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APPLICATION OF A SECOND ORDER VSC TO NONLINEAR SYSTEMS IN MULTI-INPUT PARAMETRIC-PURE-FEEDBACK FORM

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The use of a multi-input control design procedure for uncertain nonlinear systems expressible in multi-input parametric-pure feedback form to determine the control law for a class of mechanical systems is described in this paper. The proposed procedure, based on the well-known backstepping design technique, relies on the possibility of extending to multi-input uncertain systems a second order sliding mode control approach recently developed, thus reducing the computational load, as well as increasing robustness.

1. INTRODUCTION

Recently, the attention of some researchers has been focused on the possibility of generating *higher order sliding modes* (Levant [9]) and appreciable results have been attained in case of sliding regimes [15] of the second order (i.e., $S = \dot{S} = 0$ in finite time, with only S measurable and a control discontinuous on S directly affecting $\ddot{S}$) (see Bartolini et al [3] and Levant [10]). To be more specific, a second order sliding mode control (SOSMC) problem is that of steering to zero asymptotically the state of the uncertain system described by

$$\begin{cases} \dot{x}_i &= x_{i+1}, \quad i = 1, \dots, n-1 \\ \dot{x}_n &= \phi_0(x_1, \dots, x_n) + \beta_0(x_1, \dots, x_n)\, u \end{cases} \tag{1}$$

with $\phi_0(\cdot)$, $\beta_0(\cdot)$ uncertain scalar functions with known upper and lower bounds ($\beta_0(\cdot)$ with known sign), and unmeasurable x_n. Yet, if the system to control instead of being expressible in the form (1) has uncertainties of more general type, for instance appearing at each state equation, the solution procedure suggested in Bartolini et al [2] is no more directly applicable. In case the system, though nonlinear and with some degree of uncertainty, is expressible in the so-called parametric-strict or parametric-pure feedback forms [6, 11, 12], then a combined backstepping/SOSMC design procedure can be conceived to solve the problem, as indicated in Bartolini et al [1].

The aim of this paper is to extend the results of Bartolini et al [1] to the case of multi-input nonlinear systems with parameter uncertainties, making reference, in particular, to some common mechanical systems typically expressible in the multi-input parametric-pure feedback form. As outlined in Kokotović et al [8], the adaptive backstepping design can be easily extended to this class of systems provided that the matrix which pre-multiplies the control vector (i. e., the control matrix) is nonsingular and known. In particular cases of a certain applicative significance, the solution to the multi-input SOSMC problem appears to be particularly simple [4] and suitable to be exploited within a backstepping framework. It is the case of positive (or negative) definite and dominant diagonal control matrix.

The overall control design procedure for multi-input uncertain nonlinear systems we propose consists in a modified state transformation which retains $\rho_i - 1$ transformed state equations (ρ_i being the number of equations of each block of the original system form), for each block, equal to those obtained via the backstepping procedure, coupling them with two auxiliary equations, obtained by selecting, for each block, a suitably sliding manifold, and considering its first and second derivative. By grouping the auxiliary equations associated with each block, an uncertain second order multi-input nonlinear auxiliary system is obtained belonging to the class of systems to which the extension to the multi-input case of the SOSMC strategy indicated in Bartolini et al [4] is applicable. In particular, if the control matrix, apart from being positive definite, is also dominant diagonal, then m single-input SOSM control signals (m being the number of blocks) need to be used to attain the finite time reaching of the origin of the auxiliary system state space, and the same convergence results as in the multi-input purely backstepping design are obtained, even if the type of uncertainty dealt with is more general than that tractable via the backstepping procedure.

To show the effectiveness of the proposed control design procedure, the application to a two-link robotic arm with flexible joints is dealt with in this paper [5]. Such a system turns out to be expressible in multi-input parametric-pure feedback form. If some uncertainties, apart from the parameters vector components, are allowed, the basic backstepping procedure does not apply, while the procedure proposed in this paper can be used, provided some bounds on the relevant uncertain quantities are determinable. Note that the resulting control vector signal is not affected by the chattering effect since it results in being continuous, though with discontinuous derivatives of its components.

2. PROBLEM STATEMENT

A multi-input dynamical system can be described by the system of differential equations

$$\dot{x}(t) = f(x(t), t) + \phi_x(x(t), t)^T\theta + (b(x(t), t) + q(x(t), t)^T\overline{\theta})\, u(t) \qquad (2)$$

where $x(t) \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $f(x(t), t) \in \mathbb{R}^n$, $b(x(t), t) \in \mathbb{R}^{n\times m}$, and $\phi_x(x(t), t)$, $q(x(t), t)$ belonging to $\mathbb{R}^{p\times n}$ are known smooth matrix functions, while the constant vector $\theta \in \mathbb{R}^p$ represents some parametric uncertainties; $\overline{\theta} = [\theta \dots \theta] \in \mathbb{R}^{p\times m}$.

The general form can be transformed into an equivalent one, more suitable for the determination of the solution to the control problem, using an almost algorithmic procedure by Su et al [14]. The transformed state vector can be suitably partitioned in m subsystems (blocks), each of order ρ_i, $\sum_{i=1}^{m} \rho_i = n$. Thus, the multi-input pure-feedback form can be expressed as

$$\left\{ \begin{array}{rcl} \dot{x}_{\gamma_i+j} & = & x_{\gamma_i+j+1} + \phi^T_{\gamma_i+j}(x_1, \ldots, x_{\rho_1-\rho_i+j+1}, \ldots, \\ & & \quad x_{\gamma_{m+1}}, \ldots, x_{\gamma_{m+1}-\rho_i+j+1})\theta \\ \dot{x}_{\gamma_{i+1}} & = & \sum_{j=1}^{m} (\beta_{i,j}(\boldsymbol{x}) + q^T_{i,j}(\boldsymbol{x})\,\theta)\, u_j + \phi^T_{\gamma_{i+1}}(\boldsymbol{x})\,\theta \end{array} \right. \tag{3}$$

with $i = 1, \ldots, m$, $\gamma_k = \sum_{l=1}^{k-1} \rho_l$, where $\boldsymbol{x}(t) = [x_1(t), \ldots, x_{\gamma_m}(t)]^T \in I\!R^n$, $\theta = [\theta_1, \ldots, \theta_p]^T \in I\!R^p$ vector of constant unknown parameters, and $\phi_{\gamma_{i+1}}(\boldsymbol{x}(t)) \in I\!R^p$. Let us define

$$\beta(\boldsymbol{x}(t), v) = \left[\begin{array}{lcl} \beta_{1,1} + q^T_{1,1} v & \ldots & \beta_{1,m} + q^T_{1,m} v \\ \vdots & \ldots & \vdots \\ \beta_{m,1} + q^T_{m,1} v & \ldots & \beta_{m,m} + q^T_{m,m} v \end{array} \right] \in I\!R^{m \times m}$$

where $\beta_{i,j}$ are known smooth nonlinear functions (note that in the definition of $\beta(\boldsymbol{x}(t), v)$ it has been used the variable $v \in I\!R^p$ to be substituted in the sequel either by θ or by the adapted vector $\hat{\theta}$). Moreover $\beta(\boldsymbol{x}(t), \theta)$ is non singular. Note also that the assumption of perfect knowledge of $\beta_{i,j}$ will be dispensed with later.

The control objective is to make the output signals $\nu_i(t) = x_{\gamma_i+1}(t)$, $i = 1, \ldots, m$, track the smooth reference trajectories $y_{i,r}(t)$ (tracking objective).

3. SOME PRELIMINARIES ON THE MULTI-INPUT BACKSTEPPING DESIGN PROCEDURE

The backstepping design procedure in the case of multi-input systems and with reference to a tracking objective consists in the step-by-step construction of a transformed system with state

$$z_{\gamma_i+j+1} = x_{\gamma_i+j+1} - y^{(j)}_{i,r} - \alpha_{\gamma_i+j} \tag{4}$$

$i = 1, \ldots, m$, $j = 0, \ldots, \rho_i - 1$, where α_{γ_i+j} is the so-called *virtual control* signal at the design step γ_i+j, and $y^{(j)}_{i,r}$ is the derivative of order j of the signal $y_{i,r}$, which is the reference for the output $\nu_i = x_{\gamma_i+1}$ (note that, for the sake of brevity, from now on the dependence on t may be sometimes omitted). With this state transformation the original tracking problem is transformed into a stabilization problem, i. e., α_{γ_i+j} is computed at step $\gamma_i + j$ to drive $\boldsymbol{z} = [z_1, \ldots, z_n]^T$ to the equilibrium state $[0, \ldots, 0]^T$. This latter is proved to be stable through a standard Lyapunov analysis.

For the reader's convenience, let us recall the relevant relationships of the backstepping procedure for systems in multi-input pure-feedback form at *step* $\gamma_i + j$ as in Kokotović et al [8], i. e.,

$$z_{\gamma_i+j+1} \quad = \quad x_{\gamma_i+j+1} - y^{(j)}_{i,r} - \alpha_{\gamma_i+j} \tag{5}$$

$$\begin{aligned}
V_{\gamma_i+j} &= \sum_{k=1}^{\gamma_i+j} \frac{1}{2} z_k^2 + \frac{1}{2}(\hat{\theta}-\theta)^T \Gamma^{-1}(\hat{\theta}-\theta) && (6)\\
\alpha_{\gamma_i+j} &= -z_{\gamma_i+j-1} - (c_{\gamma_i+j} + s_{\gamma_i+j}) z_{\gamma_i+j} + \omega_{\gamma_i+j}^T \\
&\quad \left(\sum_{k=1}^{\gamma_i+j-1} z_{k+1} \frac{\partial \alpha_k}{\partial \hat{\theta}} \Gamma - \hat{\theta} \right) + \sum_{k=1}^{m} \frac{\partial \alpha_{\gamma_i+j-1}}{\partial x_k} x_{k+1} \\
&\quad + \sum_{k=1}^{\gamma_i+j+1} \frac{\partial \alpha_{\gamma_i+j-1}}{\partial y_{i,r}^{k-1}} y_{i,r}^k + \frac{\partial \alpha_{\gamma_i+j-1}}{\partial \hat{\theta}} \tau_{\gamma_i+j} && (7)\\
s_{\gamma_i+j} &= \kappa_{\gamma_i+j} |\omega_{\gamma_i+j}|_2^2 && (8)\\
\omega_{\gamma_i+j} &= \phi_{\gamma_i+j} - \sum_{k=1}^{m} \frac{\partial \alpha_{\gamma_i+j-1}}{\partial x_k} \phi_k && (9)\\
\tau_{\gamma_i+j} &= \Gamma z_{\gamma_i+j} \omega_{\gamma_i+j} + \tau_{\gamma_i+j-1} && (10)\\
\dot{z}_{\gamma_i+j} &= -z_{\gamma_i+j-1} - c_{\gamma_i+j} z_{\gamma_i+j} + z_{\gamma_i+j+1} + \omega_{\gamma_i+j}^T \tilde{\theta} \\
&\quad + \sum_{k=1}^{j-1} z_{k+1} \frac{\partial \alpha_{\gamma_i+j}}{\partial \hat{\theta}} \Gamma \omega_{\gamma_i+j} - \frac{\partial \alpha_{\gamma_k}}{\partial \hat{\theta}} (\tau_{\gamma_i+j} - \dot{\hat{\theta}})
\end{aligned}$$

with $i = 1, \ldots, m$, $j = 0, \ldots, \rho_i - 1$, $\alpha_{\gamma_i} = 0$, $\tau_0 = [0, \ldots, 0]^T \in \mathbb{R}^p$. The terms c_{γ_i+j} are design constants, while τ_{γ_i+j} is the so-called tuning function at step $\gamma_i + j$.

Note that, each time one reaches the equation relevant to $\dot{x}_{\gamma_i+j}$ (since $\alpha_{\gamma_i} = 0$) the iterative procedure considers $z_{\gamma_i+1} = x_{\gamma_i+1} - y_{i,r}$.

4. THE PROPOSED MODIFIED STATE TRANSFORMATION

The standard multi-input backstepping procedure requires that the computations relevant to step $\gamma_i + j$ are repeated n times, so that, at *step* n, one obtains the actual control

$$u(t) = \overline{\beta}(x(t), \hat{\theta})^{-1} \overline{\alpha}(x(t)) + y_r^{(*)}(t) \tag{11}$$

$$\overline{\beta}(x(t), v) = \begin{bmatrix} \left(1 - \frac{\partial \alpha_{\rho_1-1}}{\partial x_{\rho_1}}\right) \beta_1 - \sum_{k=2}^{m} \frac{\partial \alpha_{\rho_1-1}}{\partial x_{\gamma_{k+1}}} \beta_k \\ \vdots \\ \left(1 - \frac{\partial \alpha_{n-1}}{\partial x_n}\right) \beta_m - \sum_{k=1}^{m-1} \frac{\partial \alpha_{n-1}}{\partial x_{\gamma_{k+1}}} \beta_k \end{bmatrix} \tag{12}$$

with β_i ith row of $\beta(x(t))$, $y_r^{(*)} = [y_{1,r}^{(\rho_1)}(t), \ldots, y_{m,r}^{(\rho_m)}(t)]^T$, and

$$\overline{\alpha}(x(t)) = \begin{bmatrix} \alpha_{\rho_1} - \left(\sum_{k=1}^{m} \frac{\partial \alpha_{\rho_1-1}}{\partial x_{\gamma_{k+1}}} \beta_k \right) u \\ \vdots \\ \alpha_n - \left(\sum_{k=1}^{m} \frac{\partial \alpha_{n-1}}{\partial x_{\gamma_{k+1}}} \beta_k \right) u \end{bmatrix} \tag{13}$$

In the modified design procedure we propose, the procedure (5) – (10) is instead stopped m times, each time at step $\gamma_{i+1}-1$, computing $\alpha_{\gamma_{i+1}-1}$, and the transformed state is completed to obtain $\gamma_{m+1}+m$ state variables with the $2m$ auxiliary variables

$$\begin{aligned} y_{i,1} &= x_{\gamma_{i+1}} - y_{i,r}^{(\rho_i)} - \alpha_{\gamma_{i+1}-1} + \tilde{c}_{\gamma_{i+1}-1} z_{\gamma_{i+1}-1} && (14)\\ y_{i,2} &= \dot{y}_{i,1}, \qquad i = 1, \ldots, m && (15) \end{aligned}$$

where $\tilde{c}_{\gamma_{i+1}-1}$ are constants to be suitably choosen since they affect the dynamics of $z_{\gamma_{i+1}-1}$, and, together with other constants, the stability performances of the controlled system, as it will become apparent in the sequel. With this transformation

$$\begin{aligned} \dot{z}_{\gamma_{i+1}-1} &= -z_{\gamma_{i+1}-2} - (c_{\gamma_{i+1}-1} + s_{\gamma_{i+1}-1} + \tilde{c}_{\gamma_i-1}) z_{\gamma_{i+1}-1} + y_{i,1} \\ &\quad + (\theta - \hat{\theta})^T \omega_{\gamma_{i+1}-1} - \frac{\partial \alpha_{\gamma_{i+1}-2}}{\partial \hat{\theta}} (\tau_{\gamma_{i+1}-1} - \dot{\hat{\theta}}) \\ &\quad + \sum_{k=1}^{\rho_i - 4} z_{\gamma_i+k} \frac{\partial \alpha_{\gamma_i+k}}{\partial \hat{\theta}} \Gamma \omega_{\gamma_{i+1}-1}. \qquad (16) \end{aligned}$$

This allows us to write the modified transformed system state space representation as

$$\left\{ \begin{aligned} \dot{z} &= A(z,\hat{\theta})z + W(z,\hat{\theta})^T \tilde{\theta} + D(z,\hat{\theta})^T \dot{\hat{\theta}} + \tilde{b} y \\ \dot{y} &= F(y,z,\theta,\hat{\theta},u) + B(y,z,\theta,\hat{\theta})\dot{u} \end{aligned} \right. \qquad (17)$$

where $z = [z_1, \ldots, z_{\gamma_{i+1}-1}, z_{\gamma_{i+1}+1}, \ldots, z_{\gamma_{m+1}-1}]^T$, $y = [y_{1,1}, \ldots, y_{m,1}, y_{1,2}, \ldots, y_{m,2}]^T \in \mathbb{R}^{2m}$, $A(z,\hat{\theta}) \in \mathbb{R}^{(\gamma_{m+1}-m)\times(\gamma_{m+1}-m)}$, $W(z,\hat{\theta})$, $D(z,\hat{\theta})$, $F(y,z,\theta,\hat{\theta},u)$ suitable functions vectors, $B(y,z,\theta,\hat{\theta}) = [O_{m\times m} \;\; \beta^T(x(t).\,\hat{\theta})]^T$, and

$$\tilde{b}^T = \begin{bmatrix} O_{(\rho_1-1)\times 2m} & 1 & \ldots & O_{(\rho_m-1)\times 2m} & O_{1\times(2m-1)} \\ \vdots & & & & 1 \\ & & \ldots & & 0 \end{bmatrix}$$

with $O_{l\times h}$ null matrix of dimension $l \times h$. By selecting the adaptation mechanism as $\dot{\hat{\theta}} = \tau_{\gamma_{m+1}-1} = \Gamma W(z,\hat{\theta})z$, with $\tau_{\gamma_i} = [0,\ldots,0] \in \mathbb{R}^p$, equation (17) reduces to the closed loop form $\dot{z} = A_z(z,\hat{\theta})z + W(z,\hat{\theta})^T\tilde{\theta} + \tilde{b}y$, A_z such that $A_z^T + A_z$ is a diagonal matrix.

5. THE ROLE OF MULTI-INPUT SOSMC

Now, consider system (17). Let $S := y_1 = [y_{1,1}, \ldots, y_{m,1}]^T$, $\dot{y}_1 = y_2 = [y_{1,2}, \ldots, y_{m,2}]^T \in \mathbb{R}^m$ and $\chi = [y^T, z^T, \theta^T, \hat{\theta}^T]^T$. Then, the second order equation in (17) can be written in a more compact form as

$$\left\{ \begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= F(\chi,u) + B(\chi)\,\dot{u}. \end{aligned} \right. \qquad (18)$$

The control problem can be restated as that of steering y_1, y_2 to zero in finite time in spite of the uncertainties in the vector field $F(\chi, u)$ and in the matrix $B(\chi)$, and of the non availability of the vector y_2. Note that $S = 0$ can be regarded as an m-dimensional sliding manifold. Thus, the problem is a second order sliding mode control problem, according to the definition mentioned in the Introduction.

Assume that, the vector field $F^T(\chi, u) = [F_1(\chi, u), \ldots, F_m(\chi, u)]$ is uncertain but such that its components result in being bounded by known functions in such a way that the second order sliding mode control problem relevant to the auxiliary single input system

$$\begin{cases} \dot{y}_{i,1} &= y_{i,2} \\ \dot{y}_{i,2} &= F_i(\chi, u) + \eta_i \end{cases} \tag{19}$$

(i. e., the problem of steering $y_{i,1}$ and $y_{i,2}$ to zero in finite time by measuring only $y_{i,1}$) has a solution. This problem has been dealt with in the cited papers, [2, 3] taking into account different types of uncertainty bounds, and, accordingly, different operating procedure to implement the control strategy. In this paper, to keep the treatment easier, it is assumed that

$$|F_i(\chi, u)| < \bar{F}_i \tag{20}$$

where $\bar{F}_i$ is a known constant.

Matrix $B(\chi)$ (and, consequentely, matrix $\beta(x(t))$) is assumed, from now on, to be uncertain, but with known bounds on its entries b_{ij}, and, for the sake of simplicity, positive definite. Actually, more general cases could be dealt with: at least all those indicated in Bartolini et al [4] to which the extension of SOSMC to the multi-input case is feasible. Moreover, since it is sufficient for the applications we are interested in, we suppose that $B(\chi)$ is not only positive definite but also dominant diagonal, i. e.

$$0 < \sum_{j=1, j\neq i}^{m} |b_{ij}| < b_{ii} \quad i = 1, \ldots, m. \tag{21}$$

Then, equation (18) can be rewritten, component-wise, as

$$\dot{y}_{i,2} = F_i(\chi, u) + \sum_{j=1, j\neq i}^{m} b_{ij}\dot{u}_j + b_{ii}\dot{u}_i. \tag{22}$$

In Bartolini et al [2] it is proved that, in the case of a SISO second order uncertain systems with incomplete state measure, the control $u(t)$ can be chosen as a bang-bang control, [7] switching between two values $-U_{\mathrm{Max}}$, $+U_{\mathrm{Max}}$, relying on a commutation logic based on the available state only. So, if instead of equation (22) we had

$$\dot{y}_{i,2} = F_i(\chi, u) + b_{ii}(\chi)\dot{u}_i \tag{23}$$

with $F_i(\chi, u)$ as in (20), the second order sliding mode would be attained, for instance, by means of the following algorithm, based on the assumption of having the capability of detecting the extremal values of $y_{i,1}$ (e. g., by means of peak detectors).

Algorithm 1.

i) Set $\delta_i^* \in (0,1] \cap \left(0, \frac{3B_{1_i}}{B_{2_i}}\right)$, where $B_{1_i} > 0$, $B_{2_i} \geq B_{1_i}$ are known lower and upper bounds of the quantity b_{ii}.

ii) Set $y_{i,1_{\mathrm{Max}}} = y_{i,1}(0)$.

Repeat, for any $t > 0$, the following steps.

iii) If $[y_{i,1}(t) - \frac{1}{2} y_{i,1_{\mathrm{Max}}}][y_{i,1_{\mathrm{Max}}} - y_{i,1}(t)] > 0$ then set $\delta_i = \delta_i^*$ else set $\delta_i = 1$.

iv) If $y_{i,1}(t)$ is extremal value then set $y_{i,1_{\mathrm{Max}}} = y_{i,1}(t)$.

v) Apply the control law

$$\dot{u}_i(t) = -\delta_i U_{i_{\mathrm{Max}}} \operatorname{sign}\left\{ y_{i,1}(t) - \frac{1}{2} y_{i,1_{\mathrm{Max}}} \right\}. \tag{24}$$

Until the end of the control time interval.

Note that in (24), according to Bartolini et al, [2]

$$U_{i_{\mathrm{Max}}} > \max\left(\frac{\overline{F}_{1_i}}{\delta^* B_{1_i}}; \frac{4\overline{F}_{1_i}}{3B_{1_i} - \delta^* B_{2_i}} \right). \tag{25}$$

Then, consider equation (22) and remember the assumption of diagonal dominance (21). By analogy with (24), one can assume that any control signal $\dot{u}_i$ in (22) has the form

$$\dot{u}_i = -\delta_i U_{\mathrm{Max}} \operatorname{sign}\left\{ y_{i,1}(t) - \frac{1}{2} y_{i,1_{\mathrm{Max}}} \right\}. \tag{26}$$

As a result, one obtains

$$\begin{aligned} \dot{y}_{i,2} &= F_i(\chi, u) - \sum_{j=1, j\neq i}^{m} b_{ij} \delta_j U_{\mathrm{Max}} \operatorname{sign}\left\{ y_{j,1}(t) - \frac{1}{2} y_{j,1_{\mathrm{Max}}} \right\} \\ &\quad - b_{ii} \delta_i U_{\mathrm{Max}} \operatorname{sign}\left\{ y_{i,1}(t) - \frac{1}{2} y_{i,1_{\mathrm{Max}}} \right\} \end{aligned} \tag{27}$$

or, analogously,

$$\dot{y}_{i,2} = F_i(\chi, u) - g_i(\chi)\, \delta_i U_{\mathrm{Max}} \operatorname{sign}\left\{ y_{i,1}(t) - \frac{1}{2} y_{i,1_{\mathrm{Max}}} \right\} \tag{28}$$

where $g_{i_1}(\chi) < g_i(x) < g_{i_2}(\chi)$ with

$$g_{i_1}(\chi) = b_{ii_m} - \sum_{j=1, j\neq i}^{m} |b_{ij_M}| \tag{29}$$

$$g_{i_2}(\chi) = b_{ii_M} + \sum_{j=1, j\neq i}^{m} |b_{ij_M}| \tag{30}$$

where $b_{ij_m} \leq b_{ij} \leq b_{ij_M}$, b_{ij_m}, b_{ij_M} known. Note that a value of U_{Max} valid for any $\dot{u}_i$ ($U_{\text{Max}} = \max_{1\leq i\leq m} U_{i_{\text{Max}}}$) can be derived taking into account the following expressions

$$F^*_{\text{Max}} = \max_{1\leq i\leq m} \bar{F}_i \tag{31}$$

$$\rho = \max_{1\leq i\leq m} \left\{ \max \left[\frac{1}{\delta^* g_{i_1}} ; \frac{4}{3g_{i_1} - \delta^* g_{i_2}} \right] \right\} \tag{32}$$

$$\delta^* \in (0 ; 1] \cap \left(0 ; \min_{1\leq i\leq m} \frac{3g_{i_1}}{g_{i_2}} \right) \tag{33}$$

that is $U_{\text{Max}} \geq \rho F^*_{\text{Max}}$.

Then, a control vector with components as in (26), and U_{Max} satisfying inequalities (31) – (33), is sufficient to steer the vectors y_1 and $\dot{y}_1 = y_2$ to zero in finite time. Summing up, it has been observed that if matrix $B(\chi)$ is positive definite and dominant diagonal, then the multi-input auxiliary system can be splitted into m single-input systems to which the single-input SOSMC approach described in Bartolini et al [3] can be applied.

Note: The multi-input strategy just recalled, contained in Bartolini et al [4], is applicable to systems in the double-integrator form (18). Systems in multi-input parametric-feedback form are not suitable to be controlled through that strategy, because of the presence of the unmatched uncertainties $\phi_i^T\theta$, $i = 1, \ldots, m$. The backstepping procedure is used to generate, from a multi-input parametric-feedback form, an auxiliary system that has a double-integrator form, to which Algorithm 1 is applicable.

Then, on the whole, the design procedure we propose to solve the control problem in question can be expressed in algorithmic form as follows.

Algorithm 2.

i) Stop the backstepping procedure for m times at *step* $\gamma_{i+1} - 1$ and compute the quantities $\alpha_{\gamma_{i+1}-1}$, $z_{\gamma_{i+1}-1}$, $\tau_{\gamma_{i+1}-1}$. Set $\dot{\hat{\theta}} = \tau_{\gamma_{m+1}-1} = \Gamma W(z, \hat{\theta})\, z$.

ii) Define the vectors

$$\begin{aligned}
\tilde{z} &= [z_{\rho_1-1}, \ldots, z_{\gamma_{i+1}-1}, \ldots, z_{\gamma_{m+1}-1}]^T \\
\tilde{c} &= [\tilde{c}_{\rho_1-1}, \ldots, \tilde{c}_{\gamma_{i+1}-1}, \ldots, \tilde{c}_{\gamma_{m+1}-1}] \\
\tilde{\alpha} &= [\alpha_{\rho_1-1}, \ldots, \alpha_{\gamma_{i+1}-1}, \ldots, \alpha_{\gamma_{m+1}-1}]^T \\
\tilde{x} &= [x_{\rho_1}, \ldots, x_{\gamma_{i+1}}, \ldots, x_{\gamma_{m+1}}]^T \\
\tilde{y}_r &= [y_{1,r}^{(\rho_1)}, \ldots, y_{i,r}^{(\rho_i)}, \ldots, y_{m,r}^{(\rho_m)}]^T
\end{aligned}$$

and compute $S = y_1 = \tilde{c}\tilde{z} + \tilde{x} - \tilde{y}_r - \tilde{\alpha}$.

iii) Compute the upper bounds of the relevant functions in (17) to obtain the bounds $\overline{F}_i$, g_{i_1} and g_{i_2}, $i = 1, \ldots, m$.

iv) Apply Algorithm 1 to determine each component $\dot{u}_i$ of the control vector, with $U_{i_{\mathrm{Max}}} = U_{\mathrm{Max}}$ as in (31) – (33).

If Algorithm 2 is applicable, the reaching of the origin of the auxiliary system state space is guaranteed. Now, the behaviour of the remainder of the transformed system (namely (17)) needs to be analyzed. To this end, choose, as a Lyapunov function, $V = \frac{1}{2}(z^T z + (\theta - \hat{\theta})^T \Gamma^{-1}(\theta - \hat{\theta}))$. The derivative of V with the virtual functions as synthetized in procedure (5) – (11), and $\dot{\hat{\theta}}$ replaced by $\Gamma W(z, \hat{\theta})\, z$ results

$$
\begin{aligned}
\dot{V} &= \frac{1}{2} z^T \dot{z} + \frac{1}{2} \dot{z}^T z - \frac{1}{2} \tilde{\theta}^T \Gamma^{-1} \dot{\hat{\theta}} - \frac{1}{2} \dot{\hat{\theta}}^T \Gamma^{-1} \tilde{\theta} \\
&= z^T \left(\frac{A_z + A_z^T}{2} \right) z + z^T \tilde{b} y. \qquad (34)
\end{aligned}
$$

Due to the skew-symmetry of the matrix A_z, $A_z + A_z^T$ is a negative definite diagonal matrix, whose elements are functions of c_i and $\tilde{c}_i$. Defining $c_0 = \min_i c_i$ and $\tilde{c}_0 = \min_i \tilde{c}_i$, and recalling that $\tilde{b} y = y_1$, it yields

$$
\dot{V} \leq -c_0 |z|_2^2 - \tilde{c}_0 |\tilde{z}|_2^2 + \tilde{z}^T y_1 \qquad (35)
$$

where $|\cdot|_2$ is the Euclidean norm. Then, there exists a ball centered at the origin of the z-state space, of radius $\frac{|y_1|_2}{\tilde{c}_0}$, out of which $\dot{V}$ is surely negative [8] (note that, for sufficiently high c_i, $i = 1, \ldots, n$, the first derivative of V could be negative in all the state space). In the case of $\beta(\boldsymbol{x}(t))$ positive definite and dominant diagonal, vector y_1 is guaranteed to converge to zero in finite time. So, in such a case, the ball will collapse to the origin in finite time as well.

6. COMPUTATIONAL LOAD

The overall number of steps required by Algorithm 2 are n, as in the standard backstepping procedure, but the total number of on-line computations required is reduced. For convenience, S has been written as $\tilde{c}\tilde{z} + \tilde{x} - \tilde{y}_r - \tilde{\alpha}$, but, the reader could easily see that $\tilde{x} - \tilde{y}_r - \tilde{\alpha}$ is the expression of $[z_{\rho_1}, \ldots, z_{\gamma_{i+1}}, \ldots, z_{\gamma_{m+1}}]^T$, i.e. the z transformation of the backstepping is completely done also in the combined procedure. What differs is the computation of the control law. In the backstepping procedure, the on-line computational load required to obtain u is equivalent to that required to obtain the vector $[\alpha_{\rho_1 - 1}, \ldots, \alpha_{\gamma_{i+1}-1}, \ldots, \alpha_{\gamma_{m+1}-1}]^T$. In the combined procedure the control law is realized through a peak detector and a signum function.

The calculation previously required off-line (i.e., in the design of the control law) and on-line (i.e., to generate the control law) are now required off-line only.

7. APPLICATION OF THE CONTROL DESIGN PROCEDURE TO A TWO-LINK ROBOTIC ARM WITH JOINT FLEXIBILITY

As an example of application, in this section, we consider a two link non planar robotic arm with flexibility [13] between each joint and the corresponding actuating device as in Diong et al [5] (Figure 1), i.e., $J(x)\, \dot{x} = g(x)$,

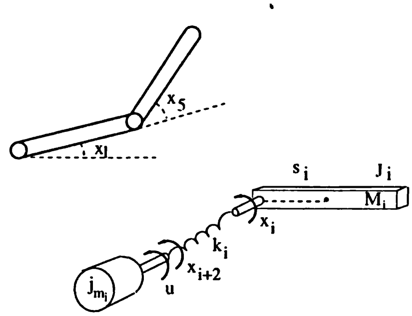

Fig. 1. Double link robotic arm with joint flexibility.

where

$$g(x) = \begin{cases} x_2 \\ -k_1 x_1 + k_1/n_1 x_3 + m_2 s_2 l_1 (x_6^2 + 2x_2 x_6) \\ \quad \sin(x_5) + m_2 s_2 g \cos(x_1 + x_5) \\ \qquad +(m_1 s_1 + m_2 l_1) g \cos(x_1) \\ x_4 \\ k_1/(j_{m_1} n_1)\, x_1 - k_1/(j_{m_1} n_1^2)\, x_3 - b_{m_1}/j_{m_1} x_4 \\ \quad +1/j_{m_1} u_1 \\ x_6 \\ -k_2 x_5 + k_2/n_2 x_7 + m_2 s_2 l_1 x_2^3 \sin(x_5) \\ \quad +m_2 s_2 g \cos(x_1 + x_5) \\ x_8 \\ k_2/(j_{m_2} n_2)\, x_5 - k_2/(j_{m_2} n_2^2)\, x_7 - b_{m_2}/j_{m_2} x_8 \\ \quad +1/j_{m_2} u_2 \end{cases} \tag{36}$$

$J_{j,j}(x) = 1, j = 1, 3, 4, 5, 7, 8$, $J_{2,2}(x) = j_1 + j_2 + m_2(s_2^2 + l_1^2) + m_1 s_1^2 + 2m_2 s_2 l_1 \cos(x_5)$, $J_{2,6}(x) = J_{6,2}(x) = j_2 + m_2 s_2^2 + m_2 s_2 l_1 \cos(x_5)$, $J_{6,6}(x) = j_2 + m_2 s_2^2$, all other terms are zero.

$$\begin{array}{lll} m_1 = 5, & m_2 = 6 & \text{link masses} \\ l_1 = 0.5, & l_2 = 0.5 & \text{link lengths} \\ s_1 = 0.25, & s_2 = 0.29167 & \text{centre of masses} \\ j_1 = 0.125, & j_2 = 0.15 & \text{link inertias} \\ j_{m_1} = 0.025, & j_{m_2} = 0.025 & \text{rotor inertias} \\ k_1 = 1000, & k_2 = 1000 & \text{joint stiffness} \\ n_1 = 10, & n_2 = 10 & \text{gear ratios} \\ b_{m_1} = 0.1, & b_{m_2} = 0.1 & \text{rotor damping} \end{array} \tag{37}$$

j_{m_1}, j_{m_2}, k_1, k_2, b_{m_1}, b_{m_2} are supposed unknown constant to the controller, but known in bounds. In this particular case the control matrix is diagonal. The transformed state vector according to the proposed design procedure is $\{z_1, z_2, z_3, z_5, z_6, z_7,$

$y_{1,1}, y_{1,2}, y_{2,1}, y_{2,2}\}$. The reference quantities, $y_{1,r}(t), \ldots, y_{2,r}^{(4)}(t)$, come from a linear reference model of suitable order. As for the variable structure part of the controller, the sliding manifold is

$$S = y_1 = \begin{bmatrix} x_4 - y_{1,r}^{(3)} + 10z_3 \\ x_8 - y_{2,r}^{(3)} + 10z_7 \end{bmatrix}. \tag{38}$$

This special choice of the manifold (i.e., the fact that x_4 and x_8 are used instead of z_4 and z_8) is motivated by the fact that $\dot{x}_3$ and $\dot{x}_7$ are not affected by uncertain terms. In Figures 2, 3, 4, 5 some signals showing the good performance of the controlled system are reported. Note that, the control signals have been zoomed to show that they are continuous.

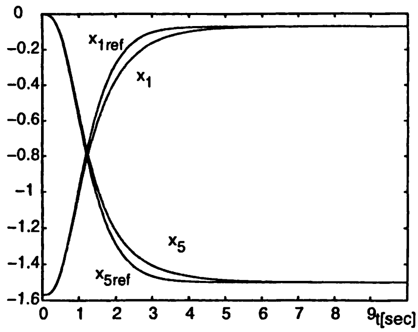

Fig. 2. x_1, x_5 **trajectories versus reference trajectories.**

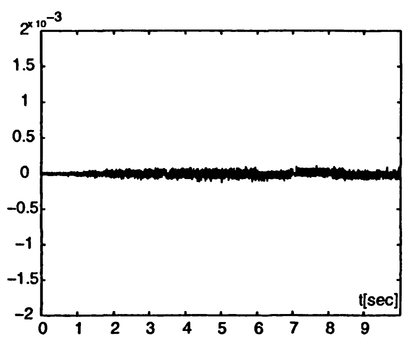

Fig. 3. S_1 **in the proposed multi-input SOSMC procedure.**

8. CONCLUSIONS

In the paper, a tracking control problem is considered consisting in forcing the m-dimensional output of a nonlinear uncertain system to track an m-dimensional ref-

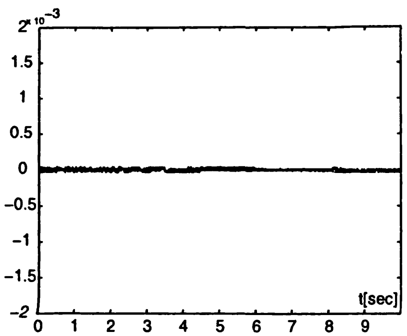

Fig. 4. S_2 in the proposed multi-input SOSMC procedure.

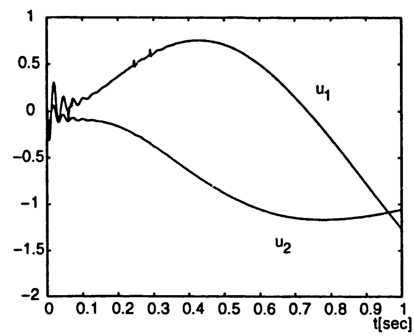

Fig. 5. u_1, u_2 signals in the proposed multi-input SOSMC procedure.

erence vector signal with the first ρ_i derivatives of each component known, bounded and piece-wise continuous. Such an objective is attained designing a suitable control vector on the basis of a procedure which goes through the construction of a transformed system characterized by $\rho_i - 1$ per block differential equations (ρ_i being the number of equations of the block of the multi-input pure-feedback form considered) analogous to those attainable via a purely multi-input backstepping design, coupled with an uncertain nonlinear multi-input second order auxiliary system. Under suitable assumptions on the control matrix, the control is chosen to be the extension to the multi-input case of a SOSMC algorithm, so as to force the transformed state variables involved in the second order auxiliary system to zero in finite time. The remainder of the transformed system turns out to be a reduced order system for which the same results valid for a purely backstepping controller still hold.

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REFERENCES

[1] G. Bartolini, A. Ferrara, L. Giacomini and E. Usai: A combined backstepping/second order sliding mode approach to control a class of nonlinear systems. In: Proc. IEEE International Workshop on Variable Structure Systems. Tokyo 1996.
[2] G. Bartolini, A. Ferrara and E. Usai: Applications of a suboptimal discontinuous control algorithm for uncertain second order systems. Internat. J. Robust Nonlin. Control *7* (1997), 299–320.
[3] G. Bartolini, A. Ferrara and E. Usai: Chattering avoidance by second-order sliding modes control. IEEE Trans. Automat. Control *34* (1998), 2, 241–246.
[4] G. Bartolini, A. Ferrara, E. Usai and V. I. Utkin: Second order chattering-free sliding mode control for some classes of multi-input uncertain nonlinear systems. In: Proc. of the 6th IEEE Mediterranean Conference on Control and Systems. Alghero 1998.
[5] B. M. Diong and J. V. Medanic: Simplex-type variable structure controllers for systems with non-matching disturbances and uncertainties. Internat. J. Control *68* (1997), 625–656.
[6] I. Kanellakopoulos, P. V. Kokotovic and A. S. Morse: Systematic design of adaptive controllers for feedback linearizable systems. IEEE Trans. Automat. Control *36* (1991), 1241–1253.
[7] D. E. Kirk: Optimal control theory. Prentice Hall, Englewood Cliffs, N.J. 1970.
[8] P. V. Kokotović, M. Krstić and I. Kanellakopoulos: Nonlinear and Adaptive Control Design. Wiley, New York 1995.
[9] A. Levant: Sliding order and sliding accuracy in sliding mode control. Internat. J. Control *58* (1993), 1247–1263.
[10] A. Levant: Higher order sliding: collection of design tools. In: European Control Conference, Bruxelles 1997.
[11] K. Nam and A. Arapostathis: A model reference adaptive control scheme for pure-feedback non-linear systems. IEEE Trans. Automat. Control *33* (1988), 803–811.
[12] D. Seto, A. M. Annaswamy, and J. Baillieul: Adaptive control of a class of nonlinear systems with a triangular structure. IEEE Trans. Automat. Control *39* (1994), 1411–1428.
[13] M. W. Spong and M. Vidyasagar: Robot Dynamics and Control. Wiley, New York 1989.
[14] R. Su and L. R. Hunt: A canonical expansion for nonlinear systems. IEEE Trans. Automat. Control *31* (1986), 670–673.
[15] V. I. Utkin: Sliding Modes in Control and Optimization. Springer–Verlag, Berlin 1992.

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