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ON GENERALIZED POPOV THEORY FOR DELAY SYSTEMS

S. I. NICULESCU, V. IONESCU, D. IVĂNESCU[1], L. DUGARD AND J.-M. DION

This paper focuses on the Popov generalized theory for a class of some linear systems including discrete and distributed delays. Sufficient conditions for stabilizing such systems as well as for coerciveness of an appropriate quadratic cost are developed. The obtained results are applied for the design of a *memoryless* state feedback control law which guarantees the (exponential) closed-loop stability with an $\mathcal{L}_2$ norm bound constraint on disturbance attenuation.

Note that the proposed results extend similar ones proposed by some of the authors [11].

1. INTRODUCTION

Control of time-delay systems is a problem of recurring interest since many physical processes involving transport phenomena (engineering or biological systems) can be modelized using delays (see, for instance Kolmanovskii and Nosov [13], Kolmanovskii and Myshkis [12] or Răsva [28] and the references therein) and the existence of a delay may induce instability or poor performances for the closed-loop schemes.

Recently, special interest has been focused on the stabilization problem of linear systems including delayed state via *memoryless controllers* (state-feedback, output feedback). A classification of such controllers function of the closed-loop stability property, which could depend or not on the delay size has been given in [20] (and the references therein).

Although the last decade has witnessed significant advances on the $\mathcal{H}_\infty$ control theory for linear systems (see [3] for a Riccati based approach or [1] for an LMI based approach), the $\mathcal{H}_\infty$ control for linear systems with delayed state have not been fully investigated. Memoryless controllers for such systems have been considered in [16] (frequency-domain, constant delay, state feedback), [19] (time-domain, supplimentary constraints on the closed-loop system, LMI approach) for systems without uncertainties. The uncertainty case have been considered in [22] (time-varying and bounded uncertainties, a Riccati equation approach, time-varying delay, uncertainty, state feedback), [30] (time-varying and bounded uncertainties, a Riccati equation approach, constant delay, output feedback) or in [17, 18] (IQC uncertainties, LMI approach, state-feedback and dynamic state feedback).

[1] All the correspondence should be sent to this author

The development is essentially based on the 'generalized' Riccati theory presented by Ionescu and Weiss [7] which is an extension of the famous Popov's positivity theory [27] to the indefinite sign case, usually encountered in game-theory situations. To be more specific, our results are based on the necessary and sufficient condition for the existence of the stabilizing solution to an adequate Kalman – Yakubovich – Popov system (KYPS) of indefinite sign, called KYPS in '*J-form*'. This approach, combined with the Krasovskii theory for time-delay systems, leads to explicit representation formulae. The delay system class considered includes discrete and distributed delay terms and can be seen as a special case of infinite-dimensional systems described by distributional convolutions (see [20] and the references therein). Note that the results presented here extend previous results obtained by some of the authors [9, 10, 11, 23], and can be further extended to multiple delays and/or structured uncertainty.

The novelty of the approach lies in the way to interpret the Lyapunov – Krasovskii functional as a *quadratic index* for an appropriate linear time-invariant system. Note that an overview of some *finite-dimensional interpretations* and related (closed-loop) stability results for linear time-delay systems can be found in [20]. Further comments can be also found in [24, 29].

The paper is organized as follows: in Section 2 the problem statement is given; Section 3 addresses some basic results on Popov theory. The main results are presented in Section 4. The $\mathcal{H}^\infty$ problem is considered in Section 5. A numerical example is presented in Section 6. Some concluding remarks end the paper. For the sake of clarity, the proofs of some results are included in Appendix.

2. PROBLEM STATEMENT

Consider the state-delayed system including 'discrete' and 'distributed' delays (following the terminology proposed in Kolmanovskii and Myshkis [12]) of the form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_1 x_t(-\tau_1) + \int_0^{\tau_2} A_2(\theta)\, x_t(-\theta)\, \mathrm{d}\theta \\ &\quad + B_1 u_1(t) + B_2 u_2(t), \end{aligned} \tag{1}$$

where x_t represents the translation operator $x_t(\theta) = x(t+\theta)$, and $A_2(\theta)$ is a piecewise continuous function, $x(t) \in \boldsymbol{R}^n$ is the state, $u_1(t) \in \boldsymbol{R}^{m_1}$, $u_2(t) \in \boldsymbol{R}^{m_2}$ are the disturbance and control inputs, A, A_1, B_i $i,j = 1,2$ are constant matrices of appropriate dimensions; τ_1 is the 'discrete' delay (to reconstruct the state at t we need information only in a 'point' of the interval $[t-\tau_1, t]$) and τ_2 corresponds to the 'distributed' delay due to the integral 'action' (information on all interval $[t-\tau_2, t]$).

We are interested in finding a *memoryless controller* of the form:

$$u_2(t) = F_2 x(t) \tag{2}$$

that *simultaneously*

- *stabilizes* the system (1), i. e. the closed-loop is exponentially stable and
- *achieves the* γ – *attenuation property*, that is

$$\|T_{y_1 u_1}\| \;<\; \gamma,$$

where $T_{y_1 u_1}$ is the L^2-linear bounded input-output operator defined by the closed-loop configuration obtained by coupling.

Several discussions on the delays τ_1 and τ_2 will be also considered. Note also that the proposed approach can be easily extended to multiple discrete and/or distributed delays by an appropriate construction of the corresponding Lyapunov – Krasovskii functional. Note also that the *idea* used here is to construct a Lyapunov candidate that allows, in some sense, to *'decouple'* present state (seen in the usual sense) from delayed state (see also the guided tour proposed in [24] for discrete delays). Such functional (defined on an appropriate product space) allows the use of appropriate finite-dimensional techniques (matrix pencils, LMIs, Riccati equations) for deriving sufficient conditions.

Unlike the techniques previously mentioned, our interest is directed towards the tools offered by the generalized Popov theory. The interest on such techniques is twofold: firstly, we may give some *alternative computational schemes* for the analysis and synthesis of some class of delay systems (by using appropriate *finite-dimensional* interpretations), and secondly, we may extend the *generalized Popov theory* developed for linear systems to some classes of infinite-dimensional systems (i. e. time-delay systems). The main interest of the authors is to propose some *tractable* methods (in fact, easy to compute) for the analysis and design of some classes of (time-varying) delay systems.

Notations. In the sequel, we will drop the explicit time dependence of $x(t)$, $u_1(t)$ and $u_2(t)$ on t for brevity.

3. SOME BASIC RESULTS ON THE GENERALIZED POPOV THEORY

In this section, several basic notions and results concerning matrix pencil techniques applied to general Riccati theory are presented. The present developement is essentially based on the theory exposed in Ionescu and Weiss [7] or in Ionescu, Oară and Weiss [8]. Some comments on the delay-independent stability of linear time-delay systems have been considered in Niculescu and Ionescu [21], or in Niculescu [20].

Definition 1. Call $\Sigma = (A, B; P)$ where $A \in \boldsymbol{R}^{n \times n}$, $B \in \boldsymbol{R}^{n \times m}$ and

$$P = \begin{bmatrix} Q & L \\ L^T & R \end{bmatrix} = P^T \in \boldsymbol{R}^{(n+m)\times(n+m)}$$

a Popov triplet.

Frequently, the extensive notation $\Sigma = (A, B; Q, L, R)$ will be used.

Let $\Sigma = (A, B; Q, L, R)$ be a Popov triplet and let

$$J = \begin{bmatrix} -I_{m_1} & \\ & I_{m_2} \end{bmatrix}, \quad m_1 + m_2 = m \tag{3}$$

be an arbitrary sign matrix. Associated with Σ the following two objects:

(1) The Kalman – Popov – Yakubovich system in J form (KPYS(Σ, J))

The following nonlinear system with unknown X, V, W:

$$\begin{aligned} R &= V^T J V \\ L + XB &= W^T J V \\ Q + A^T X + XA &= W^T J W \end{aligned} \tag{4}$$

is usually denoted as the KPYS(Σ, J).

(2) The extended Hamiltonian pencil EHP(Σ) $\lambda M - N$ where

$$\begin{cases} M = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ \\ N = \begin{bmatrix} A & 0 & B \\ -Q & -A^T & -L \\ L^T & B^T & R \end{bmatrix}, \\ \\ M, N \in \boldsymbol{R}^{(2n+m)\times(2n+m)}. \end{cases} \tag{5}$$

Definition 2. Any triplet (X, V, W) for which (4) is fulfilled and in addition $X = X^T$, V is nonsingular and of lower-left block triangular form

$$V = \begin{bmatrix} V_{11} & 0 \\ V_{21} & V_{22} \end{bmatrix} \tag{6}$$

partitioned in accordance with J in (3) and $A + BF$ is exponentially stable for

$$F = -V^{-1}W, \tag{7}$$

called the stabilizing feedback gain, is called a stabilizing solution to the KPYS(Σ, J).

Definition 3. The EHP(Σ) is said disconjugate if it has a stable proper deflating subspace $\mathcal{V}$ of dimension n and, in addition, if

$$V = \begin{array}{c} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \end{array} \begin{array}{l} n \\ n \\ m \end{array}$$

is any basis matrix for $\mathcal{V}(\mathcal{V} = \langle V \rangle)$, then V_1 is nonsingular.

Recall that $\mathcal{V}$ is said to be a *stable proper deflating subspace* [7, 26] of an arbitrary matrix pencil $\lambda M - N$ if $NV = MVS$, MV is monic, S is Hurwitzian and $\mathcal{V} = \langle V \rangle$. A relevant result of the generalized Popov theory is [8]:

Theorem 1. Let $\Sigma = (A, B; Q, L, R)$ be a Popov triplet and J any sign matrix as in (3). Then the following statements are equivalent:

1. R is nonsingular and the KPYS(Σ, J) has a stabilizing solution (X, V, W);
2. The EHP(Σ) is regular and disconjugate and, in addition, if R is partitioned in accordance with J in (3), i.e.,

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{bmatrix} \tag{8}$$

then

$$R_{22} > 0, \ \mathrm{sgn} R = J. \tag{9}$$

If 2 is true, then (see Definition 3) $X = V_2 V_1^{-1}$ and $F = V_3 V_1^{-1}$.

4. MAIN RESULTS

In a first step, let us consider the following delay system:

$$\begin{cases} \dot{x} = Ax + A_1 x_t(-\tau_1) + \int_0^{\tau_2} A_2(\theta)\, x_t(-\theta)\,\mathrm{d}\theta + Bu, \\ \quad x = \phi \text{ on } [-\tau, 0] \end{cases} \tag{10}$$

where $x \in \boldsymbol{R}^n$ is the state, $u \in \boldsymbol{R}^m$ is the input, $A, A_1 \in \boldsymbol{R}^{n\times n}$, $B \in \boldsymbol{R}^{n\times m}$, $A_2(\cdot)$ is a piecewise continuous function, $\tau = \max\{\tau_1, \tau_2\}$ is the delay and ϕ is any continuous n-valued function on $[-\tau, 0]$.

Let $\Sigma = (A, B; Q, L, R)$ be a Popov triplet where the entries A and B coincide with A and B in (10). Let $R_{d1} \in \boldsymbol{R}^{n\times n}$ and consider the *extended* time-varying Popov triplet

$$\Sigma_v = \left(A, [A_1 \ \ A_2(\cdot) \ \ B]; Q, [0 \ \ 0 \ \ L], \begin{bmatrix} R_{d1} & 0 & 0 \\ 0 & R_{d2}(\cdot) & 0 \\ 0 & 0 & R \end{bmatrix} \right), \tag{11}$$

(v from time-varying) associated to (10), where R_{d2} is a continuous time-varying function with some sign constraints.

Such extended Popov triplet allows to reduce the control problem of a time-varying delay system to a time-varying system free of delay. For the sake of simplicity we shall not address such problem here.

Remark 1. If A_2 is a constant matrix, we recover the *time-invariant* Popov triplet used in [23] with R_{d2} a symmetric and strictly negative-definite matrix. Furthermore, if $A_2 \equiv 0$, we recover the Popov triplet proposed in [11].

The approach considered here is based on the following *extended* time-invariant (parametrized) Popov triplet

$$\Sigma_e = \left(A, [A_1 \ \ M^{\frac{1}{2}}(S) \ \ B]; Q, [0 \ \ 0 \ \ L], \begin{bmatrix} R_{d1} & 0 & 0 \\ 0 & -I_n & 0 \\ 0 & 0 & R \end{bmatrix} \right), \tag{12}$$

where:

$$M = \int_0^{\tau_2} A_2(\theta)\,(S + \varepsilon I)^{-1} A_2(\theta)^T\,\mathrm{d}\theta, \quad S = S^T \tag{13}$$

for some *matrix* S (seen as a parameter).

We shall see later how the considered control problem for (10) is reduced to some algebraic properties of the extended triplet Σ_e. The *idea* is to interpret such problem as a control problem of an appropriate system free of delay. Furthermore, the corresponding Lyapunov – Krasovskii functional will be interpreted as an appropriate quadratic index for the associated system.

Let also the following (extended) sign matrix

$$J_e = \left[\begin{array}{c|c} -I_{2n} & \\ \hline & J \end{array}\right] = \left[\begin{array}{c|cc} -I_{2n} & & \\ \hline & -I_{m_1} & \\ & & I_{m_2} \end{array}\right], \quad (m_1 + m_2 = m) \tag{14}$$

be considered. Let B, L and R be partitioned in accordance with J in (14), i.e.,

$$B = [B_1 \;\; B_2], \quad L = [L_1 \;\; L_2], \quad R = \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{bmatrix}. \tag{15}$$

The basic result of this section is

Theorem 2. Assume that the KPYS(Σ_e, J_e) has a stabilizing solution (X, V_e, W_e). Let the stabilizing feedback gain F_e be partitioned in accordance with J_e in (14), that is,

$$F_e = -V_e^{-1} W_e = \begin{bmatrix} F_d \\ F_1 \\ F_2 \end{bmatrix}. \tag{16}$$

Let also u be split in accordance with B in (15), i.e.,

$$u = \begin{array}{c} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \end{array} \begin{array}{c} m_1 \\ m_2 \end{array}.$$

Let S be a symmetric and positive-definite matrix and ε a positive scalar. Assume further that

$$X \geq 0 \tag{17}$$

$$R_{11} < 0 \tag{18}$$

$$R_{d2}(\tau_2) < 0 \tag{19}$$

$$\tilde{Q} + R_{d1} + R_{d2}(0) > 0 \tag{20}$$

where

$$R_{d2}(\theta) = R_{d2}(0) + (S + \varepsilon I_n)\,\theta, \quad \theta \in [0, \tau_2] \tag{21}$$

$$\tilde{Q} = Q + L_2 F_2 + F_2^T L_2^T + F_2^T R_{22} F_2. \tag{22}$$

Then the state feedback

$$u_2 = F_2 x \tag{23}$$

stabilizes (10), i. e.,

$$\begin{cases} \dot{x} = \tilde{A}x + A_1 x_t(-\tau_1) + \int_0^{\tau_2} A_2(\theta)\, x_t(-\theta)\, \mathrm{d}\theta, \\ x = \phi \quad \text{on } [-\tau, 0] \end{cases} \tag{24}$$

defines an exponentially stable solution for all ϕ. Here $\tilde{A} = A + B_2 F_2$.

The complete proof is given in Appendix and makes use of the following Liapunov – Krasovkii functional:

$$\begin{aligned} V(x_t) &= x^T(t)\, X x(t) + \int_{t-\tau_1}^{t} x^T(\theta)\, (-R_{d1})\, x(\theta)\, \mathrm{d}\theta \\ &\quad + \int_{t-\tau_2}^{t} x^T(\theta)\, (-R_{d2}(t-\theta))\, x(\theta)\, \mathrm{d}\theta, \end{aligned} \tag{25}$$

where $X = X^T \geq 0$ and $R_{d1} = R_{d1}^T < 0$ are given before; the time-varying matrix function R_{d2} is constructed according (21). Note that since the inequality (19) is satisfied and S and ε are positive, it follows that $-R_{d2}(\xi)$ is a symmetric and positive-definite matrix for each $\xi \in [0, \tau_2]$, etc.

As specified, the *idea* is to see (25) as a quadratic index for an appropriate *time-invariant* linear system free of delay, and, thus to apply the generalized Popov theory to such system. Note also the particular construction of the matrix function $R_{d2}(\cdot)$, which simplifies such interpretation.

Remark 2. Since the Liapunov – Krasovskii functional (25) is very general, one may construct various *S–parametrizations* (not only linear!) of the time-varying matrix function $R_{d2}(\cdot)$, for which Theorem 2 is still true.

Thus, due to the particular form of the *distributed delay*, if, for example, R_{d2} is a continuous increasing (decreasing) function, we need "strong" conditions only in 2 points: 0 and τ_2, etc.

Remark 3. Using the results developed in [21], it follows that one may relax the condition $R_{d1} < 0$ to $R_{d1} \leq 0$, and thus to use more general forms for the corresponding J matrix. For the sake of simplicity, we have not presented such analysis here.

Remark 4. It is easy to see that if τ_1 is a continuous time-varying function, with bounded derivative as in [22], i. e.

$$\dot{\tau}_1(t) \leq \beta_1 < 1, \quad \beta \in \boldsymbol{R}, \tag{26}$$

then the Theorem holds if one changes R_{d1} by $\frac{1}{1-\beta_1} R_{d1}$. Note that the corresponding Liapunov – Krasovskii functional changes similarly.

Similar technique can be used if we assume that τ_2 is a continuous time-varying function.

For the sake of simplicity, we shall not develop such extension here.

A natural consequence of this Theorem is given by the following:

Corollary 1. If all the conditions in the statement of Theorem 2 hold, then

$$\begin{cases} \dot{x} = \tilde{A}x + A_1 x_t(-\tau_1) + \int_0^{\tau_2} A_2(t-\theta)\, x_t(-\theta)\, \mathrm{d}\theta, \\ x = 0 \text{ on } [-\tau, 0] \end{cases} \tag{27}$$

defines a linear bounded input-state operator from L_+^{2,m_1} into $L_+^{2,n}$.

P r o o f. By $L_+^{2,r}$ we mean the Hilbert space of norm square integrable C^r-valued functions defined on $[0,\infty)$. The proof is a trivial consequence of the exponentially stable evolution defined by (24) (see also [6, 28]). □

Taking into account Theorem 1, an equivalent form of Theorem 2 can be stated as follows:

Theorem 3. Assume that the EHP(Σ_e) is disconjugate. Assume also that

$$S > 0, \quad R_{22} > 0, \quad \mathrm{sgn} R = J, \quad R_{d1} < 0. \tag{28}$$

If

$$V_2 V_1^{-1} \geq 0 \tag{29}$$

and both (18) and (20) hold, then (23) stabilizes (10). Here

$$\begin{array}{cc} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} & \begin{array}{c} n \\ n \\ n+m \end{array} \end{array}$$

is any basis matrix for the maximal stable proper deflating subspace of the EHP(Σ_e) and

$$F_e = V_3 V_1^{-1}$$

partitioned as in (16).

Remark 5. Theorem 3 provides easy checkable sufficient conditions for the stabilizability of the state-delayed system (10) (see also [21]) in terms of algebraic properties of the associated matrix pencil.

Let $\hat{Q}$ be any $n \times n$ symmetric matrix. Let $\Sigma = (A, B; \hat{Q}, L, R)$ be the Popov triplet constructed with $\hat{Q}$ and with entries of Σ_e. Associate with Σ the "usual" Popov index [4]

$$J_\Sigma(\phi, u) = \left\langle \begin{bmatrix} x \\ u \end{bmatrix}, \begin{bmatrix} \hat{Q} & L \\ L^T & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right\rangle \tag{30}$$

where $(x, u) \in L_+^{2,n} \times L_+^{2,m}$ and x and u are linked via (10) for some ϕ.

Then we have (for the proofs, see the Appendix):

Proposition 1. Let us consider a symmetric matrix $\hat{Q}$ satisfying

$$\begin{bmatrix} \hat{Q} & L_2 \\ L_2^T & R_{22} \end{bmatrix} \geq 0. \tag{31}$$

Assume also that all the conditions in the statement of Theorem 2 hold except (20) which is modified as

$$\tilde{Q} + R_{d1} + R_{d2}(0) > \tilde{\hat{Q}} \tag{32}$$

where

$$\tilde{\hat{Q}} = \hat{Q} + L_2F_2 + F_2^T L_2^T + F_2^T R_{22} F_2. \tag{33}$$

If the controller (23) stabilizes the delay system and $\phi = 0$, i. e., (10) becomes (27), then there exists $\bar{\zeta} > 0$ such that

$$J_\Sigma(0, u_1) \leq -\bar{\zeta}\|u_1 - F_1 x\|_2^2, \quad \forall\, u_1 \in L_+^{2,m_1} \tag{34}$$

where

$$J_\Sigma(0, u_1) := J_\Sigma(0, u)|_{u_2 = F_2 x}\,. \tag{35}$$

Proposition 2. Assume that all conditions in the statement of Theorem 2 hold. Assume additionally that

$$\bar{Q} + R_{d1} + R_{d2}(0) > 0 \tag{36}$$

where

$$\bar{Q} := Q + LF + F^T L^T + F^T RF.$$

Then

$$\begin{cases} \dot{x} = \tilde{A}x + A_1 x_t(-\tau_1) + \int_0^{\tau_2} A_2(\theta)\, x_t(-\theta)\, \mathrm{d}\theta + B_1 u_1 \\ v_1 = -F_1 x \qquad\qquad\qquad\qquad\qquad\qquad\qquad + u_1 \end{cases}$$

(with $x = 0$ on $[-\tau, 0]$) defines a linear boundedly invertible operator on L_+^{2,m_1}.

Using all the results presented before, we shall state and prove the main result of this paper.

Theorem 4. Let (10) together with the quadratic cost defined by the right-hand side of (30) be given. For arbitrary m_1, m_2 such that $m_1 + m_2 = m$, let B, L and R be partitioned as in (15).

Assume that there exists three $n \times n$ symmetric matrices $S > 0$, Q, R_{d1} and a positive scalar ε such that the KPYS(Σ_e, J_e), where Σ_e and J_e are defined by (11) and (14), respectively, has a stabilizing solution (X, V_e, W_e) and let the stabilizing feedback F_e be partitioned in accordance with (16).

Assume also that the following conditions all hold:

1. $X \geq 0$
2. $\begin{bmatrix} \hat{Q} & L_2 \\ L_2^T & R_{22} \end{bmatrix} \geq 0$

3. $R_{11} < 0$
4. $R_{d2}(\tau_2) < 0$
5. $\tilde{Q} + R_{d1} + R_{d2}(0) > \tilde{\hat{Q}}$
6. $\bar{Q} + R_{d1} + R_{d2}(0) > 0$

where

$$\begin{aligned}\tilde{Q} &= Q + L_2F_2 + F_2^TL_2^T + F_2^TR_{22}F_2\\ \tilde{\hat{Q}} &= \hat{Q} + L_2F_2 + F_2^TL_2^T + F_2^TR_{22}F_2\\ \bar{Q} &= Q + LF + F^TL^T + F^TRF\\ R_{d2}(\theta) &= R_{d2}(0) + (S + \varepsilon I_n)\theta, \quad \theta \in [0, \tau_2].\end{aligned}$$

Then

a. $u_2 = F_2x$ stabilizes (10)

b. There exists $c_0 > 0$ such that

$$J_\Sigma(0, u_1) \leq -c_0\|u_1\|_2^2 \quad \forall u_1 \in L_+^{2,m_1}$$

where $J_\Sigma(0, u_1)$ has been defined by (35), (30).

P r o o f. a. follows directly from Theorem 2 combined with 2 and 4 in the statement (see the proof of Proposition 1).

b. From Proposition 2 it follows that there exists $\zeta_1 > 0$ such that

$$\|v_1\|_2^2 = \|u_1 - F_1x\|_2^2 \geq \zeta_1\|u_1\|_2^2. \tag{37}$$

Using Proposition 1 the conclusion follows by substituting (37) in (34) and putting $c_0 = \zeta_1\bar{\zeta}$. □

5. $\mathcal{H}^\infty$-CONTROL

In this section the theory developed in Section 4 will be applied for solving the $\mathcal{H}^\infty$-control problem formulated for state-delayed systems. Such a problem is stated as follows.

Let the system

$$\begin{cases} \dot{x} = Ax + A_1x_t(-\tau_1) + \int_0^{\tau_2} A_2(\theta)\, x_t(-\theta)\,\mathrm{d}\theta + B_1u_1 + B_2u_2 \\ y_1 = C_1x + D_{11}u_1 + D_{12}u_2 \end{cases} \tag{38}$$

(where $x = 0$ on $[-\tau, 0]$) be given. Here x is the state (in the usual sense), u_1 and u_2 are the disturbance and control inputs, respectively, and y_1 is the output to be controlled. The state x is assumed to be accessible for measurement. We are looking for a state feedback law

$$u_2 = F_2 x \tag{39}$$

which stabilizes (38) and achieves γ-attenuation property for the closed-loop system, i.e., there exists $c_0 > 0$ such that

$$-\gamma^2 \|u_1\|_2^2 + \|y_1\|_2^2 \leq -c_0 \|u_1\|_2^2 \quad \forall\, u_1 \in L_+^{2,m_1} \tag{40}$$

or equivalently the system

$$\begin{cases} \dot{x} = (A + B_2 F_2)\, x \; + A_1 x_t(-\tau_1) + \int_0^{\tau_2} A_2(\theta)\, x_t(-\theta)\, \mathrm{d}\theta \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + B_1 u_1 \\ y_1 = (C_1 + D_{12} F_2)\, x \qquad\qquad\qquad\qquad + D_{11} u_1 \end{cases} \tag{41}$$

(where $x = 0$ on $[-\tau, 0]$) defines a γ-strictly contractive input-output map. Here γ is a prescribed tolerance for the attenuation level.

Introduce

$$\begin{array}{l} \qquad\quad m_1 \quad m_2 \\ B = [B_1 \quad B_2] \\ \hat{Q} = C_1^T C_1, \quad L = [L_1 \quad L_2] = C_1^T [D_{11} \quad D_{12}] \end{array} \tag{42}$$

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{bmatrix} = \begin{bmatrix} -\gamma^2 I + D_{11}^T D_{11} & D_{11}^T D_{12} \\ D_{12}^T D_{11} & D_{12}^T D_{12} \end{bmatrix} \tag{43}$$

Then we have

Theorem 5. Assume that there exist two $n \times n$ symmetric matrices Q and R_d such that all the conditions of Theorem 4 hold with respect to the particular data (42). Then for F_2 given in Theorem 4, (39) is a solution to the H^∞-control problem stated above.

Proof. Let $\Sigma = (A, B; \hat{Q}, L, R)$. Then

$$J_\Sigma = -\gamma^2 \|u_1\|_2^2 + \|y_1\|_2^2 \tag{44}$$

as directly follows by simple computation from (42). Apply Theorem 4 to (44) and the conclusion follows trivially. □

Note that $\tilde{\hat{Q}}$ in Theorem 4 reads now as $\tilde{\hat{Q}} = C_{1F_2}^T C_{1F_2}$ where $C_{1F_2} = C_1 + D_{12} F_2$.

6. AN EXAMPLE

In this section a numerical example is presented to illustrate our approach. It should be pointed out that there are few results in the literature addressed to this problem (Dugard, Verriest eds., [2]).

Let the following unstable distributed state-delayed system given by:

$$\begin{aligned} A &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad A_2(\theta) = \begin{bmatrix} \theta & 0 \\ 0 & \theta \end{bmatrix} \\ B_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 1 \end{bmatrix} \quad D_{12} = \begin{bmatrix} 1 \end{bmatrix}. \end{aligned}$$

The problem is to find a memoryless controller

$$u_2 = F_2 x \tag{45}$$

that achieves simultaneously closed-loop stability and γ-attenuation.

The prescribed tolerance is

$$\gamma = 0.5 \tag{46}$$

Choose $R_d = \begin{bmatrix} -11 & 0 \\ 0 & -11 \end{bmatrix}$.Then the extended Popov triplet (see (12)) is

$$\begin{aligned} \Sigma_e &= \Bigg(A, [A_1 \ M(\tau_2)^{\frac{1}{2}} \ B_1 \ B_2]; Q, [0 \ 0 \ L_1 \ L_2], \\ &\qquad \begin{bmatrix} R_d & 0 & 0 & 0 \\ 0 & -I_2 & 0 & 0 \\ 0 & 0 & -\gamma^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\Bigg) \end{aligned} \tag{47}$$

and (see (14))

$$J_e = \left[\begin{array}{c|cc} -I & & \\ \hline & -1 & \\ & & 1 \end{array}\right]. \tag{48}$$

As we consider here $\epsilon = 0$ and taking $S = 2 * I$, the equation (13) yields $M(\tau_2) = \operatorname{diag}\left(0.5\frac{\tau^3}{3},\ 0.5\frac{\tau^3}{3}\right)$. But (47) is equivalent to the algebraic Riccati equation (ARE) associated with Σ_e, that is,

$$\begin{aligned} &A^T X + XA + Q - ([0 \ 0 \ L] + X[A_1 \ M(\tau_2)^{\frac{1}{2}} \ B]) \\ &\begin{bmatrix} R_d & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & R \end{bmatrix}^{-1} \left(\begin{bmatrix} 0 \\ 0 \\ L^T \end{bmatrix} + \begin{bmatrix} A_1^T \\ M(\tau_2)^{\frac{1}{2}} T \\ B^T \end{bmatrix} X \right) = 0. \end{aligned}$$

Solving this Riccati equation we find that $X = f(Q,\ M(\tau_2))$. The stabilizing solution X is still positive for $\tau_{2\,\max} = 1.13$. Let us check the conditions stated in Theorem 4. The first condition is fulfilled when $Q > \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. For the next five ones, we obtain

2) and 3) are obviously fulfilled

4) $R_{d2}(\tau_2) = R_{d2}(0) + S\tau_2 < 0$. That means $R_{d2}(0) \leq -\begin{bmatrix} 0.7 & 0 \\ 0 & 0.7 \end{bmatrix}$.

5) and 6) are fulfilled whenever $Q > -R_{d2}(0) - R_{d1} + \hat{Q}$.

Therefore all of the conditions stated in Theorem 30 hold for $Q > \begin{bmatrix} 13.7 & 0 \\ 0 & 13.7 \end{bmatrix}$. $R_{d2}(0) \leq -\begin{bmatrix} 0.7 & 0 \\ 0 & 0.7 \end{bmatrix}$,and consequently: $u_2 = [-0.62 \quad -8.56]\,x$ is the desired feedback law.

7. CONCLUSIONS

An extension of the generalized Popov theory to the case of delay system with discrete and distributed delays has been done. Our interest has been focused on the memoryless controller design for $\mathcal{H}^\infty$-control problem stated for systems described by retarded functional differential equations. As future research directions we suggest: a) the analysis of the general time-varying case; b) statement of *necessary* solvability conditions in terms of *signature condition*; c) investigation of observer-based compensation technique.

APPENDIX

A. Proof of Theorem 2

Since (X, V_e, W_e) is a stabilizing solution to the KPYS(Σ_e, J_e), it has the following form:

$$\begin{aligned} \begin{bmatrix} R_{d1} & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & R \end{bmatrix} &= V_e^T J_e V_e \\ [0 \;\; 0 \;\; L] + X\left[A_1 \;\; M(\tau_2)^{\frac{1}{2}} \;\; B\right] &= W_e^T J_e V_e \\ Q + A^T X + XA &= W_e^T J_e W_e. \end{aligned} \tag{49}$$

Taking into account Definition 2 in conjugation with (14), the first equation in (49) leads to the following structure for V_e:

$$V_e = \left[\begin{array}{c|c} V_{dp} & \\ \hline & V \end{array}\right] = \left[\begin{array}{c|cc} V_{dp} & & \\ \hline & V_{11} & 0 \\ & V_{21} & V_{22} \end{array}\right], \tag{50}$$

where $V_{dp} = \begin{bmatrix} V_{d1} & \\ & V_{d2} \end{bmatrix}$.

Let W_e be partitioned accordingly, i.e.,

$$W_e = \left[\begin{array}{c} W_{dp} \\ \hline W \end{array}\right] = \left[\begin{array}{c} W_{dp} \\ \hline W_1 \\ W_2 \end{array}\right], \tag{51}$$

where $W_{dp}^T = [W_{d1}^T \;\; W_{d2}^T]$. Substituting (50), (51) in (49), leads to the following form:

$$[R_{d1} \;\; -I \;\; R] \;\; = \;\; [-V_{d1}^T V_{d1} \;\; -V_{d2}^T V_{d2} \;\; V^T J V]$$

$$\begin{aligned}\left[XA_1 \;\; XM(\tau_2)^{\frac{1}{2}}\right] &= \left[-W_{d1}^TV_{d1} \;\; -W_{d2}^TV_{d2}\right] \\ L+XB &= W^TJV \\ Q+A^TX+XA &= -W_{d1}^TW_{d1}-W_{d2}^TW_{d2}+W^TJW.\end{aligned} \tag{52}$$

Using (16) one gets

$$F_e=\left[\frac{F_{dp}}{F}\right]=\left[\frac{-V_{dp}^{-1}W_{dp}}{-V^{-1}W}\right]=\left[\frac{F_{dp}}{\begin{matrix}F_1\\F_2\end{matrix}}\right]=\left[\frac{-V_{dp}^{-1}W_{dp}}{\begin{matrix}-V_{11}^{-1}W_1\\ V_{22}^{-1}V_{21}V_{11}^{-1}W_1-V_{22}^{-1}W_2\end{matrix}}\right] \tag{53}$$

where the structure (6) has been also taken into account and where

$$A+[A_1 \;\; M(\tau_2)^{\frac{1}{2}} \;\; B]\begin{bmatrix}F_{d1}\\F_{d2}\\F\end{bmatrix}.$$

is uniformly asymptotically stable.

Let now

$$\tilde{F}_{e,2}:=\left[\frac{0}{\tilde{F}_2}\right]=\left[\frac{0}{\begin{matrix}0\\F_2\end{matrix}}\right] \tag{54}$$

and let $\tilde{\Sigma}_{e,2}$ be the $\tilde{F}_{e,2}$-equivalent of Σ_e in (11). Following Theorem 1 and taking into account the zero structure of $\tilde{F}_{e,2}$ in (54), the updated form of the last equation in (52) corresponding to $\tilde{\Sigma}_{e,2}$ is

$$\tilde{A}^TX+X\tilde{A}=-\tilde{Q}-W_{d1}^TW_{d1}-W_{d2}^TW_{d2}+(W+V\tilde{F}_2)^TJ(W+V\tilde{F}_2). \tag{55}$$

Furthermore

$$\begin{aligned}W+V\tilde{F}_2 &= \begin{bmatrix}W_1\\W_2\end{bmatrix}+\begin{bmatrix}V_{11}&0\\V_{21}&V_{22}\end{bmatrix}\begin{bmatrix}0\\V_{22}^{-1}V_{21}V_{11}^{-1}W_1-V_{22}^{-1}W_2\end{bmatrix} \\ &= \begin{bmatrix}W_1\\V_{21}V_{11}^{-1}W_1\end{bmatrix}\end{aligned}$$

from which

$$\begin{aligned}(W+V\tilde{F}_2)^TJ(W+V\tilde{F}_2) &= -W_1^TW_1+W_1^TV_{11}^{-T}V_{21}^TV_{21}V_{11}^{-1}W_1 \\ &= W_1^TV_{11}^{-T}(-V_{11}^TV_{11}+V_{21}^TV_{21})V_{11}^{-1}W_1.\end{aligned} \tag{56}$$

Taking into account (15), the second equation in (52) yields, by equating the $(1,1)$ entries

$$R_{11}=-V_{11}^TV_{11}+V_{21}^TV_{21}. \tag{57}$$

With (57) in (56) and then with (56) in (55), one gets eventually

$$\tilde{A}^TX+X\tilde{A}=-\tilde{Q}-W_{d1}^TW_{d1}-W_{d2}^TW_{d2}-W_1^TV_{11}^{-T}\tilde{R}_{11}V_{11}^{-1}W_1 \tag{58}$$

where

$$\tilde{R}_{11} := -R_{11} > 0 \tag{59}$$

as follows from (18).

We introduce the Lyapunov functional

$$\begin{aligned} V(x_t) &= x^T(t)\,Xx(t) + \int_{t-\tau_1}^{t} x^T(\theta)\,(-R_{d1})\,x(\theta)\,\mathrm{d}\theta \\ &\quad + \int_{t-\tau_2}^{t} x^T(\theta)(-R_{d2}(t-\theta))\,x(\theta)\,\mathrm{d}\theta, \end{aligned} \tag{60}$$

where $X = X^T \geq 0$ and $R_{d1} = R_{d1}^T < 0$ are given before; the time-varying matrix function $R_{d2}(\cdot)$ is given in (21). Note that since the inequality (19) is satisfied, it follows that $-R_{d2}(\xi)$ is a symmetric and positive-definite matrix for each $\xi \in [0, \tau_2]$. Simple computations prove that there exist two positive numbers d_1, d_2 such that

$$d_1 \parallel x(t) \parallel^2 \leq V(t, x_t) \leq d_2 \sup_{\theta\in[t-\tau,t]} \parallel x(\theta) \parallel^2 . \tag{61}$$

Taking the Lyapunov derivative of (60) with respect to (24), one obtains:

$$\begin{aligned} \dot{V}(x_t) &= x^T(\tilde{A}^T X + X\tilde{A})\,x + x(t-\tau_1)^T A_1^T Xx + x^T XA_1 x(t-\tau_1) \\ &+ \left(\int_0^{\tau_2} A_2(\theta)\,x(t-\theta)\,\mathrm{d}\theta\right)^T Xx + x^T X\left(\int_0^{\tau_2} A_2(\theta)\,x(t-\theta)\,\mathrm{d}\theta\right) \\ &+ x^T(-R_{d1})\,x - x^T(t-\tau_1)\,(-R_{d1})\,x(t-\tau_1) + x^T(-R_{d2}(0))\,x \\ &- x^T(t-\tau_2)\,(-R_{d2}(\tau_2))\,x(t-\tau_2) \\ &+ \int_{-\tau_2}^{0} x^T(\theta+t)\left(\frac{\mathrm{d}R_{d2}(\theta)}{\mathrm{d}\theta}\right) x(\theta+t)\,\mathrm{d}\theta. \end{aligned} \tag{62}$$

Since

$$\begin{aligned} & 2x^T X \int_0^{\tau_2} A_2(\theta)\,x(t-\theta)\,\mathrm{d}\theta \\ \leq\; & x^T X\left(\int_0^{\tau_2} A_2(\theta)\,(S+\varepsilon I)^{-1} A_2^T(\theta)\,\mathrm{d}\theta\right) Xx \\ +\; & \int_0^{\tau_2} x^T(t+\theta)\,(S+\varepsilon I)\,x(t+\theta)\,\mathrm{d}\theta, \end{aligned} \tag{63}$$

and since we have (52), we can write the corresponding equations of the KPYS$(\tilde{\Sigma}_{e,2}, J_e)$ (see the structure of $\tilde{F}_{e,2}$ in (54)), and (62) becomes:

$$\begin{aligned} \dot{V}(x_t) &\leq -x^T\tilde{Q}x - x^T W_{d1}^T W_{d1} x - x^T W_{d2}^T W_{d2} - x^T W_1^T V_{11}^{-T} \tilde{R}_{11} V_{11}^{-1} W_1 x \\ &- x^T W_{d1}^T V_{d1} x(t-\tau_1) - x(t-\tau_1)^T V_{d1}^T W_{d1} x - x(t-\tau_1)^T V_{d1}^T V_{d1}(t-\tau_2) \\ &+ x^T R_{d1} x - x^T R_{d2}(0)\,x - x^T(t-\tau_2)\,(-R_{d2}(\tau_2))\,x(t-\tau_2) \\ &+ x^T X M^{1/2} M^{1/2} X x \end{aligned} \tag{64}$$

where both (20) and (58) have been used. With (61) the proof is completed via the Krasovskii stability theorem [6].

B. Proof of Proposition 1

In order to obtain a criterium for γ – attenuation for the distributed time-delay system, we shall associate an index defined from $\mathcal{C}([-\tau, 0] \times L_{+}^{2,m}$ to $\boldsymbol{R}$ by

$$J_{\Sigma_e}(\phi, u) := \left\langle \begin{bmatrix} x \\ x(t-\tau_1) \\ x \\ u \end{bmatrix}, \begin{bmatrix} Q & 0 & 0 & L \\ 0 & R_{d1} & 0 & 0 \\ 0 & 0 & R_{d2} & 0 \\ L^T & 0 & 0 & R \end{bmatrix} \begin{bmatrix} x \\ x(t-\tau_1) \\ x \\ u \end{bmatrix} \right\rangle \tag{65}$$

which will be called an *Extended Popov index*.

Here (x, u) is any pair satisfying (1) and in addition it belongs to $L_{+}^{2,n} \times L_{+}^{2,m}$. Here $\langle \cdot, \cdot \rangle$ stands for the L_{+}^{2}-inner product.

Assume that the KPYS(Σ_e, J_e) has a stabilizing solution (X, V_e, W_e). Assume also that there exists a pair $(x, u) \in L_{+}^{2,n} \times L_{+}^{2,m}$ such that (1) is fulfilled for some ϕ. Then the following evaluation holds:

$$\begin{aligned} J_{\Sigma_e}(\phi, u) \quad \leq \quad & -\|W_d x + V_d x(t-\tau_1)\|_2^2 + \langle Wx + Vu, J(Wx + Vu) \rangle + x_0^T X x_0 \\ & (x_0 = x(0)). \end{aligned} \tag{66}$$

The proof of 66 is straightforward using only algebraic manipulations
Notice now that:

$$\begin{bmatrix} \tilde{\hat{Q}} & \tilde{L}_2 \\ \tilde{L}_2^T & R_{22} \end{bmatrix} = \begin{bmatrix} I & F_2^T \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{Q} & L_2 \\ L_2^T & R_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ F_2 & I \end{bmatrix} \tag{67}$$

As (31) holds, (67) shows that $\tilde{\hat{Q}} \geq 0$. Hence (32) implies (20) and consequently Corollary 1 works and $J_\Sigma(0, u_1)$ defined by (35) makes sense.

Let

$$J_{\Sigma_e}(0, u_1) := J_{\Sigma_e}(0, u)|_{u_2 = F_2 x} \tag{68}$$

which makes sense as well. Since

$$J_\Sigma(0, u_1) = \left\langle \begin{bmatrix} x \\ u_1 \end{bmatrix}, \begin{bmatrix} \tilde{\hat{Q}} & \tilde{L}_1 \\ \tilde{L}_1^T & R_{11} \end{bmatrix} \begin{bmatrix} x \\ u_1 \end{bmatrix} \right\rangle \tag{69}$$

and

$$J_{\Sigma_e}(\phi, u) := \left\langle \begin{bmatrix} x \\ x(t-\tau_1) \\ x \\ u \end{bmatrix}, \begin{bmatrix} \tilde{Q} & 0 & 0 & L \\ 0 & R_{d1} & 0 & 0 \\ 0 & 0 & R_{d2}(0) & 0 \\ L^T & 0 & 0 & R \end{bmatrix} \begin{bmatrix} x \\ x(t-\tau_1) \\ x \\ u \end{bmatrix} \right\rangle \tag{70}$$

it follows from (69) and (70) and (66) that $J_\Sigma(0, u_1)$ and $J_{\Sigma_e}(0, u_1)$ are linked by

$$\begin{aligned} J_\Sigma(0, u_1) \quad = \quad & J_{\Sigma_e}(0, u_1) - \langle x(t-\tau_1), R_{d1} x(t-\tau_1) \rangle \\ & - \langle x, \tilde{Q} x \rangle + \langle x, \tilde{\hat{Q}} x \rangle - \langle x, R_{d2}(0)\, x \rangle. \end{aligned} \tag{71}$$

Set $u_2 = F_2 x$ in the right-hand side of (66) and obtain with (54)

$$
\begin{aligned}
& J_{\Sigma_e}(0,u_1) \leq \\
\leq\ & -\|W_d x + V_d x(t-\tau_1)\|_2^2 \\
& + \left\langle \begin{bmatrix} W_1 x + V_{11}u_1 \\ W_2 x + V_{21}u_1 + V_{22}u_2 \end{bmatrix}, \begin{bmatrix} -(W_1 x + V_{11}u_1) \\ W_2 x + V_{21}u_1 + V_{22}u_2 \end{bmatrix} \right\rangle p \\
=\ & -\|W_d x + V_d x(t-\tau_1)\|_2^2 + \langle u_1 - F_1 x, (-V_{11}^T V_{11} + V_{21}^T V_{21})(u_1 - F_1 x)\rangle \\
=\ & -\|W_d x + V_d x(t-\tau_1)\|_2^2 - \langle u_1 - F_1 x, \tilde{R}_{11}(u_1 - F_1 x)\rangle \qquad (72)
\end{aligned}
$$

where $u_2 = (V_{22}^{-1}V_{21}V_{11}^{-1}W_1 - V_{22}^{-1}W_2)\,x$.

Here (58), (60) have been considered.

Substituting (72) in (72), one obtains (see also (60)) after some algebraic computations:

$$
\begin{aligned}
J_\Sigma(0,u_1) \leq & -\|W_d x + V_d x(t-\tau_1)\|_2^2 - \langle x, (\tilde{Q} + R_{d1} - \tilde{\bar{Q}} + R_{d2}(0))\,x\rangle \\
& - \int_0^\infty \dot{V}(x_t)\,\mathrm{d}t + \langle x(t-\tau_2), R_{d2}(\tau_2)\,x(t-\tau_2)\rangle - \|\tilde{R}_{11}^{\frac{1}{2}}(u_1 - F_1 x)\|_2^2. \quad (73)
\end{aligned}
$$

As (32) holds, (34) follows from (73), where $\bar{\zeta} > 0$ is the least eigenvalue of $\tilde{R}_{11}$.

C. Proof of Proposition 2

According to Corollary 1, $u_1 \mapsto v_1$ is an L_+^2 operator. Let

$$
\begin{aligned}
\dot{x} &= \bar{A}x + A_1 x(t-\tau_1) + \int_0^{\tau_2} A_2(t-\theta)\,x_t(-\theta)\,\mathrm{d}\theta + B_1 v_1 \\
u_1 &= F_1 x + v_1 \qquad (74)
\end{aligned}
$$

be the inverse system of (37). Here $\bar{A} = \tilde{A} + B_1 F_1 = A + B_1 F_1 + B_2 F_2$. Let us show that

$$
\dot{x} = \bar{A}x + A_1 x(t-\tau_1) + \int_0^{\tau_2} A_2(t-\theta)\,x_t(-\theta)\,\mathrm{d}\theta, \quad x = \phi \text{ on } [-\tau, 0]
$$

defines an exponentially stable evolution for all ϕ. Proceed similarly to the proof of Theorem 2 and consider $\tilde{\Sigma}_{e,1}$ – the $\tilde{F}_{e,1}$-equivalent of Σ_e where

$$
\tilde{F}_{e,1} = \left[\frac{0}{F}\right] = \begin{bmatrix} 0 \\ \hline F_1 \\ F_2 \end{bmatrix}.
$$

Then, as $W + VF = 0$, the updated form of the last equation in (53) is

$$
\bar{A}^T X + X\bar{A} = -\bar{Q} - W_{d1}^T W_{d1} - W_{d2}^T W_{d2}
$$

and (64) becomes now:

$$\dot{V}(x_t) \leq -x^T(\bar{Q}+R_{d1}+R_{d2}(0))\,x - \|W_d x - V_d x(t-\tau_1)\|^2 \leq -\hat{\zeta}\|x\|^2, \quad \hat{\zeta}>0$$

as follows from (36). Thus the proof ends.

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Dr. Silviu Iulian Niculescu, HEUDIASYC, Université de Technologie de Compiègne, BP 20529, Centre de Recherche de Royallieu, 60205, Compiègne. France.
e-mail: silviu@hds.utc.fr

Prof. Vlad Ionescu, Department of Automatic Control, University 'Politehnica' of Bucharest, 313, Splaiul Independenţei, Sector 6, Bucharest. Romania.
e-mail: vlad.ionescu@starnets.ro

PhD Dan Ivănescu, Prof. Luc Dugard and Prof. Jean-Michel Dion, Laboratoire d'Automatique de Grenoble, ENSIEG, BP 46, 38402, Saint-Martin-d'Herès, cedex. France.
e-mails: dan,dugard,dion@lag.ensieg.inpg.fr