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PLANAR ANISOTROPY REVISITED

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The paper concerns estimation of anisotropy of planar fibre systems using the relation between the rose of directions and the rose of intersections. The discussion about the properties of the Steiner compact estimator is based on both theoretical and simulation results. The approach based on the distribution of the Prokhorov distance between the estimated and true rose of directions is developed. Finally the curved test systems are investigated in both Fourier and Steiner compact analysis of anisotropy.

1. INTRODUCTION

Several methods have been suggested for the estimation of the rose of directions of a planar fibre system. We discuss those of them which are based just on counting the number of intersections with a test line system, cf. Hilliard [4], Digabel [2], Mecke [8], Kanatani [5], Rataj and Saxl [11]. Some of those papers are written in the design-based approach with deterministic structure and random probes, others in the model-based approach. We use the latter approach in this paper assuming that a stationary random fibre process (Stoyan et al [14]) is investigated by means of a linear probe. The results of both approaches are comparable and have the same value for practical stereological applications.

The basic integral equation which relates the rose of directions of a stationary fibre process to its rose of intersections has been used for the estimation of anisotropy in basically three ways. First a direct solution of integral equation is available under some assumptions, which requires to estimate the second derivative of the rose of intersections from discrete data. Secondly the Fourier analysis may be applied which also arises some statistical difficulties of the problem. The most recent is the Steiner compact method which makes use of convex geometry and the relation between symmetric convex bodies and finite measures on the unit semicircle. However, very little is known about the statistical properties of this estimator since in fact a measure is estimated and a suitable probability metric has to be used for the quantification of the properties. The main aim of the present paper is to develop the investigation of the Steiner compact method. The consistency with respect to Hausdorff metric proved by Rataj and Saxl [11] is a qualitative asymptotic result. From practical stereology we have the experience that besides asymptotic theory also the small sample properties of estimators are desired. Since these are generally hardly tractable, we obtain some quantitative results at least for a special type of the fibre process, namely the anisotropic Poisson line process. The approach is different, finally the distribution of the Prokhorov distance between the estimator and the true rose of directions is obtained by means of computer simulations. The procedure moreover contributes to the theory and practice of spatial sampling.

In the second half of the paper the attention is paid to the question whether the curved test systems may have some advantage against straight line test systems. This important problem has not yet been investigated in detail.

2. BACKGROUND

Consider $Z = [0, \pi)$ with addition modulo π. This addition may be interpreted as a rotation of straight lines around origin in the plane R^2 equipped with the Borel σ-algebra $\mathcal{B}^2$. Thus for $z_1, z_2 \in Z$ corresponding to angles (with $x-$axis) of given two lines, $z_1 + z_2$ is the sum of angles. Denote by $\mathcal{M}$, $\mathcal{P}$ the system of finite measures, probability measures on the borel σ-algebra $\mathcal{B}$ of subsets of Z, respectively. Let Φ be a stationary fibre process in R^2 (Stoyan [14]), L_A its length density, $\mathcal{R} \in \mathcal{P}$ its rose of directions, i.e. the distribution of fibre tangent orientations. Let $P_L(z)$, $z \in Z$ be the rose of intersections, i.e. the mean number of points $\Phi \cap l(z)$ per unit length of a test straight line $l(z)$ with orientation z. It holds

$$P_L(z) = L_A \mathcal{G}_{\mathcal{R}}(z), \tag{1}$$

where we denote the sine transform

$$\mathcal{G}_{\mathcal{R}}(z) = \int_0^\pi |\sin(z-u)| \mathcal{R}(\mathrm{d}u). \tag{2}$$

The aim is to estimate $\mathcal{R}$ given $P_L(z_j)$, $z_j \in Z$, $j = 1, \dots n$. If a continuous probability density ρ of $\mathcal{R}$ exists we have

$$P_L''(z) + P_L(z) = 2L_A \rho(z),$$

which yields an explicit solution. This is in practice hardly tractable since the second derivative P_L'' has to be evaluated from discrete data. The formula helps in the use of parametric models of $\mathcal{R}$, cf. Digabel [2].

Several authors showed that for the Fourier images

$$\widehat{\mathcal{R}}(k) = \int_0^\pi e^{2kiz} \mathcal{R}(\mathrm{d}z) \tag{3}$$

and $\widehat{P}_L(k) = \int_0^\pi P_L(z) e^{2kiz}\,\mathrm{d}z$, $k = \dots -1, 0, 1 \dots$ it holds

$$\widehat{\mathcal{R}}(k) = \frac{1}{2L_A}(1-4k^2)\,\widehat{P}_L(k), \quad k = \dots, -1, 0, 1, \dots \tag{4}$$

When getting $\widehat{P}_L(k)$ from data and using (4) the variances of $\widehat{\mathcal{R}}(k)$ may tend to infinity. We return to this estimator in Section 5.

The third approach to the estimation of the rose of direction is based on the notion of a Steiner compact set. Let $\mathcal{K}_1$ be the system of all compact convex sets in R^2. In convex geometry (Schneider [13]) elements of $\mathcal{K}_1$ are called convex bodies. Let $S = [0, 2\pi)$, for $s \in S$ let $\overline{s}$ be the corresponding point $(\cos s, \sin s)$ on the unit circle in R^2. If $K \in \mathcal{K}_1$ then for each $s \in S$ there is exactly one number $p_K(s)$ such that the line

$$\{x \in R^2 : \langle x, \overline{s}\rangle - p_K(s) = 0\}$$

intersects K and $\langle x, \overline{s}\rangle - p_K(s) \leq 0$ for each $x \in K$. This line is called the support line and the function $p_K(s)$, $s \in S$, the support function of K.

If $U = [-u, u]$ is a line segment in R^2, we have $p_U(s) = |\langle \overline{s}, u\rangle|$ (the absolute value of the scalar product of vectors). For the Minkowski sum $K = \oplus_{i=1}^{n}[-u_i, u_i]$ of line segments (which is a convex polygon) it holds $p_K(s) = \sum_{i=1}^{n} |\langle \overline{s}, u_i\rangle|$. Then for a centrally symmetric convex body $K \subset R^2$ we have the representation

$$p_K(s) = \int |\langle \overline{s}, u\rangle| \eta(\mathrm{d}u) \tag{5}$$

for a finite Borel measure η on S. Each convex body K can be considered as a limit of convex polygons with respect to Hausdorff metric

$$d(K, L) = \inf\{\varepsilon > 0 : \ K \subset L^{\varepsilon},\ L \subset K^{\varepsilon}\},$$

where $K^{\varepsilon} = K \oplus b(0, \varepsilon)$, $b(0, a)$ is a ball with radius a centred is 0 and K, $L \in \mathcal{K}_1$.

The Hausdorff metric on $\mathcal{K}_1$ can be expressed equivalently by means of support functions as

$$d(K, L) = \sup\{|p_K(s) - p_L(s)|,\ s \in S\}, \quad K, L \in \mathcal{K}_1.$$

In fact if $d(K, L) < \alpha$ then $K \subset L^{\alpha}$ and $p_K(s) < p_L(s) + \alpha$, by reversing this argument we get the formula.

Let $T_K(s)$ be the intersection point of the support line with K (if the intersection is a line segment, $T_K(s)$ will be the endpoint with respect to the anticlockwise orientation of the boundary ∂K of K). If x, y are two points of ∂K by $l_K(x, y)$ the length of the corresponding arc of ∂K is denoted. Denote $\mathcal{K} = \{K \in \mathcal{K}_1,\ K \text{ is centrally symmetric}\}$ and for $\mathcal{R} \in \mathcal{M}$ let $\overline{\mathcal{R}}$ be a measure on S which satisfies $\overline{\mathcal{R}}(B) = \overline{\mathcal{R}}(B + \pi) = \frac{1}{2}\mathcal{R}(B)$ for any $B \in \mathcal{B}$. The following result was obtained by Matheron [7]) in a more general setting.

Proposition 1. There is a one-to-one correspondence between the elements $\mathcal{R} \in \mathcal{M}$ and $K \in \mathcal{K}$ given by

$$\overline{\mathcal{R}}((s, t]) = l_K(T_K(s), T_K(t)), \quad s, t \in S.$$

The weak convergence on $\mathcal{M}$ is equivalent to d-convergence on $\mathcal{K}$.

In the situation of Proposition 1, K is called the Steiner compact set corresponding to $\mathcal{R}$. For $K \in \mathcal{K}$ the support function is uniquelly determined by its values on

Z. Stoyan et al [14] call this restriction of p_K to Z the modified support function, we will denote it again p_K. Thus for a stationary fibre process Φ and the Steiner compact K associated to the rose of directions $\mathcal{R}$ of Φ it holds

$$p_K(z) = \frac{1}{2} L_A \mathcal{G}_{\mathcal{R}}(z), \quad z \in Z. \tag{6}$$

i.e. comparing with (1) $2p_K(z) = P_L(z)$, $z \in Z$. In fact the measure η in (5) is here interpreted as an L_A-multiple of the rose of directions, rotated by $\frac{\pi}{2}$, cf. (2).

Rataj and Saxl [11] suggested a graphical method of estimation of the rose of directions by means of its related Steiner compact set. Let $p_i = \frac{1}{2}\frac{n_i}{h}$ be the estimators of support function values at orientations $z_i \in \mathcal{Z}$, $i = 1, \ldots, n$, where n_i is the number of intersections of the studied fibre system (realization of a fibre process) with test segment of length h and orientation z_i. Then the convex polygon

$$K_n = \{x : \langle x, z_i \rangle \leq p_i, \ i = 1, \ldots, 2n\} \tag{7}$$

provides an estimator of the Steiner compact K related to $\mathcal{R}$. The measure $\mathcal{R}_n$ corresponding to K_n according to Proposition 1 is

$$\mathcal{R}_n = \sum_{i=1}^{n} h_i \delta_{z_i}, \tag{8}$$

where h_i are the lengths of edges of the polygon K_n and δ_z is the Dirac measure concentrated at z. The h_i's have outer normals z_i, in fact K_n may have less edges than $2n$ if $h_i = 0$ for some i. The relation between p_i and h_i follows (we denote $a_+ = \max(a, 0)$):

Lemma 1. It holds

$$h_i = \left(\min_{-\pi < \beta_{ij} < 0} \frac{p_i \cos \beta_{ij} - p_j}{\sin \beta_{ij}} - \max_{0 < \beta_{ij} < \pi} \frac{p_i \cos \beta_{ij} - p_j}{\sin \beta_{ij}} \right)_+, \quad i = 1, \ldots, n, \tag{9}$$

where β_{ij} are anticlockwise oriented angles between z_i and z_j.

Proof. Fix i and consider the support line l_i of p_i and the unique point $x \in l_i$ with distance p_i from origin. On the right hand side of (9) there are two terms. The ratio in the first term corresponds to the signed distance between x and the intersection point of support lines corresponding to p_i, p_j. The ratio in the second term has the same interpretation for p_j's with positive angles β_{ij} between z_i and z_j. Clearly the difference between appropriate extremes of these terms in (9) yields the edge length which may be zero if the difference is negative. □

The Hausdorff d-convergence of K_n to K is investigated by Rataj and Saxl [11]. Since the weak convergence on $\mathcal{M}$ is metrized by the Prokhorov metric, according to Proposition 1 this is a convenient metric to describe the convergence of corresponding $\mathcal{R}_n$ to $\mathcal{R}$. The Prokhorov distance between measures $Q, T \in \mathcal{M}$ is defined as

$$r(Q,T) = \inf\{\varepsilon > 0;\ Q(C) \leq T(C^\varepsilon) + \varepsilon,\ T(C) \leq Q(C^\varepsilon) + \varepsilon \text{ for all closed } C \subset Z\}.$$

This definition is equivalent (Rachev [10]) to restricted condition which we use in the form

$$r(\mathcal{R}_n, \mathcal{R}) = \inf\{\varepsilon > 0;\ \mathcal{R}_n(C) \leq \mathcal{R}(C^\varepsilon) + \varepsilon \text{ for all closed } C \subset Z\}. \tag{10}$$

In our case $\mathcal{R}_n$ is discrete with finite support $supp\,\mathcal{R}_n \subset \{z_1, \dots, z_n\}$ so we have the following reduction to finitely many conditions.

Lemma 2. It holds

$$r(\mathcal{R}_n, \mathcal{R}) = \inf\{\varepsilon > 0;\ \mathcal{R}_n(C) \leq \mathcal{R}(C^\varepsilon) + \varepsilon \text{ for all } C \subset supp\,\mathcal{R}_n\}. \tag{11}$$

Proof. Rewrite (10) as $r = \inf \mathcal{D}_1$ and (11) as $\rho = \inf \mathcal{D}_2$. Since $\mathcal{D}_1 \subset \mathcal{D}_2$ we have $\rho \leq r$. If it were $\rho < r$, then there is an $\varepsilon > 0,\ \varepsilon < r$ such that $\mathcal{R}_n(C) \leq \mathcal{R}(C^\varepsilon) + \varepsilon$ for all $C \subset supp\,\mathcal{R}_n$. Since $\varepsilon < r$, there exists a $C \subset Z$ closed with $\mathcal{R}_n(C) > \mathcal{R}(C^\varepsilon) + \varepsilon$. Let $C_1 = C \cap supp\,\mathcal{R}_n$, then $C_1 \subset C$ and $\mathcal{R}_n(C_1) = \mathcal{R}_n(C) > \mathcal{R}(C^\varepsilon) + \varepsilon \geq \mathcal{R}(C_1^\varepsilon) + \varepsilon$, a contradiction. Thus $r = \rho$. □

3. EXPLICIT RESULTS FOR POISSON LINES

The most tractable model of a fibre process in the plane is a stationary line process Φ with the line density L_A and the rose of directions $\mathcal{R}$. Any straight line $l(x)$ can be represented by a point $x = (z, y)$ in the parametric space formed by a set $\mathcal{C}_1 = (0, \pi] \times (-\infty, \infty)$. Here z is the orientation of the line and y its oriented distance from the origin. We have d positive, negative for lines intersecting the positive, negative semiaxis x, respectively. If $z = 0$, y is positive for lines in the upper half plane. We can thus represent a stationary line process Φ by means of a point process Ψ on $\mathcal{C}_1$, such that the intensity measure Λ of the process Ψ is (Stoyan et al [14])

$$\Lambda(d(y, z)) = L_A\, \mathrm{d}y \mathcal{R}\,(\mathrm{d}z). \tag{12}$$

If the stationary line process Φ is Poisson then the point process Ψ is Poisson stationary with respect to y coordinate. Conversely, a random point process on $\mathcal{C}_1$ stationary in d-coordinate defines a stationary line process in R^2.

We will investigate the intersections of a stationary Poisson line process with test segments of the same length h and of varying orientations. To each segment s a subset $A(s) \subset \mathcal{C}_1$ can be found such that $x = (z, y) \in A(s)$ if and only if the line $l(x)$ hits s. If the test segment is parametrized by its center (x_s, y_s), orientation $\beta \in Z$ and length $h > 0$, and the line (z, y) has slope $k = \tan z,\ z \neq \frac{\pi}{2}$, the hitting condition is

$$\frac{kx_s - y_s + y\sqrt{k^2+1}}{\sin\beta - k\cos\beta} \in \left[-\frac{h}{2}, \frac{h}{2}\right]. \tag{13}$$

More generally we shall consider n test segments s_i with varying orientations $\beta_i,\ i = 1, \dots, n$ and the same length h. Then for any subset $I = \{i_1, \dots, i_m\}$ of $\{1, \dots, n\}$ denote

$$A(s_I) = A(s_{i_1}, \dots, s_{i_m}) = \bigcap_{k=1}^{m} A(s_{i_k}) \cap \bigcap_{j \in I^c} A^c(s_j) \tag{14}$$

the subset of $\mathcal{C}_1$ corresponding to lines which intersect exactly m given test segments and not any other. The corresponding $A(s_I)$, $A(s_J)$ are disjoint for different I, J.

When Φ is a stationary Poisson line process, then the number of lines $N(I) = N(i_1, \ldots, i_m)$ of Φ intersecting exactly m given test segments is a random variable with Poisson distribution $Po(\lambda)$ with parameter

$$\lambda = L_A \int_{A(s_{i_1}, \ldots, s_{i_m})} \mathrm{d}y \mathcal{R}(\mathrm{d}z), \tag{15}$$

cf. (12). For different I, J the corresponding $N(I), N(J)$ are independent. From a realization of the process Φ we get estimators of support function values

$$p_j = \frac{1}{2h} \sum_{m=1}^{n} \sum_{\substack{\{i_1,\ldots,i_m\}=I \\ j \in I}} N(i_1, \ldots, i_m), \quad j = 1, \ldots, n.$$

Using the transfomation formula (9) we get from here the estimators of edge lengths h_i of the Steiner compact K_n and from (8) the desired $\mathcal{R}_n$. Given the true $\mathcal{R}$ the ultimate goal is to evaluate the distribution of the Prokhorov distance $r(\mathcal{R}_n, \mathcal{R})$. We use formula (11) and search for ε in discrete steps $\varepsilon = j\nu$, $j = 1, 2, \ldots$, $\nu > 0$, where only finitely many conditions have to be verified (over all subsets of $supp\,\mathcal{R}_n$ in each step). Distribution of the Prokhorov distance is finally obtained by means of the Monte–Carlo simulation of intersection counts. The whole procedure is demonstrated in the following situation.

Consider the unit semicircle $x = \cos\beta$, $y = \sin\beta$, $\beta \in [-\pi, \pi]$. Denote $\alpha_n = \frac{\pi}{2n}$ and define the test system $\mathcal{T}$ of n segments s_i inscribed in the semicircle, see Figure 1a. The segments have centres (x_j, y_j), $x_j = \cos\beta_j \cos\alpha_n$, $y_j = \sin\beta_j \cos\alpha_n$, normal orientations $\beta_j = (2j - n - 1)\alpha_n$, $j = 1, \ldots, n$. The segments have equal lengths $h = 2\sin\alpha_n$. The total length of $\mathcal{T}$ converges to π with $n \to \infty$. Any straight line in the plane has at most two intersections with the test system $\mathcal{T}$ so we need at most two-point subsets $I \subset \{1, \ldots, m\}$ and denote by A_i, A_{ij} the subsets of $\mathcal{C}_1$ corresponding to lines which intersect exactly one, two segments, respectively. In Figure 1b these subsets are drawn in the case of $n = 3$.

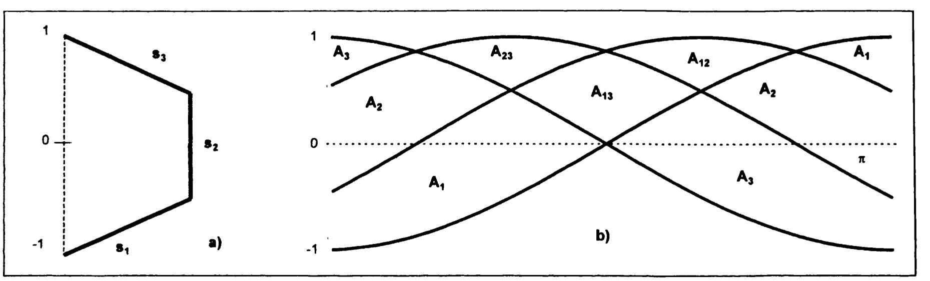

Fig. 1. The test system $\mathcal{T}$ for $n = 3$ (a), the corresponding subsets A_i, A_{ij}, $i, j = 1, \ldots, n$, $i < j$, (b).

Consider a stationary Poisson line process Φ with intensity L_A and rose of directions $\mathcal{R}$. Denote N_i, N_{ij} the Poisson distributed random variables with parameters λ_i, λ_{ij}, respectively, corresponding to numbers of intersections of Φ with given one or two segments.

Lemma 3. For the test system $\mathcal{T}$ it holds

$$\lambda_i = 2L_A\Big\{\sin\alpha_n\Big[\int_0^{(n-i)\alpha_n}\cos(\gamma-\beta_i)\,\mathcal{R}\,(\mathrm{d}\gamma)-\int_{(2n-i+1)\alpha_n}^{\pi}\cos(\gamma-\beta_i)\,\mathcal{R}\,(\mathrm{d}\gamma)\Big]$$
$$+\sin(2\alpha_n+\beta_i-\beta_n)\int_{(n-i)\alpha_n}^{(n-i+1)\alpha_n}\cos\frac{1}{2}(2\gamma-\beta_n-\beta_i)\,\mathcal{R}\,(\mathrm{d}\gamma)$$
$$+\sin(\beta_i-\beta_1-2\alpha_n)\int_{(2n-i)\alpha_n}^{(2n-i+1)\alpha_n}\cos\frac{1}{2}(2\gamma-\beta_1-\beta_i)\,\mathcal{R}\,(\mathrm{d}\gamma)\Big\},\quad i=1,\dots,n, \tag{16}$$

and

$$\lambda_{ij} = 2L_A\Big[\sin\frac{\beta_j-\beta_i}{2}\int_{(2n-i-j)\alpha_n}^{(2n-i-j+1)\alpha_n}\cos\Big(\gamma+\alpha_n-\frac{\beta_i+\beta_j}{2}\Big)\,\mathcal{R}\,(\mathrm{d}\gamma) \tag{17}$$
$$+\sin\frac{\beta_i-\beta_j}{2}\int_{(2n-i-j+1)\alpha_n}^{(2n-i-j+2)\alpha_n}\cos\Big(\gamma-\alpha_n-\frac{\beta_i+\beta_j}{2}\Big)\,\mathcal{R}\,(\mathrm{d}\gamma),\quad i,j=1,\dots,n,\ i<j.$$

For

$$p_i = \frac{1}{2h}(N_i+\sum_{j\neq i}N_{ij}),\ i=1,\dots,n \tag{18}$$

it holds

$$cov(p_i,p_j) = \frac{1}{4h^2}\operatorname{var}N_{ij}.$$

P r o o f. The first part needs the expression for boundaries between A_{ij}, A_i, cf. Figure 1b, which are essentialy shifted sine curves. The inner integral in (15) is evaluated and transformed to (16) using elementary trigonometry. The second part follows from the independence properties of N_{ij}, N_i : It holds $cov(p_i,p_j) = \frac{1}{4h^2}[E(N_i+\sum_{k\neq i}N_{ik})(N_j+\sum_{l\neq j}N_{jl}) - E(N_i+\sum_{k\neq i}N_{ik})\,E(N_j+\sum_{l\neq j}N_{jl})] = \frac{1}{4h^2}(EN_{ij}^2-(EN_{ij})^2)$. □

Instead of theoretical formulas (16), (17) we may use the Monte–Carlo approach for evaluation of parameters λ_i, λ_{ij}. It consists in simulation of a large number m of points (d,α) in $[-1,1]\times[0,\pi]\subset\mathcal{C}_1$ so that the d coordinate is uniform random and α coordinate is simulated from distribution $\mathcal{R}$. Then $\lambda_i \approx 2L_A\frac{m_i}{m}$, $\lambda_{ij}\approx 2L_A\frac{m_{ij}}{m}$, where m_i, m_{ij} are numbers of points in corresponding subsets A_i, A_{ij}. These numbers are obtained using the hitting conditions (13).

From a simulation of N_i, N_{ij}, where only one-dimensional Poisson random variables are required, we get a realization of the random variable $PD = r(\mathcal{R}_n,\mathcal{R})$

using subsequently formulas (18), (9), (11) and the discrete step approximation for getting infimum in (11). Repeating this step independently we obtain the desired distribution of Prokhorov distance $PD = r(\mathcal{R}_n, \mathcal{R})$. A computer program has been developed for the test system $\mathcal{T}$ to investigate the changes of distribution of PD with respect to the following variables: a) the intensity L_A, b) the rose of directions (corresponding to uniform distribution $\mathcal{U}$, a unimodal and a bimodal distribution), c) the number of segments n. It is not the aim of this paper to present many simulation results, two typical graphs are in Figure 2 (probability densities f obtained by smoothing the computed discrete distribution).

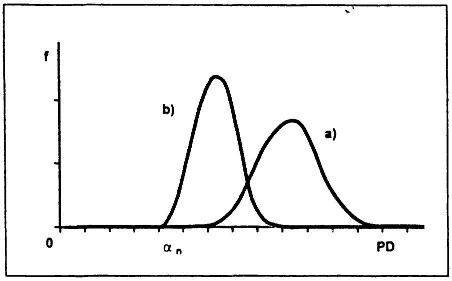

Fig. 2. Estimated probability densities of PD for $\mathcal{R} = \mathcal{U}$, $n = 8$ and $L_A = 50$ (a), $L_A = 1000$ (b).

Since the distance between a discrete and continuous distribution is measured in Figure 2, we observe that the distribution of PD is not concentrated near zero. Among the discrete distributions $\mathcal{R}_n$ on $[0, \pi)$ with support τ cardinality at most n the uniform discrete distribution $\mathcal{U}_n$ (with exactly n equidistant atoms) is nearest to $\mathcal{U}$ in the sense of Prokhorov distance. It holds $r(\mathcal{U}_n, \mathcal{U}) = \frac{\pi}{2n+\pi}$ since the worst case in (10) is

$$1 = \mathcal{U}_n(\tau) \le \mathcal{U}(\tau^\varepsilon) + \varepsilon = \frac{2n\varepsilon}{\pi} + \varepsilon.$$

For $n < 6$ we obtain a larger lower bound under a supplementary condition.

Proposition 2. For the test system $\mathcal{T}$, an isotropic fibre process and the Steiner compact estimator $\mathcal{R}_n$ of $\mathcal{R} = \mathcal{U}$ it holds that the Prokhorov distance

$$r(\mathcal{R}_n, \mathcal{R}) \ge \frac{4\alpha_n}{\pi + 2}$$

under the condition $A = [h_i = 0 \text{ for some } i]$.

Proof. Let i be the index which satisfies A, assume that $r(\mathcal{R}_n, \mathcal{U}) < \frac{4\alpha_n}{\pi+2}$. Then there is a $\delta > 0$ such that $r(\mathcal{R}_n, \mathcal{U}) = \frac{4\alpha_n}{\pi+2} - \delta$. We use the opposite equivalent definition of Prokhorov distance

$$r(\mathcal{R}_n, \mathcal{U}) = \inf\{\varepsilon > 0;\ \mathcal{U}(C) \le \mathcal{R}_n(C^\varepsilon) + \varepsilon,\ C \text{ closed}\}.$$

Put

$$C = \left[\beta_i - \frac{2\pi\alpha_n}{\pi+2},\ \beta_i + \frac{2\pi\alpha_n}{\pi+2}\right],$$

then $\mathcal{U}(C) = \frac{4\alpha_n}{\pi+2}$ and for $\varepsilon = \frac{4\alpha_n}{\pi+2} - \delta$ we have $C^\varepsilon = [\beta_i - 2\alpha_n + \delta,\ \beta_i + 2\alpha_n - \delta]$ and $\mathcal{R}_n(C^\varepsilon) = 0$. Altogether $\mathcal{R}_n(C^\varepsilon) + \varepsilon = \frac{4\alpha_n}{\pi+2} - \delta < \mathcal{U}(C)$, which leads to a contradiction. □

A lower bound for $Pr(A)$ is $\sum_i Pr(B_i) - \sum_{i<j} Pr(B_i \cap B_j)$, where $B_i = [p_{i-1} + p_{i+1} - 2p_i \cos\frac{\pi}{n} < 0]$.

4. CURVED TEST SYSTEMS

In stereology the curved test systems became popular, e. g. the cycloidal arcs for the estimation of surface area (Baddeley [1]). We shall investigate the role of curved test systems in the estimation of the rose of directions of a planar fibre process. Consider a test system t of arcs with finite total length h and $t(B)$, $B \in \mathcal{B}^2$, the corresponding length measure of t in B. Assume that almost surely (w.r.t. the length measure) the tangent orientation $w(x)$ of t at x is defined. Then the orientation distribution Q of t on Z is given by

$$\int f(\alpha)\, Q(d\alpha) = \frac{1}{h}\int_t f(w(x))\, t\,(\mathrm{d}x)$$

valid for any $f \geq 0$ measurable on Z. Denote by $t(z)$ the rotation of $t = t(0)$ by an angle $z \in Z$.

Mecke [8] points out that if the test system is formed by curved lines with tangent orientation distribution $Q \in \mathcal{P}$, then

$$P_L^Q(z) = L_A \mathcal{G}_{\mathcal{R}*Q_-}(z), \tag{19}$$

where $P_L^Q(z)$ is the rose of intersections $\Phi \cap t(z)$. Further Q_- is the reflection of Q, i. e. $\int f(z)\, Q_-(\mathrm{d}z) = \int f(\pi - z)\, Q\,(\mathrm{d}z)$ for any nonnegative measurable function f on Z, and $\mathcal{R} * Q_-$ is convolution of measures defined by $\int f(x)\mathcal{R} * Q_-(\mathrm{d}x) = \int\int f(x+y)\mathcal{R}\,(\mathrm{d}x)\, Q_-(\mathrm{d}y)$. In particular for $Q = \mathcal{U}$ uniform it follows from (19) that $P_L^{\mathcal{U}}(z) = \frac{2}{\pi}L_A$, $z \in Z$, is a constant denoted $P_L^{\mathcal{U}}(z) = P_L$.

Generally, comparing (1) and (19) we see that if there is a statistical method for estimating $\mathcal{R}$ from (1), the same method estimates $\mathcal{R} * Q_-$ from (19) when using a curved test system. Unfortunately, the system $\mathcal{P}$ with convolution operation does not posses natural inverse element to solve equation $\mathcal{R} * Q = Q_1$ for an unknown $\mathcal{R}$, cf. Heyer [3]. For the Dirac measure δ_0 concentrated in $0 \in Z$ it holds $Q * \delta_0 = Q$ for any $Q \in \mathcal{P}$. Using the complex Fourier transform $\widehat{Q}(k)$, cf. (3), we get from $Q * Q^{-1} = \delta_0$ that $\widehat{Q}(k)\widehat{Q}^{-1}(k) = 1$ for all $k = \ldots -1, 0, 1, \ldots$ Thus a necessary condition for the existence of Q^{-1} would be that there does not exist an integer k such that $\int_0^\pi \cos 2kzQ\,(\mathrm{d}z) = 0$ and $\int_0^\pi \sin 2kzQ\,(\mathrm{d}z) = 0$, which is obviously not fulfilled by many elements of $\mathcal{P}$, e. g. by uniform $\mathcal{U}$ and discrete symmetric measures $Q = \frac{1}{n}\sum_{k=1}^n \delta_{\frac{k\pi}{n}+z}$, $n \in N$, $z \in Z$. Moreover, for an absolutely continuous measure $Q \in \mathcal{P}$ with density q with finite expectation, the Fourier coefficients tend to zero with $k \to \infty$ (Kufner and Kadlec [6]) so that the Fourier coefficients of Q^{-1} should tend to infinity.

Elements $\delta_z \in \mathcal{P}$, $z \in Z$ provide rotation $Q(z) = Q * \delta_z$ of a given measure $Q \in \mathcal{P}$. The effect of the convolution operation of measures on Steiner compact sets (corresponding by Proposition 1) may be observed most easily when both measures are discrete: $\mathcal{R} = \sum_{i=1}^{n} a_i \delta_{u_i}$, $Q = \sum_{j=1}^{m} b_j \delta_{v_j}$, $\sum a_i = \sum b_j = 1$, $a_i, b_j > 0$ $u_i, v_j \in Z$. Then the convolution $\mathcal{R} * Q$ is again a discrete measure with support $\{z = u_i + v_j;\, i = 1, \dots, n,\ j = 1, \dots, m\}$. The atom in $u_i + v_j$ has size $a_i b_j$. Now the Steiner compact associated with a discrete measure is Minkowski sum of centrally symmetric segments in R^2 corresponding to atoms. These segments are $[-c_{ij},\, c_{ij}]$, where c_{ij} are vectors in R^2 with orientations $u_i + v_j$ and lengths $a_i b_j$.

The following result comes from Mecke [8].

Proposition 3. For the Fourier images $\widehat{R}(k)$, $\widehat{Q}(k)$ defined by (3) and for $\widehat{P}_L^Q(k) = \int_0^\pi P_L^Q(z) e^{2kiz} \mathrm{d}z$ it holds

$$\widehat{\mathcal{R}}(k)\widehat{Q}(-k) = \frac{1}{2L_A}(1 - 4k^2)\widehat{P}_L^Q(k), \quad k = \dots, -1, 0, 1, \dots \tag{20}$$

Proof. Let f be a π-periodic twice continuously differentiable function then $\int_0^\pi f(z)\mathcal{R}(\mathrm{d}z) = \frac{1}{2}\int_0^\pi \mathcal{G}_{\mathcal{R}}(z)[f(z) + f''(z)]\,\mathrm{d}z$ using two-fold integration by parts. Then putting $f(z) = e^{2kzi}$ we get formula (4). Using the same idea to $\mathcal{R} * Q_-$ and using the fact that the Fourier transform of convolution is a product of Fourier transforms we get (20). □

Alternatively we may use the real Fourier series on $Z = [0, \pi)$ in the form

$$P_L^Q(z) = \frac{a_0^P}{2} + \sum_{k=1}^{\infty} \left(a_k^P \cos 2kz + b_k^P \sin 2kz\right), \tag{21}$$

where

$$a_k^P = \frac{2}{\pi}\int_0^\pi P_L^Q(z)\cos 2kz\,\mathrm{d}z, \qquad b_k^P = \frac{2}{\pi}\int_0^\pi P_L^Q(z)\sin 2kz\,\mathrm{d}z, \quad k = 0, 1, \dots$$

Lemma 4. It holds $a_0^P = 2P_L$ for any $Q \in \mathcal{P}$.

Proof. Using (19) we have $a_0 = \frac{2}{\pi}\int_0^\pi P_L^Q(z)\,\mathrm{d}z = \frac{2}{\pi}\int_0^\pi L_A \mathcal{F}_{\mathcal{R}*Q_-}(z)\,\mathrm{d}z = \frac{4}{\pi}L_A = 2P_L$. □

For any (signed) measure X on Z denote

$$a_k^X = \frac{2}{\pi}\int_0^\pi \cos 2kzX\,(\mathrm{d}z), \qquad b_k^X = \frac{2}{\pi}\int_0^\pi \sin 2kzX\,(\mathrm{d}z).$$

Then analogously to (20) we have two equations for real coefficients

$$a_k^P = \frac{\pi L_A}{1 - 4k^2}(a_k^{\mathcal{R}} a_k^Q + b_k^{\mathcal{R}} b_k^Q) \text{ and } b_k^P = \frac{\pi L_A}{1 - 4k^2}(a_k^{\mathcal{R}} b_k^Q - a_k^Q b_k^{\mathcal{R}}). \tag{22}$$

5. FOURIER ANALYSIS OF THE ROSE OF DIRECTIONS

Curved test systems appeared first in Philofsky and Hilliard [9], who aimed to estimate Fourier coefficients $a_k^{\mathcal{R}}$, $b_k^{\mathcal{R}}$ in (22). They tried to find for each fixed $n \in N$ a test system with orientation distribution Q_n such that $a_k^{Q_n} = 0$ for all k not equal to n, and $b_k^{Q_n} = 0$ for all k. Then the density (with respect to uniform distribution $\mathcal{U}$) q_n of Q_n is $q_n(z) = a_n^{Q_n} \cos 2nz$, and each Q_n is a signed measure. Finally for $n = 0$ it holds $P_L^{Q_n}(0) = a_0^{\mathcal{R}}$ and for nonzero n it is

$$P_L^{Q_n}(0) = \frac{\pi a_n^{\mathcal{R}}}{2(1-4n^2)}. \tag{23}$$

It is observed that the number of intersections yields directly the n−th Fourier coefficient of the rose of directions. Here the test system is not rotated but we have for each coefficient a different test curve, analogously for $b_n^{\mathcal{R}}$. The test system is constructed by means of parametric equations in the plane

$$x(z) = \int_0^z q_n(u) \cos u \, \mathrm{d}u, \quad y(z) = \int_0^z q_n(u) \sin u \, \mathrm{d}u.$$

Since q_n correspond to signed measures the number of intersections on arcs where q_n are negative has to be subtracted from the number of intersections on arcs where the q_n are positive to obtain the desired $P_L^{Q_n}(0)$.

We can formulate a variant of this idea in which the Fourier coefficients of the rose of intersections $P_L(z)$ are obtained from intersection counts on curved test systems. The use of signed measures is avoided, instead of subtraction of intersection counts on test lines we subtract after counting, see (24).

Proposition 4. Let specially $Q = \delta_0$ in (21), i.e. $P_L^Q(z) = P_L(z)$. Then it holds

$$a_m^P = 2(P_L^{Q_{ma}} - P_L) \quad \text{and} \quad b_m^P = 2(P_L^{Q_{mb}} - P_L), \quad m = 1, 2, \ldots \tag{24}$$

where $P_L^{Q_{ma}} = \frac{1}{\pi}\int_0^\pi (1 + \cos 2mz) P_L(z)\,\mathrm{d}z$ and $P_L^{Q_{mb}} = \frac{1}{\pi}\int_0^\pi (1 + \sin 2mz) P_L(z)\,\mathrm{d}z$. Here Q_{ma}, Q_{mb} are orientation distributions of test lines given parametrically as

$$\begin{aligned} x_m(z) &= x_0 + \frac{h_m}{\pi}\int_0^z \cos\theta(1+\cos(2m\theta))\,\mathrm{d}\theta \\ y_m(z) &= y_0 + \frac{h_m}{\pi}\int_0^z \sin\theta(1+\cos(2m\theta))\,\mathrm{d}\theta \end{aligned} \tag{25}$$

for Q_{ma} and

$$\begin{aligned} x_m(z) &= x_0 + \frac{h_m}{\pi}\int_0^z \cos\theta(1+\sin(2m\theta))\,\mathrm{d}\theta, \\ y_m(z) &= y_0 + \frac{h_m}{\pi}\int_0^z \sin\theta(1+\sin(2m\theta))\,\mathrm{d}\theta \end{aligned}$$

for Q_{mb}, $0 < z \leq \pi$, where (x_0, y_0) is an arbitrary point in R^2, h_m is the total length of the test line.

Remark. Here we write P_L^Q instead of $P_L^Q(0)$ for $Q = Q_{ma},\ Q_{mb}$. The test systems are not rotated, there is a different test curve shape for estimation of each coefficient $a_m,\ b_m,\ m = 1, 2 \dots$

Proof. It holds $P_L = \frac{1}{\pi}\int_0^\pi P_L(z)\,\mathrm{d}z$, therefore we can write the coefficients (21) as

$$\begin{aligned} a_m^P &= 2[\frac{1}{\pi}\int_0^\pi (1+\cos 2mz)P_L(z)\,\mathrm{d}z - P_L], \qquad (26)\\ b_m^P &= 2[\frac{1}{\pi}\int_0^\pi (1+\sin 2mz)P_L(z)\,\mathrm{d}z - P_L] \end{aligned}$$

to get (24) if the integrals in (26) are interpreted in terms of intersection counts pertaining to the test lines of specific shapes given by equations (25). □

Once the Fourier coefficients $a_k^P,\ b_k^P$ are estimated, the Fourier coefficients for the rose of directions are obtained from (22) as

$$a_k^{\mathcal{R}} = D_k(a_k^P a_k^Q + b_k^P b_k^Q), \qquad b_k^{\mathcal{R}} = D_k(a_k^P b_k^Q - a_k^Q b_k^P), \qquad (27)$$

where

$$D_k = \frac{1-4k^2}{\pi L_A[(a_k^Q)^2 + (b_k^Q)^2]}.$$

In the case of Proposition 4 it is $Q = \delta_0$ so that $a_k^Q = \frac{2}{\pi},\ b_k^Q = 0$ for all k, so we have specialy

$$a_k^{\mathcal{R}} = \frac{1-4k^2}{2L_A}a_k^P, \qquad b_k^{\mathcal{R}} = -\frac{1-4k^2}{2L_A}b_k^P. \qquad (28)$$

The structure of the coefficients is similar to (4) and to that of Philofski and Hilliard [9] obtained by an alternative method. If the density ρ of $\mathcal{R}$ exists its Fourier series is

$$\begin{aligned} \rho(z) &= \frac{a_0^{\mathcal{R}}}{2} + \sum_{k=1}^\infty a_k^{\mathcal{R}}\cos 2kz + b_k^{\mathcal{R}}\sin 2kz \qquad (29)\\ &= \frac{1}{\pi} + \frac{1}{2L_A}\sum_{k=1}^\infty (1-4k^2)[a_k^P\cos 2kz - b_k^P\sin 2kz], \qquad z\in[0,\pi). \end{aligned}$$

We observe a phenomenon typical for some stereological problems, namely the sum (29) with random coefficients $a_k^P,\ b_k^P$ may have infinite variance caused by $1-4k^2$ term. By using only first few coefficients of the series we can have a finite variance of the estimator but this may lead to an unsatisfactorily biased estimator in the case of bimodal or other complex anistropies.

6. THE ARC TEST SYSTEM CURVES

In this section we consider rotating arc test systems, where arc is a differentiable curve in the plane. Since the fibre processes studied are stationary we may shift a given arc arbitrarily. Assume that $x \in R^2$, $x = (r\cos\beta, r\sin\beta)$, $r > 0$. Let $\mathcal{S}_{r,\beta}$ be the system of arcs s in R^2 with endpoints x, $-x$ such that the union $s \cup [-x, x]$ is a boundary of a convex body K_s. Since K_s is not centrally symmetric, we consider its support function p_{K_s} on $[0, 2\pi]$ modulo 2π. The intersections $s \cap l$ will be studied of $s \in \mathcal{S}_{r,\beta}$ and lines l in R^2. Typically there are at most two intersections $s \cap l$ for any given line. Any line is again represented by the point (z, y) in the parametric space $\mathcal{C}_1 = (0, \pi] \times (-\infty, \infty)$. The set $\mathcal{L} \subset \mathcal{C}_1$ of all lines in R^2 which hit s is expressed as

$$\mathcal{L} = \left\{(z, d); -p_{K_s}\left(z - \frac{\pi}{2}\right) \le y \le p_{K_s}\left(z + \frac{\pi}{2}\right)\right\}.$$

Let $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$, where $\mathcal{L}_1$, $\mathcal{L}_2$ are disjoint and correspond to lines which have one, two intersections with s, respectively.

Consider again the stationary anisotropic Poisson line process Φ characterized by the line density L_A and the rose of directions $\mathcal{R}$ and represent Φ by means of a Poisson point process Ψ on $\mathcal{C}_1$. Denote $N(\mathcal{L}_i)$ the number of points of $\Psi \cap \mathcal{L}_i$, $i = 1, 2$.

Lemma 5. Let $s \in \mathcal{S}_{r,\beta}$ has the length h and orientation distribution $Q(\beta)$. The number of intersections $\Phi \cap s$ is a random variable $N(\mathcal{L}) = N(\mathcal{L}_1) + 2N(\mathcal{L}_2)$, where $N(\mathcal{L}_i)$ are independent Poisson distributed with parameter $\lambda_i = L_A \int_{\mathcal{L}_i} \mathrm{d}y \mathcal{R}(\mathrm{d}z)$, $i = 1, 2$. It holds

$$EN(\mathcal{L}) = hP_L^Q(\beta) = L_A \left(\int_{\mathcal{L}_1} \mathrm{d}y \mathcal{R}(\mathrm{d}x) + 2\int_{\mathcal{L}_2} \mathrm{d}y \mathcal{R}(\mathrm{d}x)\right), \tag{30}$$

$$\operatorname{var} N(\mathcal{L}) = L_A \left(\int_{\mathcal{L}_1} \mathrm{d}y \mathcal{R}(\mathrm{d}x) + 4\int_{\mathcal{L}_2} \mathrm{d}y \mathcal{R}(\mathrm{d}x)\right).$$

Proof. Follows immediately from (12) and the fact that $EN = \operatorname{var} N = \lambda$ for a Poisson distributed random variable N with parameter λ. □

Rataj and Saxl [11] developed the Steiner compact estimators of $\mathcal{R}$ presented in Section 2 by means of the following smoothing. For $n \in N$ and orientations $0 < z_1 < z_2 < \ldots < z_n \le \pi$, for $r \in N$ and weights $\{c_j : j = -r, \ldots, 0, \ldots, r\}$, $c_{-j} = c_j \ge 0$, $j = 0, \ldots, r$, $\sum_j c_j = 1$ they construct polygons

$$\overline{K_n} = \{x : \langle x, z_i\rangle \le \overline{p}_i,\ i = 1, \ldots, n\}, \quad \text{where} \quad \overline{p}_i = \sum_{j=-r}^{r} c_j p_{i+j},\ i = 1, \ldots, n \tag{31}$$

and p_i are estimates of $\frac{1}{2}P_L(z_i)$. Let h_i be the lengths of edges of $\overline{K_n}$, then the estimator of $\mathcal{R}$ is $\mathcal{R}_n(B) = \sum_{i=1}^n h_i 1_B(z_i)$, $B \in \mathcal{B}$.

In Figure 3 the distribution of the Prokhorov distance $r(\mathcal{R}_n, \mathcal{R})$ is presented for exactly the same test system and Poisson line process as in Figure 2, with additional smoothing in (31). Comparing both figures we observe a smaller variance of PD after smoothing but the effect described in Proposition 2 remains apparent.

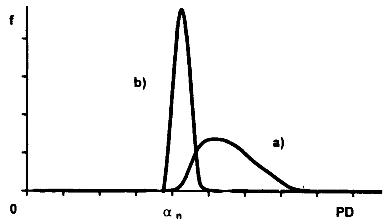

Fig. 3. Estimated probability densities of PD for $\mathcal{R} = \mathcal{U}$, $n = 8$, $L_A = 50$ (a), $L_A = 1000$ (b), smoothing with $r = 2$, $c_j = \frac{1}{2r+1}$, $j = -r, \ldots, r$.

Further we observe that the local smoothing in (31) can be expressed in terms of convolution with a discrete measure Q representing the orientation distribution of a test system.

Proposition 5. Let $Q = \sum_{i=1}^m b_i \delta_{v_i}$, $b_i > 0$, $\sum b_i = 1$, $v_i \in Z$, $i = 1, \ldots, n$. Then

$$P_L^Q(z) = \sum_{i=1}^m b_i P_L(z - \pi + v_i), \ z \in Z.$$

Proof. We have $Q_- = \sum_i b_i \delta_{\pi - v_i}$ and $\mathcal{G}_{\mathcal{R}*Q_-}(z) = \int_0^\pi |\sin(u-z)| \mathcal{R} * Q_-(\mathrm{d}u) = \sum_{i=1}^m b_i \int_0^\pi |\sin(u + \pi - v_i - z)| \mathcal{R}(\mathrm{d}u) = \sum_{i=1}^m b_i \mathcal{G}_\mathcal{R}(z - \pi + v_i)$. Then $P_L^Q(z) = L_A \mathcal{G}_{\mathcal{R}*Q_-}(z) = L_A \sum_{i=1}^m b_i \mathcal{G}_\mathcal{R}(z - \pi + v_i) = \sum_{i=1}^m b_i P_L(z - \pi + v_i)$. □

Naturally it is not necessary to restrict to discrete measures Q for local smoothing. Continuous measures correspond to curved test systems.

Example 1. Let $\mathcal{R} = \delta_0$ and Q_- has probability density $q(z) = \frac{1}{a}$ for $z \in [0, a)$ and $q(z) = 0$ elsewhere for some a, $0 < a < \frac{\pi}{2}$. Then $\mathcal{G}_\mathcal{R}(z) = \sin z$ and $\mathcal{G}_{\mathcal{R}*Q_-}(z) = \frac{\cos z - \cos(z+a)}{a}$, $z \in [0, \pi - a)$, the smoothing effect can be observed on graphs of these functions for $a > 0$ small.

We conclude that curved test systems may be useful to provide local smoothing when estimating the Steiner compact. It should be kept in mind that using the rose of intersections $P_L^Q(z)$ (i.e. using local smoothing) we get estimators of $\mathcal{R} * Q_-$ which is not exactly $\mathcal{R}$.

In Rataj and Saxl [11] the properties of Steiner compact estimator (31) were justified by the following result: For any $\varepsilon > 0$, $\alpha \in (0,1)$ there exist $n, \{z_i\}, r, \{c_j\}$ such that probability

$$Pr\{d(\overline{K}_n, K) \leq \varepsilon L_A\} \geq \alpha,$$

assuming that $p_i - \frac{1}{2}P_L(z_i)$, $i = 1, \dots, n$ are independent centred normally distributed random variables with uniformly bounded variances. We access an estimator of Steiner compact based on curved test lines and try to get quantitative results in a special case.

Consider rotations of the test arc s with orientation distribution Q around uniform angles $z_j = \frac{\pi j}{n}$, $j = 0, \dots, n-1$. Principially one could generalize Lemma 5 to this system of test arcs and use the exact approach from Section 3 to get the distribution of $PD = r(\mathcal{R}_n, \mathcal{R})$. We restrict ourselves to an example of approximation of the Hausdorff distance $d(K_n, K)$, even if it seems to be from the statistical point of view less convincing.

Let $q_j = \frac{1}{2}P_L^Q(z_j)$ be the theoretical values at z_j of the support function of Steiner compact set K corresponding to $\mathcal{R} * Q_-$ and p_j their empirical counterparts estimated from numbers of intersections on test lines. We construct convex polygons $\overline{K_n} = \{x \in R^2;\ \langle x, z_j \rangle \le p_j,\ j = 1, \dots, n\}$. According to Rataj and Saxl [11] it holds

$$d(\overline{K_n}, K_n) \le \frac{Y}{\cos\frac{\Delta}{2}}, \quad \text{and} \quad d(K_n, K) \le \frac{1}{2} L_A \tan\frac{\Delta}{2}, \tag{32}$$

where $Y = \max_j |q_j - p_j|$, $K_n = \{x \in R^2;\ \langle x, z_j \rangle \le q_j,\ j = 1, \dots, n\}$ and $\Delta = \frac{\pi}{n}$ is the discretization step. To evaluate the first bound in (32) we need to know the dependence structure of p_i, $i = 1, \dots, n$. It depends on many factors such that the mutual location and shape of test lines. In practice it is usually not possible to get p_i's from n independent realizations, very frequently we have just one observation window of the structure. Then the p_i's become observations of positively dependent random variables, cf. Lemma 3. The general sharp bounds for Y are

$$\Big(\sum_j F_j(t^-) - (n-1)\Big)_+ \le Pr(Y \le t) \le \min_j F_j(t^-), \tag{33}$$

where F_j are distribution functions of $|q_j - p_j|$, $j = 1, \dots, n$. See Rychlik [12] for further results on such bounds. While the lower bound in (33) is not much useful here, in the situation with strong positive dependence $Pr(Y \le t)$ is close to the upper bound.

Example 2. Let s be a circular arc with radius r and length $h = r\alpha$, where α is the central angle of the arc. The minimum on the right hand side of (33) is realized by the p_j with largest variance. Consider the extremal case $\mathcal{R} = \delta_0$, $z_j = \frac{\pi}{2}$, when Φ are parallel lines. Then $N(\mathcal{L}) = N(\mathcal{L}_1)$, $N(\mathcal{L}_1)$ is Poisson distributed with parameter $2L_A p_{K_s}(0)$ and $\operatorname{var} N(\mathcal{L}) = 2L_A p_{K_s}(0) \approx hL_A$, assuming that r is large. Then we get that

$$\operatorname{var}(p_j - q_j) = \frac{1}{4h^2}\operatorname{var} N(\mathcal{L}) \approx \frac{L_A}{4h}$$

depends on length intensity of Φ and the length of test probes. We can fix say $h = 2500L_A^{-1}$ and estimate $Pr(Y \le 0.02L_A) \approx 0.95$ using the Gaussian approximation. Put $n = 18$, $z_j = \frac{\pi j}{18}$, we obtain $Pr(d(\overline{K}_{18}, K_{18}) \le 0.021L_A) \approx 0.95$. Together with $d(K_{18}, K) \le 0.04L_A$ according to the second inequality in (32) we obtain that the distance $d(\overline{K}_{18}, K)$ will not exceed 6.1 % of L_A with probability 0.95 in the case of strongly positively dependent data.

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