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DETECTION AND ACCOMMODATION OF SECOND ORDER DISTRIBUTED PARAMETER SYSTEMS WITH ABRUPT CHANGES IN THE INPUT TERM: EXISTENCE AND APPROXIMATION

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The purpose of this note is to investigate the existence of solutions to a class of second order distributed parameter systems with sudden changes in the input term. The class of distributed parameter systems under study is often encountered in flexible structures and structure-fluid interaction systems that use smart actuators. A failure in the actuator is modeled as either an abrupt or an incipient change of the actuator map whose magnitude is a function of the measurable output. A Galerkin-based finite approximation for the adaptive diagnostic observer and its on-line approximator is proposed in order to facilitate the numerical implementation of the aforementioned diagnostic observer.

1. INTRODUCTION

In this paper we present a theoretical investigation of the existence of solutions to a structural distributed parameter system subject to an abrupt change in the input term. The change in the input term models a failure in the actuator which occurs at an unknown time instance. This failure term is modeled as an additive perturbation of the actuator dynamics that is often encountered in flexible structures utilizing smart actuators. The time profile of the actuator failure can be taken either as abrupt or incipient. The gain of the actuator failure has magnitude that is modeled as a nonlinear function of the measurable output signal.

In order to detect and diagnose such an actuator failure which will eventually be used for accommodating this failure, a *model-based* fault diagnosis scheme is presented. This scheme consists of a detection/diagnostic observer and an on-line estimator of the actuator failure term. This failure monitoring scheme, as shown via a Lyapunov stability argument, can detect the occurrence of the anticipated actuator failure and via the on-line estimator can diagnose the nature of this actuator failure. Since the proposed detection/diagnostic scheme is infinite dimensional, its implementation necessitates a finite dimensional approximation and thus an approximation scheme is presented along with a summary of the convergence results. Results from a numerical study are summarized along with a presentation of the corresponding simulation results and a discussion of the findings.

2. A GALERKIN APPROACH TO EXISTENCE FOR SYSTEMS WITH FAILURES

We consider the nonlinear equation

$$w_{tt} + \kappa_1 w_{xxxx} + \kappa_2 w_{xxxxt} = [\beta(t,x)\, g(y)]_{xx} + f(t,x), \tag{2.1}$$

with boundary and initial conditions given by

$$\begin{aligned} &w_x(0,t) = w(0,t) = 0, \quad && w_x(1,t) = w(1,t) = 0, \\ &w(\cdot,0) = \phi_0 \in H_0^2(0,1), \quad && w_t(\cdot,0) = \phi_1 \in L^2(0,1), \end{aligned} \tag{2.2}$$

where the function y denotes the *output signal.* In equation (2.1) the output function y satisfies

$$y(t) = \int_0^1 k_s \chi_{[x_1,x_2]}(x)\, w_{xxt}(x,t)\, \mathrm{d}x$$

with $0 \leq x_1 < x_2 \leq 1$. The unknown function $w(t,x)$ and the forcing $f(t,x)$ are defined for $x \in [0,1]$, $t \geq 0$. The constants κ_1, κ_2 and k_s are positive and $g(\cdot)$ is a continuous function. In the context of the flexible structure encountered in Demetriou and Polycarpou [6], κ_1 denotes the *stiffness* parameter, κ_2 the *damping* parameter and k_s the sensor piezoceramic constant which is a piezoceramic material and geometry related quantity, see Banks *et al.* [4] and Dosch *et al.* [8].

The system given by (2.1) is a general form of the system studied by Demetriou and Polycarpou [6, 7]. Indeed, when the actuator (input) failure term $\beta(t,x)\, g(y)$ is written as

$$\beta(t,x)\, g(y) = \beta_1(t)\, \left(k_a \chi_{[x_1,x_2]}(x)\, u(t)\right) g(y)$$

with the *time profile* (Polycarpou and Helmicki [9]) of the failure given by

$$\beta_1(t) = \begin{cases} 0 & \text{if } t < T_f \\ 1 - e^{-\lambda(t-T_f)} & \text{if } t \geq T_f \end{cases}, \quad \lambda > 0, \tag{2.3}$$

and the nominal forcing (actuator) term given by

$$f(t,x) = [k_a \chi_{[x_1,x_2]}(x)\, u(t)]_{xx}, \qquad k_a > 0,$$

then equation (2.1) has exactly the same form as the beam equation considered in Demetriou and Polycarpou [6]. The time T_f denotes the unknown instance of the failure occurrence and the signal u denotes the input voltage to the patch. Similarly, k_a denotes the actuator piezoceramic constant, see Banks *et al.* [4]. Therefore, the above describe the dynamics of a flexible cantilevered beam before $(t < T_f)$ and after $(t \geq T_f)$ the occurrence of an anticipated actuator failure commencing at an unknown time T_f. In view of the above, the plant equation (2.1) can now be written as

$$w_{tt} + \kappa_1 w_{xxxx} + \kappa_2 w_{xxxxt} = \left[k_a \chi_{[x_1,x_2]}(x)\, u(t)\right]_{xx} + \beta_1(t) \left[k_a \chi_{[x_1,x_2]}(x)\, u(t)\, g(y(t))\right]_{xx}.$$

We begin by imposing the following assumptions on the parameters in problem (2.1) - (2.2):

(A_β)

$$\beta \in L^\infty\left(0, T, L^2(0,1)\right), \quad \sup_{t\in[0,\infty)} \|\beta(t)\| \le L. \tag{2.4}$$

(A_g) *There are positive constants* $\widetilde{C}_j$, $j=1,2$, with $\widetilde{C}_1 < \kappa_2/L$, *such that*

$$|g(\xi)| \le \frac{\widetilde{C}_1}{k_s}|\xi| + \widetilde{C}_2, \quad \text{for all } \xi \in \mathbb{R}. \tag{2.5}$$

(A_f) *The forcing term* f *satisfies*

$$f \in L^\infty\left(0, T, H^{-2}\right). \tag{2.6}$$

Our primary concern is to investigate the existence of a weak solution to (2.1) - (2.2). Our approach is in the spirit of Banks *et al.* [2, 3]. To this end, we define the notion of a weak solution as follows.

Definition 2.1. We denote by $\mathcal{L}_T$ the Banach space of functions defined on the rectangle $Q_T = [0,1]\times[0,T]$ and having the following properties:

1. If $w \in \mathcal{L}_T$, then
$$w \in C\left([0,T], H_0^2(0,1)\right). \tag{2.7}$$

2. For any $w \in \mathcal{L}_T$ there exists the weak derivative
$$w_t \in C\left([0,T], L^2(0,1)\right) \cap L^2\left(0,T,H^2(0,1)\right). \tag{2.8}$$

The norm in this space is given by

$$\|w\|_{\mathcal{L}_T} = \max_{t\in[0,T]}\{\|w_t(t)\| + \|w_{xx}(t)\|\} + \|w_{xxt}\|_{L^2(Q_T)}, \tag{2.9}$$

where $\|\cdot\|$ denotes the $L^2(0,1)$ norm.

Definition 2.2. A function $w \in \mathcal{L}_T$ is a weak solution of (2.1) - (2.2) if it satisfies the following identity for every $t \in [0,T]$:

$$\int_{Q_t}\left(-w_\tau\eta_\tau + \kappa_1 w_{xx}\eta_{xx} + \kappa_2 w_{xx\tau}\eta_{xx}\right) \mathrm{d}x\,\mathrm{d}\tau + \int_0^1 w_t(x,t)\,\eta(x,t)\,\mathrm{d}x = \int_0^1 \phi_1(x)\,\eta(x,0)\,\mathrm{d}x + \int_{Q_t} f\eta\,\mathrm{d}x\,\mathrm{d}\tau + \int_{Q_t}\beta g(y)\,\eta_{xx}\,\mathrm{d}x\,\mathrm{d}\tau \tag{2.10}$$

for all $\eta \in \mathcal{L}_T$. Here $Q_t = [0,1]\times[0,t]$. In addition, w must also satisfy

$$w(x,0) = \phi_0(x), \ \text{a.e. } x \in [0,1].$$

Next we will prove that if a solution exists then it must satisfy a certain a priori estimate.

Theorem 2.3. The following a priori estimate holds for the problem (2.1) – (2.2):

$$\|w_t(t)\|^2 + \kappa_1 \|w_{xx}(t)\|^2 + \epsilon \int_0^t \|w_{xx\tau}(\tau)\|^2 \, \mathrm{d}\tau \leq C, \tag{2.11}$$

where

$$C \equiv C\left(\|\phi_1\|, \|(\phi_0)_{xx}\|, \|f\|_{L^\infty(0,T,H^{-2})}, T\right).$$

Proof. Taking the L^2-inner product of (2.1) with w_t we get

$$\begin{aligned} &\langle w_{tt}(t), w_t(t)\rangle + \kappa_1\langle w_{xx}(t), w_{xxt}(t)\rangle + \kappa_2\langle w_{xxt}(t), w_{xxt}(t)\rangle \\ = \;& \langle \beta(t) g(y(t)), w_{xxt}(t)\rangle + \langle f(t), w_t(t)\rangle \end{aligned} \tag{2.12}$$

for almost all $t \in [0, T]$. Hence,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\|w_t(t)\|^2 + \frac{\kappa_1}{2}\|w_{xx}(t)\|^2\right] + \kappa_2 \|w_{xxt}(t)\|^2 \\ = \;& \langle f(t), w_t(t)\rangle + \langle \beta(t)\, g(y(t)), w_{xxt}(t)\rangle, \end{aligned}$$

which gives us

$$\begin{aligned} &\|w_t(t)\|^2 + \kappa_1 \|w_{xx}(t)\|^2 + 2\,\kappa_2 \textstyle\int_0^t \|w_{xx\tau}(\tau)\|^2 \, \mathrm{d}\tau = \|\phi_1\|^2 \\ &+\kappa_1 \|(\phi_0)_{xx}\|^2 + 2\displaystyle\int_0^t \langle \beta(\tau)\, g(y(\tau)), w_{xx\tau}(\tau)\rangle \mathrm{d}\tau + 2\int_0^t \langle f(\tau), w_\tau(\tau)\rangle \mathrm{d}\tau. \end{aligned} \tag{2.13}$$

Now, using the assumption (A_f) above, the fourth term on the right hand side of (2.13) can be bounded as follows:

$$\left|\int_0^t \langle f(\tau), w_\tau(\tau)\rangle \mathrm{d}\tau\right| \leq \frac{\delta}{2}\int_0^t \|w_{xx\tau}(\tau)\|^2 \, \mathrm{d}\tau + \frac{1}{2\delta}\int_0^t \|f(\tau)\|_{H^{-2}}^2 \, \mathrm{d}\tau.$$

Furthermore, the third term on the right hand side of (2.13) satisfies the following estimate:

$$\begin{aligned} &\left|\int_0^t \langle \beta(\tau)\, g(y(\tau)), w_{xx\tau}(\tau)\rangle \mathrm{d}\tau\right| \\ \leq \;& \int_0^t \|\beta(\tau)\| \, |g(y(\tau))| \, \|w_{xx\tau}(\tau)\| \, \mathrm{d}\tau \\ \leq \;& \int_0^t \left(\frac{\widetilde{C}_1}{k_s}|y(\tau)| + \widetilde{C}_2\right) \|\beta(\tau)\| \, \|w_{xx\tau}(\tau)\| \, \mathrm{d}\tau \\ \leq \;& \int_0^t \left(\frac{\widetilde{C}_1}{k_s}k_s \|w_{xx\tau}(\tau)\| + \widetilde{C}_2\right) L \, \|w_{xx\tau}(\tau)\| \, \mathrm{d}\tau \\ \leq \;& \int_0^t L\widetilde{C}_1 \|w_{xx\tau}(\tau)\|^2 \, \mathrm{d}\tau + \int_0^t L\widetilde{C}_2 \|w_{xx\tau}(\tau)\| \, \mathrm{d}\tau \end{aligned}$$

$$\leq L\widetilde{C}_1 \int_0^t \|w_{xx\tau}(\tau)\|^2 \,d\tau + \frac{1}{2\delta}\int_0^t (L\widetilde{C}_2)^2 \,d\tau + \frac{\delta}{2}\int_0^t \|w_{xx\tau}(\tau)\|^2 \,d\tau.$$

Now choose δ such that

$$\delta = \frac{1}{4}(\kappa_2 - L\widetilde{C}_1). \tag{2.14}$$

Then

$$\|w_t(t)\|^2 + \kappa_1 \|w_{xx}(t)\|^2 + \left(2\kappa_2 - 2\delta - 2L\widetilde{C}_1\right)\int_0^t \|w_{xx\tau}(\tau)\|^2 \,d\tau$$
$$\leq \|\phi_1\|^2 + \kappa_1 \|(\phi_0)_{xx}\|^2 + \tfrac{1}{\delta}(L\widetilde{C}_2)^2 T + \tfrac{1}{\delta}T\|f\|_{L^\infty(0,T;H^{-2})}.$$

Hence the proof is complete. □

For the rest of this section we let $\{\lambda_j\}_{j=1}^\infty$ and $\{\psi_j\}_{j=1}^\infty$ be the eigenvalues and eigenfunctions of the strictly positive self adjoint operator $A = \frac{d^4}{dx^4}$ with the dense domain in $L^2(0,1)$ given by

$$\mathcal{D}(A) = \{\phi \in H^4(0,1) : \phi'(0) = \phi(0) = 0,\ \phi'(1) = \phi(1) = 0\},$$

respectively. Note that the eigenvalues λ_j are simple and that the set of eigenfunctions $\{\psi_j\}$ form a complete orthonormal system in $L^2(0,1)$. Furthermore, for any $\phi \in L^2(0,1)$ we have

$$\phi = \sum_{j=1}^\infty \phi_j \psi_j, \quad \phi_j = \langle \phi, \psi_j \rangle,$$

$$\mathcal{D}(A) = \left\{\phi \in L^2(0,1) : \sum_{j=1}^\infty \lambda_j^2 |\phi_j|^2 < \infty\right\},$$

and for $\phi \in \mathcal{D}(A)$

$$A\phi = \sum_{j=1}^\infty \lambda_j \phi_j \psi_j.$$

For more details on the properties of A we refer the reader to [2].

We seek to approximate the solution of equation (2.1) via the following Galerkin approximation:

$$w^N(x,t) = \sum_{k=1}^N C_k^N(t)\,\psi_k(x), \quad C_k^N(t) = \langle w^N(t), \psi_k\rangle. \tag{2.15}$$

According to the Galerkin procedure we seek $\{C_k^N(t)\}$ such that

$$\frac{d^2}{dt^2}C_k^N(t) + \kappa_1\lambda_k C_k^N(t) + \kappa_2\lambda_k \frac{d}{dt}C_k^N(t) = \langle \beta(t) g\left(y^N(t)\right), (\psi_k)_{xx}\rangle + \langle f(t), \psi_k\rangle \tag{2.16}$$

for $k = 1, \ldots, N$, and

$$C_k^N(0) = \langle \phi_0, \psi_k \rangle, \qquad \frac{\mathrm{d}}{\mathrm{d}t} C_k^N(0) = \langle \phi_1, \psi_k \rangle. \tag{2.17}$$

The above equation is equivalent to

$$\begin{aligned} & \frac{\mathrm{d}^2}{\mathrm{d}t^2} \langle w^N(t), \psi_k \rangle + \kappa_1 \lambda_k \langle w^N(t), \psi_k \rangle + \kappa_2 \lambda_k \frac{\mathrm{d}}{\mathrm{d}t} \langle w^N(t), \psi_k \rangle \\ = \; & \langle \beta(t) g\left(y^N(t)\right), (\psi_k)_{xx} \rangle + \langle f(t), \psi_k \rangle. \end{aligned} \tag{2.18}$$

Multiplying (2.18) by $\frac{\mathrm{d}}{\mathrm{d}t} C_k^N(t)$ and summing over $k = 1, \ldots, N$ and then integrating over $[0, t]$ and using the proof of Theorem 2.3 we arrive at an estimate similar to (2.11) for the Galerkin approximates. Namely, we have

$$\begin{aligned} & \left\| w_t^N(t) \right\| + \kappa_1 \left\| w_{xx}^N(t) \right\|^2 + \epsilon \int_0^t \left\| w_{xxt}^N(\tau) \right\|^2 \mathrm{d}\tau \\ \leq \; & C \equiv C\left(\|\phi_1\|, \|(\phi_0)_{xx}\|, \|f\|_{L^\infty(0,T,H^{-2})}, T \right) \end{aligned} \tag{2.19}$$

for all $N = 1, 2, \ldots$ and $t \in [0, T]$.

Notice that in the derivation of (2.19) we used the fact that $\left\| (\phi_0)_{xx}^N \right\| \leq \|(\phi_0)_{xx}\|$ and $\left\| \phi_1^N \right\| \leq \|\phi_1\|$ and that C is a monotone increasing function of its arguments. Using (2.19) and following the arguments in Banks *et al.* [2], we can prove that there exists a subsequence denoted again by $w^N(t)$ that satisfies the following: $w^N(t) \to w(t)$ weakly in $H_0^2(0, 1)$ uniformly on $[0, T]$, and $w_{xxt}^N \to w_{xxt}$ weakly in $L^2(Q_t)$ for all $t \in [0, T]$.

Lemma 2.4. For any fixed $k = 1, 2, \ldots$ the set of functions

$$\left\{ \frac{\mathrm{d}^2}{\mathrm{d}t^2} C_k^N(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2} \langle w^N(t), \psi_k \rangle \right\}_{N=k}^{\infty}$$

is uniformly bounded in $L^2(0, T)$.

Proof. From the proof of Lemma 6.6 in Banks *et al.* [2] it follows that there exists a constant $C_3 > 0$ such that

$$\begin{aligned}
\int_0^T \left| \frac{\mathrm{d}^2}{\mathrm{d}t^2} \langle w^N(t), \psi_k \rangle \right|^2 \mathrm{d}t &\leq \int_0^T \left(C_3 + \left| \langle \beta(t)\, g\left(y^N(t)\right), (\psi_k)_{xx} \rangle \right| \right)^2 \mathrm{d}t \\
&= C_3^2 T + 2C_3 \int_0^T |\langle \beta(t) g(y^N(t)), (\psi_k)_{xx} \rangle| \, \mathrm{d}t \\
&\quad + \int_0^T |\langle \beta(t) g(y^N(t)), (\psi_k)_{xx} \rangle|^2 \, \mathrm{d}t \\
&\leq C_3^2 T + 2LC_3 \max\nolimits_{x \in [0,1]} |(\psi_k)_{xx}(x)| \int_0^T |g(y^N(t))| \, \mathrm{d}t \\
&\quad + L^2 \left(\max\nolimits_{x \in [0,1]} |(\psi_k)_{xx}(x)| \right)^2 \int_0^T |g(y^N(t))|^2 \, \mathrm{d}t.
\end{aligned}$$

Now using assumption (A_g) we get

$$\begin{aligned}
\int_0^T \left| g(y^N(t)) \right|^2 \mathrm{d}t &\leq \int_0^T \left(\frac{\widetilde{C}_1}{k_s} \left| y^N(t) \right| + \widetilde{C}_2 \right)^2 \mathrm{d}t \\
&\leq \int_0^T \left(\widetilde{C}_1 \left\| w^N_{xxt}(t) \right\| + \widetilde{C}_2 \right)^2 \mathrm{d}t \\
&\leq 2 \int_0^T \left(\widetilde{C}_1^2 \left\| w^N_{xxt}(t) \right\|^2 + \widetilde{C}_2^2 \right) \mathrm{d}t \\
&\leq 2\widetilde{C}_1^2 \int_0^T \left\| w^N_{xxt}(t) \right\|^2 \mathrm{d}t + 2\widetilde{C}_2^2 T.
\end{aligned}$$

From the above bound it immediately follows that $\int_0^T |g(y^N(t))| \, \mathrm{d}t$ is bounded as well. Hence, using the estimate (2.19) we see that there exists a constant $C_4 > 0$ such that

$$\int_0^T \left| \frac{\mathrm{d}^2}{\mathrm{d}t^2} \langle w^N(t), \psi_k \rangle \right|^2 \mathrm{d}t \leq C_4,$$

and the result is established. □

Now using Lemma 2.4, the compact embedding of $H^2(0,T) \subset C^1[0,T]$, see Adams [1], Theorem 5.4, and arguments similar to those presented in Corollary 6.7, Lemma 6.8 and Lemma 6.10 of Banks *et al.* [2] we can show that the function w has a weak derivative $w_t(t) \in L^2(0,1)$ for all $t \in [0,T]$, and that the subsequence $w_t^N(t) \to w_t(t)$, weakly in $L^2(0,1)$ uniformly in $t \in [0,T]$, and $w_t^N(t) \to w_t(t)$ strongly in $L^2(Q_T)$.

Lemma 2.5. The set of functions

$$\left\{ g\left(y^N(t)\right) \right\}_{N=1}^{\infty}$$

is bounded and therefore relatively weakly compact in $L^2(0,T)$.

Proof. From the proof of Lemma 2.4 we see that there exists a $C_5 > 0$ such that

$$\|g(y^N)\|^2_{L^2(0,T)} = \int_0^T \left|g(y^N(t))\right|^2 \,\mathrm{d}t \le C_5$$

and the result follows. □

The next result immediately follows from Lemma 2.5.

Corollary 2.6. There exists a function $\widetilde{g} \in L^2(0,T)$ such that

$$g(y^N) \to \widetilde{g} \text{ weakly in } L^2(0,T),$$

along a subsequence.

We denote by P_M $(M = 1, 2, \ldots)$ the class of functions $\eta(x,t)$ which can be written in the form

$$\eta(x,t) = \sum_{k=1}^{M} a_k(t)\,\psi_k(x), \tag{2.20}$$

where $a_k(t)$ are arbitrary C^1 smooth functions on $[0,T]$. Let

$$P = \bigcup_{M=1}^{\infty} P_M. \tag{2.21}$$

Clearly the set P is dense in the class $\mathcal{L}_T$.

Now multiply (2.18) by an arbitrary smooth function $a_k(t)$, take the sum from $k = 1$ to M and integrate over the rectangle Q_t. Integrating by parts with respect to t in the first term and with respect to x (twice) in the second and third terms and taking into account the initial condition (2.17) and the boundary conditions for ψ_k we get

$$\begin{aligned}\int_{Q_t} \left[-w^N_\tau \eta_\tau + \kappa_1 w^N_{xx} \eta_{xx} + \kappa_2 w^N_{xx\tau} \eta_{xx}\right] \mathrm{d}x\,\mathrm{d}\tau + \int_0^1 w^N_\tau \eta\,\mathrm{d}x\bigg|_{\tau=0}^{\tau=t} \\ = \int_{Q_t} \beta g(y^N)\,\eta_{xx}\,\mathrm{d}x\,\mathrm{d}\tau + \int_{Q_t} f\eta\,\mathrm{d}x\,\mathrm{d}\tau,\end{aligned} \tag{2.22}$$

which is satisfied for any $\eta \in P_M$ with $M \le N$, i. e., η has the form (2.20). Now we fix $\eta \in P_M$ with $M \le N$. Using the above results we can pass to the limit $N \to \infty$ in (2.22) and obtain

$$\begin{aligned}\int_{Q_t} \left[-w_\tau \eta_\tau + \kappa_1 w_{xx} \eta_{xx} + \kappa_2 w_{xx\tau} \eta_{xx}\right] \mathrm{d}x\,\mathrm{d}\tau + \int_0^1 w_\tau \eta\,\mathrm{d}x\bigg|_{\tau=0}^{\tau=t} \\ = \int_{Q_t} \beta\widetilde{g}\eta_{xx}\,\mathrm{d}x\,\mathrm{d}\tau + \int_{Q_t} f\eta\,\mathrm{d}x\,\mathrm{d}\tau\end{aligned} \tag{2.23}$$

for all $t \in [0,T]$. Here (2.23) is satisfied for all $\eta \in P_M$, where M is an arbitrary positive integer, and hence for any $\eta \in \mathcal{L}_T$ because P is dense in $\mathcal{L}_T$.

Remark 2.7. The only difference between (2.23) and the Definition 2.2 of the weak solution is that in (2.23) we have a $\widetilde{g}$ (a certain unknown function in $L^2(0,T)$) instead of $g(y)$ in (2.10). Therefore, the proof of the existence theorem will be complete if we prove that $g(y(t)) = \widetilde{g}(t)$ a. e. in $(0,T)$. We hope to discuss this result as well as the uniqueness of solutions to problem (2.1) – (2.2) in a forthcoming paper. This may require additional assumptions on the function g, for example, monotonicity (cf. [2, 3]).

3. ESTIMATOR AND FAILURE DETECTION

In this section we present the diagnostic observer that is used to monitor the plant for fault detection. A model-based state observer and an adaptive parameter estimator comprises this diagnostic observer which when viewed in a variational weak form yields

$$\begin{aligned} \langle \widehat{w}_{tt}, \eta\rangle + \sigma_2(\widehat{w}_t, \eta) + \sigma_1(\widehat{w}, \eta) &= \langle Bu, \eta\rangle + \langle BZ^T\widehat{\theta}u, \eta\rangle \\ &= \langle Bu, \eta\rangle + b(\widehat{\theta}; Zu, \eta) \end{aligned} \tag{3.1}$$

$$\langle \widehat{\theta}_t, \psi\rangle = b(\psi; Zu, w_t - \widehat{w}_t), \qquad \psi \in I\!R^q \tag{3.2}$$

as was presented in Demetriou and Polycarpou [6], where:

(i) The input term is given by $\langle f, \eta\rangle = \langle Bu, \eta\rangle$, where u denotes the input signal, and $B : I\!R^1 \to H^{-2}(0,1)$ is the associated input operator.

(ii) The failure function $g(y)$ is assumed to satisfy $g(y) = \sum_{i=1}^{q} \theta_i Z_i(y) = \theta^T Z(y)$, where the weights θ_i are unknown parameters and the $Z_i(y)$ are assumed to be known nonlinear functions of the output signal y that satisfy assumption (A_g).

(iii) The θ-parametrized bilinear form $b(\theta; \psi, \eta)$ is given by

$$\begin{aligned} b(\theta; Zu, \eta) &= \int_0^1 \left(k_a \chi_{[x_1,x_2]}(x)\, u(t)\right) \theta^T Z(y)\, \eta_{xx}\, \mathrm{d}x \\ &= \theta^T \int_0^1 \left(k_a \chi_{[x_1,x_2]}(x)\, Z(y)\, u(t)\right) \eta_{xx}\, \mathrm{d}x. \end{aligned}$$

(iv) The sesquilinear forms $\sigma_i(\cdot,\cdot) : H_0^2(0,1) \times H_0^2(0,1) \to I\!R^1$, $i = 1, 2$, are given by

$$\sigma_1(w, \eta) = \kappa_1 \int_0^1 w_{xx}(x,t)\eta_{xx}(x)\, \mathrm{d}x,$$

$$\sigma_2(w_t, \eta) = \kappa_2 \int_0^1 w_{txx}(x,t)\eta_{xx}(x)\, \mathrm{d}x.$$

(v) The function $\widehat{w}(x,t)$ denotes the estimate of the plant state $w(x,t)$ and $\widehat{\theta}(t)$ is the on-line (adaptive) parameter estimate of the unknown vector of coefficients θ.

We assume that we have the following bounds on the bilinear forms

$$\begin{array}{ll} \alpha_1^l\|\eta\|^2 \le \sigma_1(\eta,\eta), & \sigma_1(\eta,\phi) \le \alpha_1^u\|\eta\|\|\phi\| \\ \alpha_2^l\|\eta\|^2 \le \sigma_2(\eta,\eta), & \sigma_2(\eta,\psi) \le \alpha_2^u\|\eta\|\|\phi\| \end{array}, \quad \eta,\phi \in H_0^2(0,1),$$

$$b(\psi; Zu,\eta) \le \beta|\psi|\|\eta\|, \qquad \eta \in H_0^2(0,1), \quad \psi \in \mathbb{R}^q.$$

We expect that existence of solutions to the proposed estimator (3.1) – (3.2) can be derived in a similar fashion as in the case of the plant (2.1) – (2.2). We hope to discuss the details leading to existence-uniqueness of the solution to (3.1) – (3.2) in a future paper. The stability of the monitoring scheme is summarized below. A Lyapunov functional is used in order to derive the adaptation laws for the parameter updates. Before that, we write the system in terms of the *state error* $e = w - \widehat{w}$ and *parameter error* $\widetilde{\theta} = \widehat{\theta} - \theta$ by combining (2.1), (3.1) and (3.2):

$$\begin{aligned} \langle e_{tt},\eta\rangle + \sigma_2(e_t,\eta) + \sigma_1(e,\eta) &= \langle BZ^T(\beta_1\theta - \widehat{\theta})\,u,\eta\rangle \\ &= -\langle B\Phi Z^T\theta u,\eta\rangle - \langle BZ^T\widetilde{\theta}u,\eta\rangle \end{aligned} \qquad \eta \in H_0^2(0,1) \quad (3.3)$$

$$\langle \widetilde{\theta}_t,\psi\rangle = \mu\langle BZ^T\psi u, e_t\rangle, \qquad \psi \in \mathbb{R}^q, \quad (3.4)$$

where Φ, defined by $\Phi = 1 - \beta_1$, satisfies $\dot{\Phi} = -\lambda\Phi$ and $\mu > 0$ denotes the *adaptive gain*. The Lyapunov functional is now given by

$$V(t) = \frac{1}{2}|e_t|^2 + \frac{1}{2}\sigma_1(e,e) + \frac{1}{2\mu}|\widetilde{\theta}|^2 + \frac{1}{2}|\Phi(t)|^2.$$

The derivative of V evaluated along the trajectories of the error equations (3.3) and (3.4), produces

$$\begin{aligned} \dot{V} &= -\sigma_2(e_t,e_t) - \langle B\Phi Z^T\theta u, e_t\rangle - \lambda|\Phi|^2 \\ &\le -c_1|e_t|^2 - c_2|\Phi(t)|^2, \end{aligned}$$

for some c_1, $c_2 > 0$, where we used the fact that $\dot{\Phi} = -\lambda\Phi$. The above yields $\dot{V} \le 0$ and thus we have stability. Furthermore, by integrating the above expression over a finite interval $[T, T+t]$, we have

$$V(T+t) + \int_T^{T+t} \left\{c_1|e_\tau(\tau)|^2 + c_2|\Phi(\tau)|^2\right\} d\tau \le V(T)$$

which, via an application of Barbalat's lemma [5] for infinite dimensional systems, yields

$$\lim_{t\to\infty} |e(t)| = \lim_{t\to\infty} |e_t(t)| = 0,$$

and $\widetilde{\theta} \in L_\infty(0,\infty;\mathbb{R}^q)$. Parameter convergence can be established by imposing *persistence of excitation* [5]. From the above, it can be observed that for $t < T_f$, both the state error $e = w - \widehat{w}$ and the output error $y - \widehat{y}$ remain zero, attain a nonzero value after the failure and converge to zero as $t \to \infty$. Therefore, by simply monitoring the output error, the failure occurrence can be detected. Furthermore, by imposing the additional condition of persistence of excitation, failure diagnosis can be established via the convergence of $\widehat{\theta} - \theta$.

4. APPROXIMATION THEORY

In this section we summarize the finite dimensional approximation scheme necessary for the implementation of the diagnostic observer (3.1) – (3.2).

For each $N = 1, 2, \ldots$, let H^N be a finite dimensional subspace of $L^2(0,1)$ with $H^N \subset H_0^2(0,1)$. The Galerkin equations for $\widehat{w}^N$ and $\widehat{\theta}^N$ in H^N and $I\!R^q$ corresponding to (3.1) and (3.2) are given by

$$\begin{aligned} \langle \widehat{w}_{tt}^N, \eta^N\rangle + \sigma_2(\widehat{w}_t^N, \eta^N) + \sigma_1(\widehat{w}^N, \eta^N) &= \langle Bu, \eta^N\rangle + \langle BZ^T\widehat{\theta}^N u, \eta^N\rangle \\ &= \langle Bu, \eta^N\rangle + b(\widehat{\theta}^N; Zu, \eta^N), \end{aligned} \tag{4.1}$$

$$\begin{aligned} \langle \widehat{\theta}_t^N, \psi^N\rangle &= \langle w_t - \widehat{w}_t^N, BuZ\psi^N\rangle \\ &= b(\psi^N; Zu, w_t - \widehat{w}_t^N), \end{aligned} \tag{4.2}$$

$$\widehat{w}^N(0) \in H^N, \qquad \widehat{w}_t^N(0) \in H^N, \qquad \widehat{\theta}^N(0) \in I\!R^q. \tag{4.3}$$

We make the following standard Galerkin approximation assumptions. First define orthogonal projections $P^N : L^2(0,1) \to H^N$ of $L^2(0,1)$ onto H^N and $P_\theta^N : I\!R^q \to I\!R^q$.

(A1) The finite dimensional subspaces satisfy $H^N \subset H_0^2(0,1)$.

(A2) The functions $P^N\widehat{w}$ and $P_\theta^N\widehat{\theta}$ with $P^N\widehat{w} \in L^2(0,T,H^N)$ and $P_\theta^N\widehat{\theta} \in L^2(0,T,I\!R^q)$ are such that

- (i) $P^N\widehat{w} \to \widehat{w}$ in $C([0,T], H_0^2(0,1))$,
- (ii) $P^N\widehat{w}_t \to \widehat{w}_t$ in $C([0,T], L^2(0,1))$ and $L^2(0,T,H^2(0,1))$,
- (iii) $P_\theta^N\widehat{\theta} \to \widehat{\theta}$ in $C([0,T]; I\!R^q)$.

Using the above assumptions, we can prove the following convergence result.

Theorem 4.1. Assume that $(A1)$ and $(A2)$ hold. Let $(\widehat{w}, \widehat{\theta})$ be the solution to the initial value problem (3.1) – (3.2) and for each $n = 1, 2, \ldots$, let $(\widehat{w}^N, \widehat{\theta}^N)$ be the solution to the initial value problem (4.1) – (4.2) with

$$\widehat{w}^N(0) = P^N\widehat{w}(0), \;\; \widehat{w}_t^N(0) = P\widehat{w}_t(0), \;\; \widehat{\theta}^N(0) = P^N\widehat{\theta}(0).$$

Then

(i) $\widehat{w}^N \to \widehat{w}$ in $C([0,T], H_0^2(0,1))$,

(ii) $\widehat{w}_t^N \to \widehat{w}_t$ in $C([0,T], L^2(0,1))$ and $L^2(0,T,H_0^2(0,1))$,

(iii) $\widehat{\theta}^N \to \widehat{\theta}$ in $C([0,T], I\!R^q)$.

Proof. Define $\Delta^N = \widehat{w}^N - P^N\widehat{w}$ and $\delta^N = \widehat{\theta}^N - P^N\widehat{\theta}$. Since

$$|\widehat{w}^N - \widehat{w}| = |\widehat{w}^N - P^N\widehat{w} + P^N\widehat{w} - \widehat{w}| \le |\widehat{w}^n - P^N\widehat{w}| + |P^N\widehat{w} - \widehat{w}|.$$

It suffices, by $(A2)$, to show that

$$\Delta^N \to 0 \text{ in } C([0,T], H_0^2(0,1)),$$
$$\Delta_t^N \to 0 \text{ in } C([0,T], L^2) \text{ and } L^2(0,T,H_0^2(0,1)).$$

Similarly, for $\widehat{\theta}^N$, it suffices to show

$$\delta^N \to 0 \text{ in } C([0,T], \mathbb{R}^q).$$

We use $\eta = \Delta^N$ in (3.1) and $\eta^N = \Delta^N$ in (4.1) as the test functions. When equation (4.1) is subtracted from equation (3.1) we obtain

$$\langle \widehat{w}_{tt} - \widehat{w}_{tt}^N, \Delta^N \rangle + \sigma_2(\widehat{w}_t - \widehat{w}_t^N, \Delta^N) + \sigma_1(\widehat{w} - \widehat{w}^N, \Delta^N) = b(\widehat{\theta} - \widehat{\theta}^N; Zu, \Delta^N). \tag{4.4}$$

Similarly, use $\psi = \delta^N$ in (3.2) and $\psi^N = \delta^N$ in (4.2) and subtract equation (4.2) from equation (3.2) to obtain

$$\langle \widehat{\theta}_t - \widehat{\theta}_t^N, \delta^N \rangle = b(\delta^N; Zu, \widehat{w}_t^N - \widehat{w}_t). \tag{4.5}$$

We now consider for $\gamma > 0$ the positive functional

$$\gamma\sigma_1(\Delta^N, \Delta^N) + \gamma|\Delta_t^N|^2 + 2\langle \Delta^N, \Delta_t^N \rangle + \sigma_2(\Delta^N, \Delta^N) + \gamma|\delta^N|^2$$

which can be shown to be bounded below by $\kappa_L \Xi$, where

$$\Xi(\Delta^N, \Delta_t^N, \delta^N) \triangleq (\|\Delta^N\|^2 + |\Delta_t^N|^2 + |\delta^N|^2),$$

and κ_L is a positive constant which is a function of the parameter γ. When the above is used in the equations above, we obtain the identity

$$\begin{aligned}
&\frac{d}{dt}\left(\gamma\sigma_1(\Delta^N, \Delta^N) + \gamma|\Delta_t^N|^2 + 2\langle \Delta^N, \Delta_t^N \rangle\right) \\
&\quad + \frac{d}{dt}\left(\sigma_2(\Delta^N, \Delta^N) + \gamma|\delta^N|^2\right) + \gamma\sigma_2(\Delta_t^N, \Delta_t^N) + \sigma_1(\Delta^N, \Delta^N) \\
&= 2\gamma\sigma_1(\Delta^N, \Delta_t^N) + 2\gamma\langle \widehat{w}_{tt}^N - \widehat{w}_{tt}, \Delta_t^N \rangle + 2\gamma\langle \widehat{w}_{tt} - P^N\widehat{w}_{tt}, \Delta_t^N \rangle \\
&\quad + 2|\Delta_t^N|^2 + 2\langle \widehat{w}_{tt}^N - \widehat{w}_{tt}, \Delta^N \rangle + 2\langle \widehat{w}_{tt} - P^N\widehat{w}_{tt}, \Delta^N \rangle + 2\sigma_2(\Delta^N, \Delta_t^N) \\
&\quad + 2\gamma\langle \widehat{\theta}_t^N - \widehat{\theta}_t, \delta^N \rangle + 2\gamma\langle \widehat{\theta}_t - P^N\widehat{\theta}_t, \delta^N \rangle + 2\gamma\sigma_2(\Delta_t^N, \Delta_t^N) + 2\sigma_1(\Delta^N, \Delta^N).
\end{aligned} \tag{4.6}$$

Using the fact that $\langle \widehat{w}_{tt} - P^N\widehat{w}_{tt}, \eta \rangle = 0$ for all $\eta \in H^N$ and $\langle \widehat{\theta}_t - P^N\widehat{\theta}_t, \psi \rangle = 0$ for all $\psi \in \mathbb{R}^q$, using the bounds on the sesquilinear forms and assumptions (A1) and

(A2), we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma\sigma_1(\Delta^N,\Delta^N)+\gamma|\Delta_t^N|^2+2\langle\Delta^N,\Delta_t^N\rangle\right)\\
&\quad+\frac{\mathrm{d}}{\mathrm{d}t}\left(\sigma_2(\Delta^N,\Delta^N)+\gamma|\delta^N|^2\right)+\gamma\sigma_2(\Delta_t^N,\Delta_t^N)+\sigma_1(\Delta^N,\Delta^N)\\
&\leq \gamma\alpha_2^u\epsilon\|P^N\widehat{w}_t-\widehat{w}_t\|^2+\frac{\gamma\alpha_2^u}{4\epsilon}\|\Delta_t^N\|^2+\gamma\alpha_1^u\epsilon\|P^N\widehat{w}-\widehat{w}\|^2\\
&\quad+\frac{\gamma\alpha_1^u}{4\epsilon}\|\Delta_t^N\|^2+\alpha_2^u\epsilon\|P^N\widehat{w}_t-\widehat{w}_t\|^2+\frac{\alpha_2^u}{4\epsilon}\|\Delta^N\|^2+\alpha_1^u\epsilon\|P^N\widehat{w}-\widehat{w}\|^2\\
&\quad+\frac{\alpha_1^u}{4\epsilon}\|\Delta^N\|^2+\gamma\beta\epsilon\|P^N\widehat{\theta}-\widehat{\theta}\|^2+\frac{\gamma\beta}{4\epsilon}\|\Delta_t^N\|^2+2|\Delta_t^N|^2+\beta\epsilon\|\delta^N\|^2\\
&\quad+\frac{\beta}{4\epsilon}\|\Delta^N\|^2+\beta\epsilon\|P^N\widehat{\theta}-\widehat{\theta}\|^2+\frac{\beta}{4\epsilon}\|\Delta^N\|^2+\gamma\beta\epsilon\|\delta^N\|^2+\frac{\gamma\beta}{4\epsilon}\|P^N\widehat{w}_t-\widehat{w}_t\|^2.
\end{aligned}
\tag{4.7}
$$

Integrating the above inequality from 0 to t, and using the triangle inequality we obtain the following inequality

$$
\begin{aligned}
&\Xi(\Delta^N,\Delta_t^N,\delta^N)+\int_0^t\left(\|\Delta_s^N\|^2+\|\Delta^N\|^2\right)\mathrm{d}s\\
\leq\;& c_1\int_0^t\Xi(\Delta^N,\Delta_s^N,\delta^N)\,\mathrm{d}s+c_2\int_0^t\Theta^N(s)\,\mathrm{d}s,
\end{aligned}
\tag{4.8}
$$

where

$$\Theta^N(t)=\|P^N\widehat{w}-\widehat{w}\|^2+\|P^N\widehat{w}_t-\widehat{w}_t\|^2+\|P^N\widehat{\theta}-\widehat{\theta}\|^2$$

and c_1, c_2 are some positive constants. Assumption $(A2)$ implies that

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\int_0^t\Theta^N(s)\,\mathrm{d}s=\lim_{n\to\infty}\Theta^N(s)\,\mathrm{d}s=0, \tag{4.9}$$

which together with Gronwall's lemma yields

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\left(\|\Delta^N\|^2+|\Delta_t^N|^2+|\delta^N|^2\right)=0. \tag{4.10}$$

which proves most of the assertions of the theorem. Finally, the L^2 convergence of $\widehat{w}^N$ to $\widehat{w}$ is shown using (4.8)–(4.10). □

5. NUMERICAL RESULTS

In this section we consider the model studied by Demetriou and Polycarpou [6, 7], the details of which are given in Section 2. This is a special case of our general model (2.1). For our set of simulations, we assume that the coordinates of the patch are $x_1 = 0.45$ m and $x_2 = 0.55$ m. The stiffness $\kappa_1 = 0.491\,N\cdot\mathrm{m}^2$ with the damping

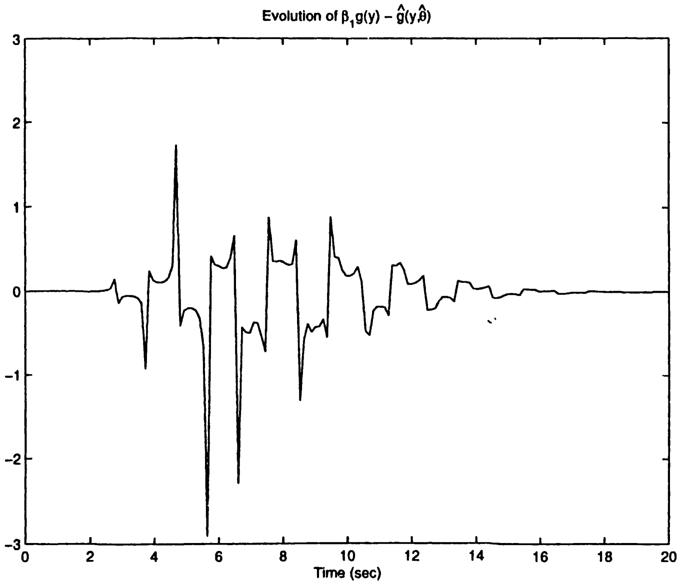

Fig. 5.1. Evolution of the difference $\beta_1(t)\, g(y) - \widehat{g}(y, \widehat{\theta}(t))$ for an incipient failure time profile.

$\kappa_2 = 0.1623 \times 10^{-3}\,\mathrm{Kg} \cdot \mathrm{m}^3/\mathrm{s}$. Cubic splines were used to discretize the spatial domain and in this case $N = 16$. The failure term

$$10\beta_1(t)\, g(y) = 10\beta_1(t)\, \frac{y}{1+y^2} = \theta Z(y)$$

and the adaptive gain $\mu = 2$. The time profile of the failure (2.3) is given by

$$\beta_1(t) = \begin{cases} 0 & \text{if } t < 2 \\ 1 - e^{-0.5(t-2)} & \text{if } t \geq 2. \end{cases}$$

The patch voltage is taken as $u(t) = 10.0 \sin(150\pi t)$. Zero initial conditions of both the plant state and the estimator state were considered for simplicity. In addition, the initial guess for $\widehat{\theta}(0)$ was also set to zero. The evolution of the difference of the failure term $\theta(t)\, g(y) - \widehat{g}(y, \widehat{\theta})$ is depicted in Figure 5.1.

The unknown parameter $\theta = \beta_1(t)10$ and its estimate $\widehat{\theta}(t)$ are depicted in Figure 5.2a and their difference (parameter error) is presented in Figure 5.2b. From both sets of plots, it can be observed that the time of failure ($T_f = 2$) is identified.

Furthermore, the evolution of the output error $y - \widehat{y}$ is depicted in Figure 5.3, where it is observed that for $t < T_f$ the output error remains at zero, attains a nonzero value at $t = 2$ (detection) and then converges to zero as $t \to \infty$.

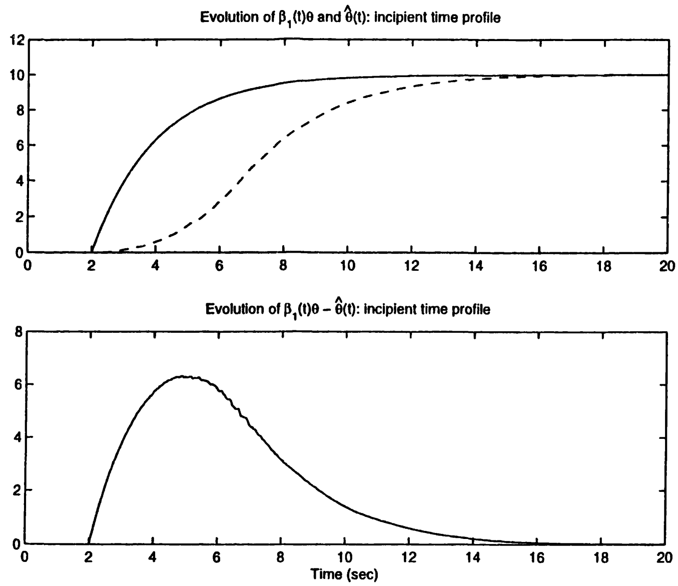

Fig. 5.2. Evolution of (a) $10\beta_1(t)$ and failure terms $\widehat{\theta}(t)$ (dashed), and (b) their difference $10\beta_1(t) - \widehat{\theta}(t)$ for an incipient failure time profile.

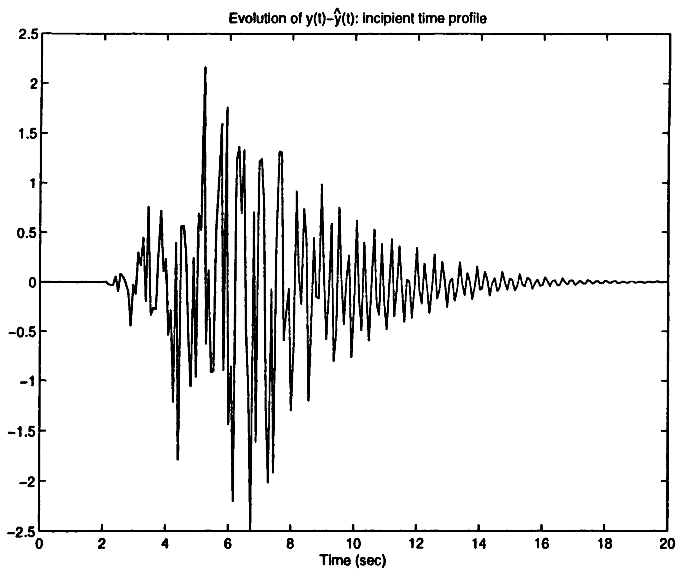

Fig. 5.3. Evolution of the output error $y(t) - \widehat{y}(t)$.

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