} \dot{\widehat{h}}(t) \\ 0 \end{bmatrix}.$$

The equation above together with the assumption of stabilizability of $(F(\cdot), G(\cdot))$ entail that $Y_0\widehat{y} = 0$. On the other hand, this conclusion and $X_0\widehat{y} = 0$ contradict the fact that rank $\begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} = n$.

3.6. Proof of Lemma 2.6

(ii) $\leftrightarrow$ (i). First notice that the pair $(F(\cdot), N(\cdot))$ is detectable, where

$$N(t) = (\gamma^2 I - E(t)'E(t))^{-1/2}(G(t)'P(t) + E(t)'H(t))$$

Indeed thanks to the stabilizing property of the solution, $F(\cdot) + L(\cdot)N(\cdot)$ is stable, with

$$L(t) = G(t)\,(\gamma^2 I - E(t)'E(t))^{-1/2}$$

This fact implies, by a well known inertia theorem [3] that $F(\cdot)$ is stable, so that the first point of (i) is proven.
Now, let $v(\theta,t) = \theta'P(t)\,\theta$ and compute the derivative along the trajectories of the system. After some easy computations it follows

$$\frac{\mathrm{d}}{\mathrm{d}t}[v(\theta,t)] = -z(t)'z(t) + \gamma^2 w(t)'w(t) - p(t)'(\gamma^2 I - E(t)'E(t))p(t)$$

where

$$p(t) = w(t) - (\gamma^2 I - E(t)'E(t))^{-1}(E(t)'H(t) + G(t)'P(t))\,x(t).$$

Hence, recalling that $F(\cdot)$ is stable and that $w(\cdot) \in L_2[\tau,\ \infty)$, integration over $[\tau,\ \infty)$ yields

$$||z||_2^2 - \gamma^2||w||_2^2 = -||p||_2^2$$

so that $||z||_2^2 - \gamma^2||w||_2^2 < 0$, $\forall\, w(\cdot) \in L_2[\tau,\ \infty)$, $p(\cdot) \neq 0$. Notice, however, that $p(\cdot) = 0$ cannot be approached by any sequence of L_2 integrable signals $w(\cdot)$ apart from the null signal $w(\cdot) = 0$. Hence, $||\mathcal{G}||_\infty < \gamma$.
(i) $\leftrightarrow$ (ii). We have only to show that the Hamiltonian matrix $W(\cdot)$ does not have any unit-modulus characteristic multipliers. Then the result will follow from Lemma 2.5. To this end assume, by contradiction, that $W(\cdot)$ has an unit-modulus characteristic multiplier, say $e^{j\omega T}$. Hence, there exists an eigenvector $[\theta'\ p']$ such that

$$\begin{bmatrix} \theta(\tau+T) \\ p(\tau+T) \end{bmatrix} = \Phi_W(\tau+T,\tau)\begin{bmatrix} \theta \\ p \end{bmatrix} = e^{j\omega T}\begin{bmatrix} \theta \\ p \end{bmatrix}.$$

Now, take the solution of the Hamiltonian system

$$\begin{bmatrix} \theta(t) \\ p(t) \end{bmatrix} = \Phi_W(t,\tau) \begin{bmatrix} \theta \\ p \end{bmatrix}$$

and define

$$\begin{aligned} w(t) &= (\gamma^2 I - E(t)'E(t))^{-1}(G(t)'p(t) + E(t)'H(t)\,\theta(t)) \\ z(t) &= (\gamma^2 I - E(t)\,E(t)')^{-1}(E(t)\,G(t)'p(t) + \gamma^2 H(t)\,\theta(t)). \end{aligned}$$

Now, it is simple to check that

$$\begin{aligned} \dot\theta &= F(t)\,\theta + G(t)\,w \\ \dot p &= -F(t)'p - H(t)'z \\ z &= H(t)\,\theta + E(t)\,w \\ w &= G(t)'p + E(t)'z \end{aligned}$$

so that all signals are EPS. Hence, for $t \in [\tau,\ \tau+T)$ it follows $z(t) = [G(\tau, e^{j\omega T})\,w]\,(t)$ and $w(t) = \gamma^{-2}[G(\tau, e^{j\omega T})^{\sim} z]\,(t)$ so that

$$[G(\tau, e^{j\omega T})\,G(\tau, e^{j\omega T})^{\sim} z]\,(t) = \gamma^2 z(t).$$

Since $z(\cdot) \neq 0$ (otherwise also $w(\cdot)$ would be identically zero) the conclusion follows that $\|G(\tau, e^{j\omega T})\| \geq \gamma$, contrary to the assumption.

3.7. Proof of Lemma 2.7

(ii) $\leftrightarrow$ (iii). This point easily follows by checking that $P(\cdot)$ and $Q(\cdot)$ are related by $P(\cdot) = \gamma^2 Q(\cdot)^{-1}$.
(iv) $\rightarrow$ (iii). This point is trivially verified by inspection.
(ii) $\rightarrow$ (i). The inequality in point (i) can be rewritten as

$$\begin{aligned} \dot Q(t) &= Q(t)\,F(t)' + F(t)Q(t) + G(t)\,G(t)' + N(t) \\ &\quad + (G(t)\,E(t)' + Q(t)\,H(t)')\,(\gamma^2 I - E(t)\,E(t)')-1(E(t)\,G(t)' + H(t)Q(t)) \end{aligned}$$

where $N(\cdot)$ is a suitable periodic positive semidefinite matrix. Stability of $F(\cdot)$ follows from the controllability of $(F(\cdot), G(\cdot))$ thanks to an inertia theorem, see [3]. Moreover, letting $P(t) = \gamma^2 Q(t)^{-1}$, simple computations show that

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t)'P(t)\,x(t)] = -z(t)'z(t) + w(t)'w(t)\gamma^2 - p(t)'(\gamma^2 I - E(t)'E(t))p(t)$$

where $z(t) = H(t)\,x(t) + E(t)\,w(t)$ and

$$p(t) = w(t) - (\gamma^2 I - E(t)'E(t))^{-1}(E(t)'H(t) + G(t)'P(t)).$$

Now recall that $F(\cdot)$ is stable and that $w(\cdot) \in L_2[\tau, \infty]$. Integrating both member with initial state $x(\tau) = 0$ we get the thesis.

(i) $\to$ (iv). Let us define

$$H_n(t) = H(t)\epsilon_n, \quad E_n(t) = E(t)\epsilon_n$$

where $\epsilon_n \in [0,1]$ monotonically tends to 1 as n tends to infinity. Hence, thanks to the assumptions in point (i), the H_∞ norm of the periodic system $\mathcal{G}_n = (F, G, H_n, E_n)$ is stricly less than γ. In view of Lemma 2.6, there exists, for each $n > 0$, a positive semidefinite stabilizing solution of the differential Riccati equation

$$\begin{aligned} -\dot{P}_n(t) &= F(t)'P_n(t) + +P_n(t)\,F(t) + H_n(t)'H_n(t) + (P_n(t)\,G(t) \\ &\quad + H_n(t)'E_n(t))\,(\gamma^2 I - E_n(t)'E_n(t))^{-1}(G(t)'P_n(t) + E_n(t)'G(t)). \end{aligned} \tag{34}$$

Such a solution is easily verified to be the smallest positive semidefinite T-periodic solution of eq. (34). Moreover, consider the periodic differential Riccati equation

$$\begin{aligned} \dot{Q}_n(t) &= F(t)Q_n(t) + Q_n(t)\,F(t)' + G(t)'G(t) \\ &\quad + (Q_n(t)\,H_n(t)' + G(t)\,E_n(t)')\,(\gamma^2 I - E_n(t)\,E_n(t)')^{-1}(H_n(t)Q_n(t) + E_n(t)\,H_n(t)'). \end{aligned}$$

Since the H_∞ norm of $\mathcal{G}_n$ is strictly less than γ and the pair $(F(\cdot), G(\cdot))$ is reachable, such an equation admits a positive definite stabilizing solution for each n, [10]. Moreover, it is easily verified that $U_n(t) := \gamma^2 Q_n(t)^{-1}$ is a positive definite solution of (34). Such a solution is the antistabiling one. It turns out that $P_n(t) \le U_n(t)$, $\forall t$. On the other hand, $Q_n(\cdot)$ is bounded from below by the reachability grammian $M(\cdot)$ of the pair $(F(\cdot), G(\cdot))$. Indeed, denoting with $\Phi_F(t,\tau)$ the transition matrix of $F(\cdot)$, it follows that

$$Q_n(t) \ge M(t) = \int_\infty^t \Phi_F(t,\sigma)\,G(\sigma)\,G(\sigma)'\Phi_F(t,\sigma)'\mathrm{d}\sigma > 0.$$

Hence $P_n(t) \le U_n(t) \le \gamma^2 M(t)^{-1}$, $\forall t$.
We now prove that $P_n(t)$ is not decreasing with respect to n, i.e. that $P_{n+1}(t) \ge P_n(t)$. To this purpose, notice that the equation of $P_n(\cdot)$ can be equivalently rewritten as

$$-\dot{P}_n(t) = \begin{bmatrix} I & P_n(t) \end{bmatrix} R_n(t) \begin{bmatrix} I \\ P_n(t) \end{bmatrix}$$

where

$$R_n(t) = \begin{bmatrix} \widehat{H}_n(t)'\widehat{H}_n(t) & \widehat{F}_n(t)' \\ \widehat{F}_n(t) & \widehat{G}_n(t)\widehat{G}_n(t)' \end{bmatrix}$$

and

$$\begin{aligned} \widehat{H}_n(t) &= (I - E_n(t)\,E_n(t)'\gamma^{-2})^{-1/2}H_n(t) \\ \widehat{F}_n(t) &= F(t) + G(t)\,(\gamma^2 I - E_n(t)'E_n(t))^{-1}E_n(t)'H_n(t) \\ \widehat{G}_n(t) &= G(t)\,(\gamma^2 I - E_n(t)'E_n(t))^{-1/2}. \end{aligned}$$

Now, it can be simply verified that

$$R_{n+1}(t) = R_n(t) + \begin{bmatrix} H(t)' \\ G(t)\,E(t)'\gamma^{-2} \end{bmatrix} S_n(t) \begin{bmatrix} H(t) & E(t)\,G(t)'\gamma^{-2} \end{bmatrix}$$

where

$$S_n(t) = (I - E_{n+1}(t)\,E_{n+1}(t)'\gamma^{-2})^{-1}\epsilon_{n+1}^2 - (I - E_n(t)\,E_n(t)'\gamma^{-2})^{-1}\epsilon_n^2.$$

Since $S_n(t) \geq 0$ it turns out that

$$R_{n+1}(t) - R_n(t) \geq 0$$

The last inequality is a celebrated monotonicity assumption which entails the monotonicity of the stabilizing solutions $P_n(\cdot)$ with respect to n. Indeed, letting $\Delta_n(t) := P_{n+1}(t) - P_n(t)$, and $F_n(t) := F(t) + G(t)\,(\gamma^2 I - E(t)'E(t))^{-1}(G(t)'P_n(t) + E(t)'H_n)$ a simple computation shows that

$$\begin{aligned}-\dot{\Delta}_n(t) &= \Delta_n(t)F_n(t) + F_n(t)'\Delta_n(t)\\ &\quad + \begin{bmatrix} I & P_{n+1}(t) \end{bmatrix}(R_{n+1}(t) - R_n(t))\begin{bmatrix} I \\ P_{n+1}(t) \end{bmatrix}.\end{aligned}$$

Since $F_n(\cdot)$ is stable, it follows from an inertia theorem on the periodic Lyapunov equation [3] that $P_{n+1}(t) \geq P_n(t)$. A similar reasoning on the equation of $Q_n(t)$ shows that $Q_{n+1}(t)^{-1} \leq Q_n(t)^{-1}$, $\forall t$, so that $U_{n+1}(t) \leq U_n(t)$. From this last inequality and the inequelities $P_n(t) \leq U_n(t) \leq \gamma^2 M(t)^{-1}$ and $P_{n+1}(t) \geq P_n(t)$ it follows that the sequence of antistabilizing positive definite T-periodic solutions $U_n(\cdot)$ of equation (18) admits the limit solution $U_\infty(t)$. Such a solution is *antistrong*, i. e.

$$F(\cdot) + G(\cdot)\,(\gamma^2 I - E(\cdot)'E(\cdot))^{-1}(G(\cdot)'U_\infty(\cdot) + E(\cdot)'H(\cdot)$$

has characteristic multipliers with modulus greater than or equal to one and satisfies

$$\begin{aligned}-\dot{U}_\infty(t) &= F(t)'U_\infty(t) + U_\infty(t)\,F(t) + H(t)'H(t)\\ &\quad + (U_\infty(t)\,G(t) + H(t)'E(t))\,(\gamma^2 I - E(t)'E(t))^{-1}(G(t)'U_\infty(t) + E(t)'H(t)).\end{aligned}$$

Moreover, since $F(\cdot)$ is stable and $U_\infty(t)$ is antistabilizing, it is easy to see that the pair

$$\left(F(\cdot),\ \begin{bmatrix} H(t) \\ (\gamma^2 I - E(t)'E(t))^{-1/2}(G(t)'U_\infty(t) + E(t)'H(t)) \end{bmatrix}\right)$$

is observable, so that $U_\infty(\cdot)$ is indeed positive definite. Finally, is just a matter of simple computations to recognize that $Q_s(t) := \gamma^2 U_\infty(t)^{-1}$ is the positive definite strong T-periodic solution of the Riccati equation in point (iv).

3.8. Proof of Lemma 2.8

First notice that, in view of Lemma 2.6, the periodic stabilizing solution $P(t)$ of (24) actually exists. By completing the squares we have that

$$\begin{aligned}-\frac{\mathrm{d}}{\mathrm{d}t}(\theta(t)'P(t)\,\theta(t)) &= \left(\gamma w(t) - \frac{G(t)'P(t)\,\theta(t)}{\gamma}\right)'\left(\gamma w(t) - \frac{G(t)'P(t)\,\theta(t)}{\gamma}\right)\\ &\quad - w(t)'w(t)\gamma^2 + z(t)'z(t).\end{aligned}$$

Integration of both members from τ to $+\infty$ leads to

$$\theta_\tau' P(\tau)\,\theta_\tau = ||\gamma w - \frac{G'P(\cdot)\,\theta}{\gamma}||_2^2 + ||z||_2^2 - \gamma^2||w||_2^2$$

so that $||z||_2^2 - \gamma^2||^2 w||_2^2 \leq \theta_\tau' P(\tau)\,\theta_\tau$. Of course, the stabilizing property of $P(\cdot)$ entails that $w(\cdot) = \gamma^{-2}G(\cdot)'P(\cdot)\,\theta(\cdot) \in L_2[\tau\ \infty)$. Then the conclusion follows.

3.9. Proof of Lemma 2.9

The periodic stabilizing solution $X(\cdot)$ exists in virtue of Lemma 2.6 applied to system $(F, G_2, H, 0)$. Actually,

$$\gamma > \sup_{w\neq 0, w\in\Psi} \frac{||M_{\rm op}w||_2}{||w||_2} \geq \sup_{w_1=0, w_2\neq 0, w_2\in L_2[\tau\ \infty)} \frac{||M_{\rm op}w||_2}{||w_2||_2}.$$

Moreover, recalling the definition of the operator $M_{\rm op}$ and the set Ψ, it follows $||\Omega_+ z||_2^2 - \gamma^2||w||_2^2 = ||\Omega_+ z||_2^2 - \gamma^2||\Omega_+ w_2||_2^2 - \gamma^2||\Omega_- w||_2^2$. The last term contributes to $||\Omega_+ z||_2$ only through the state $\theta(\tau)$. Hence, application of Lemmas 2.2 and 2.8 leads to

$$\sup_{w\in\Psi}\{||\Omega_+ z||_2^2 - \gamma^2||w||_2^2\ |\theta(\tau) = \theta_\tau\} = \theta_\tau'[X(\tau) - \gamma^2 P_2^{-1}(\tau)]\theta_\tau.$$

Hence $||M_{\rm op}w||_2 - \gamma||w||_2 < 0$ implies that $\theta_\tau'[X(\tau) - \gamma^2 P_2^{-1}(\tau)]\theta_\tau < 0$. Since τ and θ_τ are arbitrary, the thesis follows.

3.10. Proof of Lemma 2.10

First recall that, in view of the assumed stabilizability and detectability of $\mathcal{G}$, there exist two periodic matrices $K(\cdot)$ and $L(\cdot)$ such that $F(\cdot) + L(\cdot)\,H(\cdot))$ and $F(\cdot) + G(\cdot)K(\cdot)$ are stable. Consider the four stable periodic systems $\mathcal{F}_i$, $i = 1, \cdots, 4$ given by

$$\mathcal{F}_1 = \begin{cases} \dot\xi = (F(t) + G(t))K(t))\xi + G(t)v \\ z = (H(t) + E(t)K(t))\xi + E(t)v \\ w = K(t)\xi + v \end{cases}$$

$$\mathcal{F}_2 = \begin{cases} \dot\theta = F(t)\,\theta + G(t)\,w + L(t)\eta \\ \eta = H(t)\,\theta + E(t)\,w - z \end{cases}$$

$$\mathcal{F}_3 = \begin{cases} \dot\mu = (F(t) + L(t)\,H(t))\mu + (G(t) + L(t)\,E(t)))\,w - z \\ g = w - K(t)\mu \end{cases}$$

$$\mathcal{F}_4 = \begin{cases} \dot p = (F(t) + G(t)K(t))p + L(t)d \\ z = (H(t) + E(t)K(t))p - d \\ w = K(t)p \end{cases}.$$

The reason why certain input (output) variables of the various systems are the same is due to the fact that we are going to put some of them in cascade connection. In

particular, notice that system $\mathcal{F}_1$ is nothing but system $\mathcal{G}$ equipped with the periodic feedback control law $w(t) = K(t)\xi(t)) + v(t)$, whereas system $\mathcal{F}_2$ is a periodic state reconstructor for system $\mathcal{G}$.
Assume that all the initial conditions (at $t = \tau$) of all four systems are zero and consider systems $\mathcal{S}$, $\mathcal{N}$, $\mathcal{X}$, $\widehat{\mathcal{S}}$, $\widehat{\mathcal{N}}$ and $\widehat{\mathcal{X}}$ defined in the statement of the Lemma. From $\mathcal{F}_1$ it follows:

$$w(t) = [\mathcal{S}_{\mathrm{op}}(\tau)v](t), \quad z(t) = [\mathcal{N}_{\mathrm{op}}(\tau)v](t).$$

By comparing with $z(t) = [\mathcal{G}_{\mathrm{op}}(\tau)\, w](t)$ and noticing that $\mathcal{S}$ is invertible, it is

$$\mathcal{G}_{\mathrm{op}}(\tau) = \mathcal{N}_{\mathrm{op}}(\tau)\mathcal{S}_{\mathrm{op}}(\tau)^{-1}. \tag{35}$$

From system $\mathcal{F}_2$ it is

$$\eta(t) = [\widehat{\mathcal{N}}_{\mathrm{op}}(\tau)\, w](t) - [\widehat{\mathcal{S}}_{\mathrm{op}}(\tau)\, z](t). \tag{36}$$

But the cascade connection $\mathcal{F}_{2\mathrm{op}}\mathcal{F}_{1\mathrm{op}}$ entails

$$\begin{aligned}(\dot{\theta} - \dot{\xi}) &= (F(t) + L(t)\, H(t))\, (\theta - \xi)\\ \eta &= H(t)\, (\theta - \xi)\end{aligned}$$

so that $\theta(t) = \xi(t)$ and $\eta(t) = 0$, $\forall\, t$. From equation (36), the invertibility of $\widehat{\mathcal{S}}$ and $z(t) = [\mathcal{G}_{\mathrm{op}}(\tau)\, w](t)$ it then follows

$$\mathcal{G}_{\mathrm{op}}(\tau) = \widehat{\mathcal{S}}_{\mathrm{op}}(\tau)^{-1}\widehat{\mathcal{N}}_{\mathrm{op}}(\tau). \tag{37}$$

Equations (35) and (37) prove point (i) and entail

$$\widehat{\mathcal{S}}_{\mathrm{op}}(\tau)\, \mathcal{N}_{\mathrm{op}}(\tau) = \widehat{\mathcal{N}}_{\mathrm{op}}(\tau)\, \mathcal{S}_{\mathrm{op}}(\tau). \tag{38}$$

Moreover, from system $\mathcal{F}_3$ it follows

$$g = \widehat{\mathcal{X}}_{\mathrm{op}}(\tau)\, w - \widehat{\mathcal{Y}}_{\mathrm{op}}(\tau)\, z. \tag{39}$$

But the cascade connection $\mathcal{F}_{3\mathrm{op}}\mathcal{F}_{1\mathrm{op}}$ entails that

$$\begin{aligned}(\dot{\mu} - \dot{\xi}) &= (F(t) + L(t)\, H(t))\, (\mu - \xi)\\ g &= v - K(t)\, (\mu - \xi)\end{aligned}$$

so that $\mu(t) = \xi(t)$ and $g(t) = v(t)$, $\forall\, t$. From (39), it then follows

$$\widehat{\mathcal{X}}_{\mathrm{op}}(\tau)\mathcal{S}_{\mathrm{op}}(\tau) - \mathcal{Y}_{\mathrm{op}}(\tau)\mathcal{N}_{\mathrm{op}}(\tau) = I. \tag{40}$$

Analogously, from system $\mathcal{F}_4$ it follows

$$z(t) = -[\mathcal{X}_{\mathrm{op}}(\tau)\, \mathrm{d}](t), \quad w(t) = -[\mathcal{Y}_{\mathrm{op}}(\tau)\, \mathrm{d}](t). \tag{41}$$

But the cascade connection $\mathcal{F}_{2\mathrm{op}}\mathcal{F}_{4\mathrm{op}}$ entails that

$$\begin{aligned}(\dot{p} - \dot{\theta}) &= (F(t) + L(t)\, H(t))\, (p - \theta)\\ \eta &= d - H(t)\, (p - \theta)\end{aligned}$$

so that $p(t) = \theta(t)$ and $\eta(t) = d(t)$, $\forall t$. From (36), (41) it then follows

$$\widehat{\mathcal{S}}_{\text{op}}(\tau)\mathcal{X}_{\text{op}}(\tau) - \widehat{\mathcal{N}}_{\text{op}}(\tau)\mathcal{Y}_{\text{op}}(\tau) = I. \tag{42}$$

Finally, the cascade connection $\mathcal{F}_{3\text{op}}\mathcal{F}_{4\text{op}}$ entails that

$$\begin{aligned}(\dot{p} - \dot{\mu}) &= (F(t) + L(t)\,H(t))\,(p - \mu)\\ g &= K(t)\,(p - \mu)\end{aligned}$$

so that $p(t) = \mu(t)$ and $g(t) = 0$, $\forall t$. From (39) and (41) it then follows

$$\widehat{\mathcal{X}}_{\text{op}}(\tau)\,\mathcal{Y}_{\text{op}}(\tau) - \widehat{\mathcal{Y}}_{\text{op}}(\tau)\,\mathcal{X}_{\text{op}}(\tau) = 0 \tag{43}$$

Equations (35), (37), (38), (40), (42), (43) prove point (ii).

3.11. Proof of Lemma 2.11

Observe first that, since $I - \mathcal{G}_{\text{op}}(\tau)\mathcal{Q}_{\text{op}}(\tau)$ is invertible, $\mathcal{X}_{\text{op}}(\tau) - \mathcal{N}_{\text{op}}(\tau)\mathcal{Q}_{\text{op}}(\tau)$ and $\widehat{\mathcal{X}}_{\text{op}}(\tau) - \mathcal{Q}_{\text{op}}(\tau)\widehat{\mathcal{N}}_{\text{op}}(\tau)$ are invertible as well, so that the formulas in the statement are well defined. Moreover, for any $\mathcal{Q}_{\text{op}}(\tau)$, it follows

$$I = \begin{bmatrix} I & \mathcal{Q}_{\text{op}}(\tau) \\ 0 & I \end{bmatrix}\begin{bmatrix} \widehat{\mathcal{X}}_{\text{op}}(\tau) & \widehat{\mathcal{Y}}_{\text{op}}(\tau) \\ \widehat{\mathcal{N}}_{\text{op}}(\tau) & \widehat{\mathcal{S}}_{\text{op}}(\tau) \end{bmatrix}\begin{bmatrix} \mathcal{S}_{\text{op}}(\tau) & -\mathcal{Y}_{\text{op}}(\tau) \\ -\mathcal{N}_{\text{op}}(\tau) & \mathcal{X}_{\text{op}}(\tau) \end{bmatrix}\begin{bmatrix} I & -\mathcal{Q}_{\text{op}}(\tau) \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} \widehat{\mathcal{X}}_{\text{op}}(\tau) + \mathcal{Q}_{\text{op}}(\tau)\widehat{\mathcal{N}}_{\text{op}}(\tau) & \widehat{\mathcal{Y}}_{\text{op}}(\tau) + \mathcal{Q}_{\text{op}}(\tau)\widehat{\mathcal{S}}_{\text{op}}(\tau) \\ \widehat{\mathcal{N}}_{\text{op}}(\tau) & \widehat{\mathcal{S}}_{\text{op}}(\tau) \end{bmatrix} \times$$

$$\times \begin{bmatrix} \mathcal{S}_{\text{op}}(\tau) & -\mathcal{S}_{\text{op}}(\tau)\mathcal{Q}_{\text{op}}(\tau) - \mathcal{Y}_{\text{op}}(\tau) \\ -\mathcal{N}_{\text{op}}(\tau) & \mathcal{N}_{\text{op}}(\tau)\mathcal{Q}_{\text{op}}(\tau) + \mathcal{X}_{\text{op}}(\tau) \end{bmatrix}.$$

Now, letting

$$\begin{aligned}\widehat{\mathcal{V}}_{\text{op}}(\tau) &= \widehat{\mathcal{X}}_{\text{op}}(\tau) + \mathcal{Q}_{\text{op}}(\tau)\widehat{\mathcal{N}}_{\text{op}}(\tau), & \widehat{\mathcal{U}}_{\text{op}}(\tau) &= \widehat{\mathcal{Y}}_{\text{op}}(\tau) + \mathcal{Q}_{\text{op}}(\tau)\widehat{\mathcal{S}}_{\text{op}}(\tau)\\ \mathcal{V}_{\text{op}}(\tau) &= \mathcal{X}_{\text{op}}(\tau) + \mathcal{N}_{\text{op}}(\tau)\mathcal{Q}_{\text{op}}(\tau), & \mathcal{U}_{\text{op}}(\tau) &= \mathcal{Y}_{\text{op}}(\tau) + \mathcal{S}_{\text{op}}(\tau)\mathcal{Q}_{\text{op}}(\tau)\end{aligned}$$

it follows

$$\begin{bmatrix} \widehat{\mathcal{V}}_{\text{op}}(\tau) & \widehat{\mathcal{U}}_{\text{op}}(\tau) \\ \widehat{\mathcal{N}}_{\text{op}}(\tau) & \widehat{\mathcal{S}}_{\text{op}}(\tau) \end{bmatrix}\begin{bmatrix} \mathcal{S}_{\text{op}}(\tau) & -\mathcal{U}_{\text{op}}(\tau) \\ -\mathcal{N}_{\text{op}}(\tau) & \mathcal{V}_{\text{op}}(\tau) \end{bmatrix} = I \tag{44}$$

and

$$\mathcal{K}_{\text{op}}(\tau) = \mathcal{U}_{\text{op}}(\tau)\mathcal{V}_{\text{op}}(\tau)^{-1} = \widehat{\mathcal{V}}_{\text{op}}(\tau)^{-1}\widehat{\mathcal{U}}_{\text{op}}(\tau).$$

The simple verification that the feedback connection between system $\mathcal{G}$ and controller $\mathcal{K}$ is well posed is left to the reader. Now, equation (44) implies that

$$\begin{bmatrix} \mathcal{S}_{\text{op}}(\tau) & -\mathcal{U}_{\text{op}}(\tau) \\ -\mathcal{N}_{\text{op}}(\tau) & \mathcal{V}_{\text{op}}(\tau) \end{bmatrix}^{-1}$$

is stable. But this operator is exactly the input-output operator (called $\widehat{\mathcal{H}}_{\mathrm{op}}(\tau)$ in Lemma 2.13) between v and z in Figure 1. Then, application of Lemma 2.13, point (ii), allows one to conclude that the closed-loop system (namely the system in Figure 1) is (internally) stable.

Conversely, take a periodic stabilizing controller $\mathcal{K}$ and write a coprime factorization as $\mathcal{K}_{\mathrm{op}}(\tau) = \mathcal{U}_{\mathrm{op}}(\tau)\mathcal{V}_{\mathrm{op}}(\tau)^{-1}$. Then,

$$\begin{bmatrix} I & ? \\ 0 & -\widehat{\mathcal{N}}_{\mathrm{op}}(\tau)\mathcal{U}_{\mathrm{op}}(\tau)+\widehat{\mathcal{S}}_{\mathrm{op}}(\tau)\mathcal{V}_{\mathrm{op}}(\tau) \end{bmatrix}$$

$$= \begin{bmatrix} \widehat{\mathcal{X}}_{\mathrm{op}}(\tau) & \widehat{\mathcal{Y}}_{\mathrm{op}}(\tau) \\ \widehat{\mathcal{N}}_{\mathrm{op}}(\tau) & \widehat{\mathcal{S}}_{\mathrm{op}}(\tau) \end{bmatrix} \begin{bmatrix} \mathcal{S}_{\mathrm{op}}(\tau) & -\mathcal{U}_{\mathrm{op}}(\tau) \\ -\mathcal{N}_{\mathrm{op}}(\tau) & \mathcal{V}_{\mathrm{op}}(\tau) \end{bmatrix} \tag{45}$$

where ? is a block we are not interested in. Notice that the inverses of both operators at the right hand side of equation (45) are stable. The first by construction and the second in view of Lemma 2.13, point (ii). Consequently, also $\widehat{\mathcal{N}}_{\mathrm{op}}(\tau)\mathcal{U}_{\mathrm{op}}(\tau) - \widehat{\mathcal{S}}_{\mathrm{op}}(\tau)\mathcal{V}_{\mathrm{op}}(\tau)$ admits a stable inverse. Thus, we can define

$$\mathcal{W}_{\mathrm{op}}(\tau)^{-1} = -(\widehat{\mathcal{N}}_{\mathrm{op}}(\tau)\mathcal{U}_{\mathrm{op}}(\tau)+\widehat{\mathcal{S}}_{\mathrm{op}}(\tau)\mathcal{V}_{\mathrm{op}}(\tau))^{-1},$$

$$\mathcal{Q}_{\mathrm{op}}(\tau) = (\widehat{\mathcal{X}}_{\mathrm{op}}(\tau)\mathcal{U}_{\mathrm{op}}(\tau) - \widehat{\mathcal{Y}}_{\mathrm{op}}(\tau)\mathcal{V}_{\mathrm{op}}(\tau))\mathcal{W}_{\mathrm{op}}(\tau)^{-1}.$$

Thanks to what we have said before, $\mathcal{Q}_{\mathrm{op}}(\tau)$ is stable. Moreover, as can be easily seen (the check is left to the reader), the well posedeness of the closed-loop system implies the invertibility of $I+\mathcal{G}_{\mathrm{op}}(\tau)\mathcal{Q}_{\mathrm{op}}(\tau)$ and that of $\mathcal{X}_{\mathrm{op}}(\tau)+\mathcal{N}_{\mathrm{op}}(\tau)\mathcal{Q}_{\mathrm{op}}(\tau)$. Finally, an easy computation shows that

$$\begin{bmatrix} \mathcal{S}_{\mathrm{op}}(\tau) & -\mathcal{Y}_{\mathrm{op}}(\tau) \\ -\mathcal{N}_{\mathrm{op}}(\tau) & \mathcal{X}_{\mathrm{op}}(\tau) \end{bmatrix} \begin{bmatrix} I & -\mathcal{Q}_{\mathrm{op}}(\tau)\mathcal{W}_{\mathrm{op}}(\tau) \\ 0 & \mathcal{W}_{\mathrm{op}}(\tau) \end{bmatrix} = \begin{bmatrix} \mathcal{S}_{\mathrm{op}}(\tau) & -\mathcal{U}_{\mathrm{op}}(\tau) \\ -\mathcal{N}_{\mathrm{op}}(\tau) & \mathcal{V}_{\mathrm{op}}(\tau) \end{bmatrix}$$

so that

$$(\mathcal{Y}_{\mathrm{op}}(\tau)+\mathcal{S}_{\mathrm{op}}(\tau)\mathcal{Q}_{\mathrm{op}}(\tau))\mathcal{W}_{\mathrm{op}}(\tau) = \mathcal{U}_{\mathrm{op}}(\tau)$$

$$(\mathcal{X}_{\mathrm{op}}(\tau)+\mathcal{N}_{\mathrm{op}}(\tau)\mathcal{Q}_{\mathrm{op}}(\tau))\mathcal{W}_{\mathrm{op}}(\tau) = \mathcal{V}_{\mathrm{op}}(\tau)$$

implies

$$\mathcal{K}_{\mathrm{op}}(\tau) = [\mathcal{Y}_{\mathrm{op}}(\tau)+\mathcal{S}_{\mathrm{op}}(\tau)\mathcal{Q}_{\mathrm{op}}(\tau)][\mathcal{X}_{\mathrm{op}}(\tau)+\mathcal{N}_{\mathrm{op}}(\tau)\mathcal{Q}_{\mathrm{op}}(\tau)]^{-1}.$$

Analogously, it can be proven that if $\mathcal{K}_{\mathrm{op}}(\tau) = \widehat{\mathcal{V}}_{\mathrm{op}}(\tau)^{-1}\widehat{\mathcal{U}}_{\mathrm{op}}(\tau)$, then

$$\mathcal{K}_{\mathrm{op}}(\tau) = [\widehat{\mathcal{X}}_{\mathrm{op}}(\tau)+\mathcal{Q}_{\mathrm{op}}(\tau)\widehat{\mathcal{N}}_{\mathrm{op}}(\tau)]^{-1}[\widehat{\mathcal{Y}}_{\mathrm{op}}(\tau)-\mathcal{Q}_{\mathrm{op}}(\tau)\widehat{\mathcal{S}}_{\mathrm{op}}(\tau)].$$

3.12. Proof of Lemma 2.12

Let x_1, x_2 be the state variables of the two systems $\mathcal{H}_1$ and $\mathcal{H}_2$ and x that of the closed-loop system. Assume, by contradiction, that the closed-loop system has an unstable characteristic multipliers, say $\mu := e^{\lambda T}$, with $Re(\lambda) \geq 0$. Hence, letting $\Delta_{12}(t) = (I - D_1(t)\,D_2(t))$ and $\Delta_{21}(t) = (I - D_2(t)\,D_1(t))$, we have

$$x(T+\tau) = e^{\lambda T} x(\tau)$$
$$\dot{x} = A(t)\,x$$

with

$$A(t) = \begin{bmatrix} A_1(t) + B_1(t)\,D_2\Delta_{12}(t)^{-1}C_1(t) & B_1(t)\Delta_{21}(t)^{-1}C_2(t) \\ B_2(t)\Delta_{12}(t)^{-1}C_1(t) & A_2(t) + B_2(t)\Delta_{12}(t)^{-1}D_1(t)\,C_2(t) \end{bmatrix}.$$

By letting

$$z_1(t) = \Delta_{21}(t)^{-1}(D_2(t)\,C_1(t)\,x_1(t) + C_2(t)\,x_2(t))$$

and

$$z_2(t) = \Delta_{12}(t)^{-1}(D_1(t)\,C_2(t)\,x_2(t) + C_1(t)\,x_1(t))$$

it then follows that

$$\begin{cases} \dot{x}_1 = A_1(t)\,x_1 + B_1(t)\,z_1 \\ z_2 = C_1(t)\,x_1 + D_1(t)\,z_1 \\ x_1(T+\tau) = e^{\lambda T} x_1(\tau) \end{cases}$$

and

$$\begin{cases} \dot{x}_2 = A_2(t)\,x_1 + B_2(t)\,z_1 \\ z_1 = C_2(t)\,x_1 + D_2(t)\,z_2 \\ x_2(T+\tau) = e^{\lambda T} x_2(\tau) \end{cases}.$$

Now, recalling the definition of transfer function of a periodic system (equations (15), (16)), it results

$$[\mathcal{H}_1(\tau,\lambda)\mathcal{H}_2(\tau,\lambda)\,z_1]\,(t) = z_1(t).$$

Hence,

$$||\mathcal{H}_1(\tau,\lambda)\mathcal{H}_2(\tau,\lambda)|| \geq 1.$$

Finally, $Re(\lambda) \geq 0$ and the definition (22) of the H_∞ norm of a periodic system imply that $||\mathcal{H}_1\mathcal{H}_2||_\infty \geq 1$. This is a contradiction since $||\mathcal{H}_1||_\infty < 1$ and $||\mathcal{H}_2||_\infty < 1$ obviously imply $||\mathcal{H}||_\infty < 1$.

3.13. Proof of Lemma 2.13

Point (i). Obviously, if the system is internally stable then $\bar{\mathcal{H}}_{\rm op}(\tau)$ is stable. To prove the converse statement, we only have to prove that the overall system $\bar{\mathcal{H}}$ with input-output operator $\bar{\mathcal{H}}_{\rm op}(\tau)$ is reachable and observable. Letting $\Delta_{12}(t) = (I - D_1(t)\,D_2(t))$ and $\Delta_{21}(t) = (I - D_2(t)\,D_1(t))$, it follows that $\bar{\mathcal{H}} = (A, B, C, D)$ where

$$A(t) = \begin{bmatrix} A_1(t) + B_1(t)\Delta_{21}(t)^{-1}D_2(t)\,C_1(t) & B_1(t)\Delta_{21}(t)^{-1}C_2(t) \\ B_2(t)\Delta_{12}(t)^{-1}C_1(t) & A_2(t) + B_2(t)\Delta_{12}(t)^{-1}D_1(t)\,C_2(t) \end{bmatrix}$$

$$B(t) = \begin{bmatrix} B_1(t)\Delta_{21}(t)^{-1} & B_1(t)\Delta_{21}(t)^{-1}D_2(t) \\ B_2(t)\Delta_{12}(t)^{-1}D_1(t) & B_2(t)\Delta_{12}(t)^{-1} \end{bmatrix}$$

$$C(t) = \begin{bmatrix} \Delta_{21}(t)^{-1}D_2(t)\,C_1(t) & \Delta_{21}(t))^{-1}C_2(t) \\ \Delta_{12}(t)^{-1}C_1(t) & \Delta_{12}(t)^{-1}D_1(t)\,C_2(t) \end{bmatrix}$$

$$D(t) = \begin{bmatrix} \Delta_{21}(t)^{-1} & \Delta_{21}(t)^{-1}D_2(t) \\ \Delta_{12}(t)^{-1}D_1(t) & \Delta_{12}^{(}t)-1 \end{bmatrix}.$$

Now introduce the following matrices (which are invertible for all t)

$$T_O(t) = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & -C_2(t) & -\Delta_{21}(t)^{-1}D_2(t)\Delta_{12}(t) & I \\ -C_1(t) & 0 & I & -\Delta_{12}(t)^{-1}D_1(t)\Delta_{21}(t) \end{bmatrix}$$

$$T_C(t) = \begin{bmatrix} I & 0 & B_1(t) & 0 \\ 0 & I & 0 & B_2(t) \\ 0 & 0 & I & -\Delta_{21}(t)\,D_2(t)\Delta_{12}(t)^{-1} \\ 0 & 0 & -\Delta_{12}(t)\,D_1(t)\Delta_{21}(t)^{-1} & I \end{bmatrix}.$$

Recall the PBH test (equation (6)) and assume, by contradiction, that system $\bar{\mathcal{H}}$ is not reachable, i. e. that there exists a nonzero periodic solution of

$$\begin{bmatrix} \lambda I - A(t)' \\ -B(t)' \end{bmatrix} \theta = \begin{bmatrix} \dot{\theta} \\ 0 \end{bmatrix}, \quad t \geq \tau.$$

By premultiplying by $T_C(t)'$, simple computations show that

$$\begin{bmatrix} \lambda I - A_1(t)' & 0 \\ 0 & \lambda I - A_2(t)' \\ 0 & -B_2(t)' \\ -B_1(t)' & 0 \end{bmatrix} x = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ 0 \\ 0 \end{bmatrix}$$

so that reachability of the minimal systems $\mathcal{H}_1$ and/or $\mathcal{H}_2$ is violated.
Analogously, recall the PBH test (equation (7)) and assume, by contradiction, that system $\bar{\mathcal{H}}$ is not observable, i. e. that there exists a nonzero periodic solution of

$$\begin{bmatrix} -\lambda I + A(t) \\ C(t) \end{bmatrix} \theta = \begin{bmatrix} \dot{\theta} \\ 0 \end{bmatrix}, \quad t \geq \tau.$$

By premultiplying by $T_O(t)$, simple computations show that

$$\begin{bmatrix} -\lambda I + A_1(t) & 0 \\ 0 & -\lambda I + A_2(t) \\ 0 & C_2(t) \\ C_1(t) & 0 \end{bmatrix} \theta = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ 0 \\ 0 \end{bmatrix}$$

so that observability of the minimal systems $\mathcal{H}_1$ and/or $\mathcal{H}_2$ is violated.

Point (ii). We have only to prove that $\bar{\mathcal{H}}_{\mathrm{op}}(\tau)$ is stable iff $\widehat{\mathcal{H}}_{\mathrm{op}}(\tau)$ is stable. From Figure 1 it is apparent that

$$\bar{\mathcal{H}}_{\mathrm{op}}(\tau) = -\begin{bmatrix} 0 & \mathcal{H}_{21\mathrm{op}}(\tau) \\ \mathcal{H}_{12\mathrm{op}}(\tau) & 0 \end{bmatrix} \widehat{\mathcal{H}}_{\mathrm{op}}(\tau) + I$$

so that if $\widehat{\mathcal{H}}_{\mathrm{op}}(\tau)$ is stable, $\bar{\mathcal{H}}_{\mathrm{op}}(\tau)$ is stable as well.
Conversely, since the systems $(\mathcal{H}_{11\mathrm{op}}(\tau), \mathcal{H}_{12\mathrm{op}}(\tau))$ and $(\mathcal{H}_{22\mathrm{op}}(\tau), \mathcal{H}_{21\mathrm{op}}(\tau))$ are right coprime, there exist four stable periodic systems $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{Y}_1$ and $\mathcal{Y}_2$ such that

$$\begin{bmatrix} \mathcal{X}_{1\mathrm{op}}(\tau) & 0 \\ 0 & \mathcal{X}_{2\mathrm{op}}(\tau) \end{bmatrix} \begin{bmatrix} \mathcal{H}_{12\mathrm{op}}(\tau) & 0 \\ 0 & \mathcal{H}_{21\mathrm{op}}(\tau) \end{bmatrix} + \begin{bmatrix} \mathcal{Y}_{1\mathrm{op}}(\tau) & 0 \\ 0 & \mathcal{Y}_{2\mathrm{op}}(\tau) \end{bmatrix} \begin{bmatrix} \mathcal{H}_{11\mathrm{op}}(\tau) & 0 \\ 0 & \mathcal{H}_{22\mathrm{op}}(\tau) \end{bmatrix} = I.$$

Hence, a little thought leads to

$$\begin{bmatrix} \mathcal{Y}_{1\mathrm{op}}(\tau) & -\mathcal{X}_{\mathrm{op}}(\tau) \\ -\mathcal{X}_{2\mathrm{op}}(\tau) & \mathcal{Y}_{2\mathrm{op}}(\tau) \end{bmatrix} \bar{\mathcal{H}}_{\mathrm{op}}(\tau) + \begin{bmatrix} 0 & \mathcal{X}_{1\mathrm{op}}(\tau) \\ \mathcal{X}_{2\mathrm{op}}(\tau) & 0 \end{bmatrix} = \widehat{\mathcal{H}}_{\mathrm{op}}(\tau).$$

Hence, if $\bar{\mathcal{H}}_{\mathrm{op}}(\tau)$ is stable, $\widehat{\mathcal{H}}_{\mathrm{op}}(\tau)$ is stable as well.

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